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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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SOME VARIATIONS OF THE MULTIPLE-CHOICE QUESTION

H. F. S. JONAH and M. W. KELLER, Purdue University

1. Introduction. In attempting to construct multiple-choice items for objective tests in mathematics which will test the students' ability to solve particular kinds of problems certain difficulties are encountered. Thus, for some types of problems, it is possible for the student to determine the correct answer without actually solving the problem when it is stated in the conventional manner. For example, in the problem

Example 1. The roots of the equation $x^2+5x+6=0$ are (1) -2, -3 (2) 2, 3 (3) -6, 1 (4) none of the proposed answers is correct.*

the correct response can be found by substitution. Consequently, such an item does not necessarily test the ability of the student to solve a quadratic equation. To obviate this possibility some constructors state the problem in such a manner that it is necessary to solve the problem first, and then perform some operation or operations on the answer. Thus, instead of stating the problem in the form given above, it would be given in a manner similar to the following:

Example 2. The sum of the squares of the roots of the equation $x^2+5x+6=0$ is (1) -13 (2) 13 (3) 37 (4) none of the proposed answers is correct.

For problems of this type this is a simple and satisfactory solution of the difficulty. There are other types of problems, however, where such a solution is not possible.

2. Object. It is the purpose of this paper to propose for other types of problems a different variation of the multiple-choice item which the authors have devised in an effort to minimize the possibility of the student determining the correct response without solving the problem. In addition some suggestions will be made on the use of multiple-choice items for testing the students' ability to graph and sketch curves. This is a type of item which has not been used very extensively by other constructors.

The items we wish to discuss formed part of a test which the authors prepared in cooperation with the Division of Educational Reference for the Army Specialized Training Program. Hence the exact problems are restricted. Consequently, the examples which we shall use will not be the items included on this test but they will be similar. The comparison, therefore, cannot be exact. Since

* This fourth response has been used uniformly on the various tests constructed by the authors. If the student is merely guessing this does not change the theoretical probability of guessing the correct response. Since it is often difficult, however, to construct a third plausible distractor it is believed that this procedure is a satisfactory solution in such cases because the student may have devised a better distractor. This scheme permits him to use it. When the fourth response is the correct one then such a scheme permits the student to solve the problem incorrectly yet mark the correct response. It is the opinion of the authors that this weakness is more than compensated for by its advantages.

the problems will follow very closely the problems actually used it is believed that it will be possible to obtain a relatively clear picture of their effectiveness.

The test was given to all students regularly enrolled in AST Mathematics 407. The number taking the test was 385. The validity of each item, that is, how well each item discriminates between students making high grades on the test and those making low grades on the test, was determined. The method used for determining the validity index was that proposed by J. C. Flanagan.* This index is the estimate of the product moment coefficient of correlation between total test score and a given item based on the per cent obtaining the correct response to the item of those scores which were in the upper and lower 27% of the group. Since this is the product moment coefficient of correlation, an index of 0.19 is significant at the 5% level, and an index of 0.25 is significant at the 1% level. The reliability of the test was 0.90 when the short approximation form of the Richardson-Kuder formula was used. It should be pointed out that this formula tends to underestimate so that the actual value will be more than this.

3. A proposed solution. In analytical geometry when it is desired to determine if the student can find the equation of a line through two given points the objective form of this problem commonly used is

Example 3. The equation of the line passing through the points (2, 0) and (1, 2) is

(0.37; 78) (1) $2x-y=4$ (2) $2x+y=4$ (3) $2x-3y+4=0$ (4) none of the proposed answers is correct.

The first number, 0.37, in the parentheses is the validity index and the second number, 78, is the per cent of correct responses. The same notation will be used in the remainder of the examples which will be discussed.

As was mentioned previously the correct response to problems of this kind can be determined by substitution without actually solving the problem even though, in making up the distractors, the incorrect responses have one of the given points as a solution. It should be noted that the validity index of this item was definitely significant. Nevertheless one cannot be sure that the desired objective—to determine whether the student can find the equation of a line through two given points—has been achieved. In trying to devise an item which would require the student to work the problem we included on the test some experimental items. These items directed the student to find the required equation and then identify the determined coefficients with the general form. Thus, in this form Example 3 would be given:

* J. C. Flanagan, General considerations in the selection of test items and a short method of estimating the product moment coefficient from data at the tails of the distribution. Journal of Educational Psychology 30: 674–680, 1939.

Example 4. The equation of the line through (2, 0) and (1, 2) is

$$ax + by + c = 0 \quad (a \text{ is a positive integer})$$

where

(0.61; 66) $a=$(1) 2 (2) 3 (3) 1 (4) none of the proposed answers is correct,
(0.54; 63) $b=$(1) -3 (2) 1 (3) 2 (4) none of the proposed answers is correct,
(0.42; 42) $c=$(1) -4 (2) 4 (3) 6 (4) none of the proposed answers is correct.

For those topics which lend themselves to the use of this method of identification of coefficients it is believed that this type of objective item will encourage the student to work the problem. It certainly makes the trial and error method of substitution for determining the correct answer practically impossible. At the same time it has a tendency to discourage random guessing.

To machine score such a problem each part was scored separately. The correct identification of each coefficient was counted one on the total score.

The validity indexes for this problem indicates the possibility that the validity of the item is increased by this method over that of the same problem when it is given in the conventional form. Further experimentation is being conducted to determine the correctness of these tentative conclusions which were suggested by this pair of parallel problems. At least, for the ten experimental problems of this type included on the test, it can be stated that in every case each response—of which there were 37—was significant at well above the 1% level.

Because the method of determining or stating a correct answer by identification of coefficients is not generally used for solving problems in the classroom it might be thought that this would be confusing to the student, and consequently he would omit these problems. Apparently this was not the case in general since only about 1% omitted problems of this type while items of the conventional type were frequently omitted more often. There is some ambiguity in the statement of problems like Example 4 where the only restriction placed on a is that it be a positive integer. Although this might be confusing to an instructor of mathematics there was no indication during the administration of the test that it caused the student any difficulty. The statistical data also gives no such indication.

4. Some illustrative examples. In order to suggest some of the possible uses of this form of objective problem, and at the same time to give more information about the difficulty and general high validity of such items two additional examples will be given.

Example 5. The equation of the ellipse with semi-major axis of 5 on the x-axis, semi-minor axis of 3, and center at the origin is

$$ax^2 + bx + cy^2 + dy = e \quad (a \text{ is a positive integer})$$

where

(0.57; 66) $a =$ (1) 10 (2) 25 (3) 9 (4) none of the proposed answers is correct,
(0.51; 62) $b =$ (1) 0 (2) -10 (3) -6 (4) none of the proposed answers is correct,
(0.45; 57) $c =$ (1) 6 (2) 25 (3) 9 (4) none of the proposed answers is correct,
(0.72; 74) $d =$ (1) 0 (2) -10 (3) -6 (4) none of the proposed answers is correct,
(0.29; 53) $e =$ (1) 60 (2) 225 (3) 30 (4) none of the proposed answers is correct.

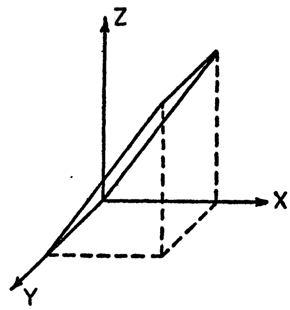


FIG. 1

Example 6. In Fig. 1 the plane PT passes through the point (3, 2, 4). The equation of the plane is

$$Ax + By + Cz + D = 0 \quad (A \text{ is a positive integer})$$

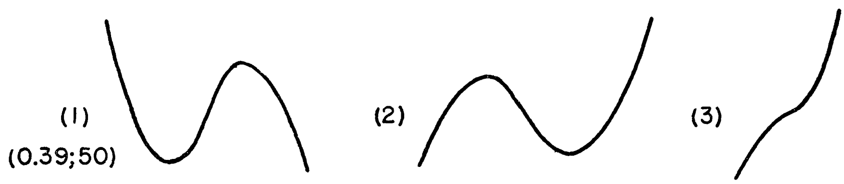
where
(0.45; 30) $A =$ (1) 4 (2) 3 (3) 1 (4) none of the proposed answers is correct,
(0.55; 33) $B =$ (1) 3 (2) 2 (3) 4 (4) none of the proposed answers is correct,
(0.34; 26) $C =$ (1) 0 (2) -3 (3) -2 (4) none of the proposed answers is correct,
(0.49; 54) $D =$ (1) 0 (2) -20 (3) -9 (4) none of the proposed answers is correct.

It should be noted that for any single problem there is considerable variation in the validity index and difficulty from response to response. This suggests the possibility of experimental investigations to determine the factors which are operating to cause these significant differences.

5. Curve sketching. Although no satisfactory method has thus far been proposed to test the ability of a student to sketch curves by the use of multiple-choice items various kinds of problems have been used to a limited extent to test certain parts of the process. Problems asking the student to determine the symmetry, extent, intercepts, *etc.*, of different equations have been used. These items are quite satisfactory to indicate whether the student understands that particular part of curve sketching. Since it is not possible to ask the student to sketch a curve when the test is an objective test using multiple-choice items it was believed that problems which require the recognition of the correct curve for a given equation and the correct equation for a given graph of a conic would help to give a more complete picture of the students' ability to sketch curves if used in conjunction with the kinds of problems previously suggested. Problems

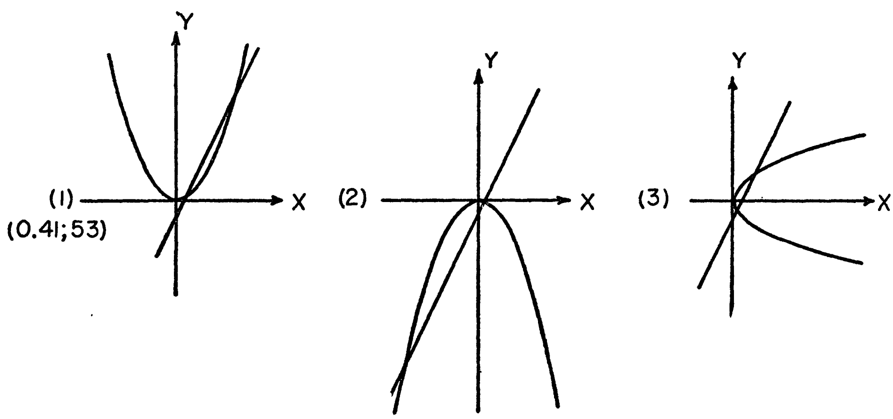
of this type were included on this test. The following examples will suggest some of the possibilities of this type of item. The statistical data included indicates the general effectiveness of such items in terms of their validity and difficulty.

Example 7. The graph of the equation $y=x^3+x^2-2x$ has the general shape



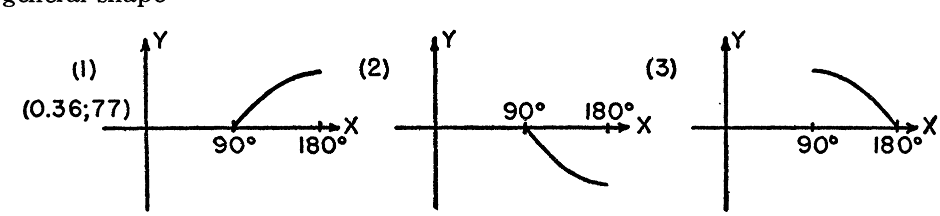
(4) none of the proposed answers is correct.

Example 8. The area bounded by the curves $y=-x^2$, and $4x-2y=1$ is sketched in



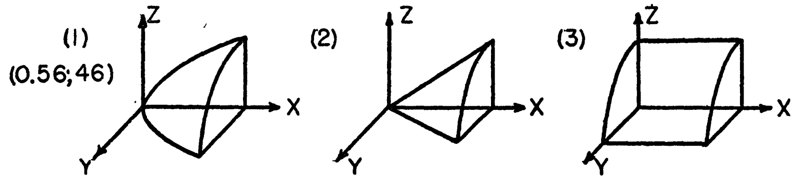
(4) none of the proposed answers is correct.

Example 9. The graph of $y=\sin x$ for values of x between 90° and 180° has the general shape



(4) none of the proposed answers is correct.

Example 10. The volume in the first octant bounded by the surfaces $x^2-y^2-z^2=0$, $x=4$, the xy-plane and the xz-plane has the general form



(4) none of the proposed answers is correct.

The reverse type of problem was also tried. One example is included.

Example 11. The equation of the conic in Fig. 2 is (0.42; 53) (1) $x^2+y^2=5$ (2) $2y^2-x=0$ (3) $x^2+4y=0$ (4) none of the proposed answers is correct.

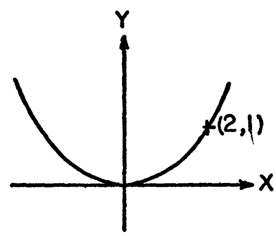


FIG. 2

The authors recognize that in several of these examples the problems are not stated with complete mathematical precision. The determining factor in each case was to make the statement as precise and exact as it seemed possible to do without confusing the student. This policy is consistent, it is believed, with that generally practiced by teachers and texts at this level of instruction.

6. Summary. In this paper we have suggested a new form of objective item for multiple-choice tests. For certain types of problems we believe it to be superior to the conventional item because it is so stated as to encourage the student to solve the problem, it does not permit the student to determine readily the answer by substitution, and at the same time it tends to minimize guessing. Since the items have such a universally high validity index for this sample their use is justified pending further investigation.

The latter part of the paper was devoted to indicating some of the possibilities for using multiple-choice questions for recognition of curves in helping to determine whether students can sketch curves. It is realized that if the student has the ability to recognize curves it does not necessarily follow that he can sketch a curve. However, we believe it does follow that a student who cannot recognize the correct curve also cannot sketch it. To that degree curve recognition does test the ability of a student to sketch curves.

The high validity, however, of these various types of items for this particular sample indicates that in so far as they do measure the ability of the student to do certain problems they discriminate well. It is hoped that others will find these suggestions helpful in constructing objective tests that are reliable and which measure more closely the desired objectives.

GEOMETRY AND EMPIRICAL SCIENCE

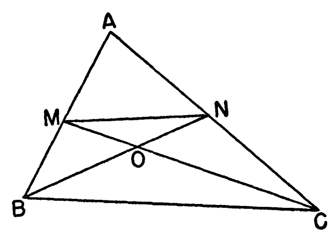
C. G. HEMPEL, Queens College

1. Introduction. The most distinctive characteristic which differentiates mathematics from the various branches of empirical science, and which accounts for its fame as the queen of the sciences, is no doubt the peculiar certainty and necessity of its results. No proposition in even the most advanced parts of empirical science can ever attain this status; a hypothesis concerning "matters of empirical fact" can at best acquire what is loosely called a high probability or a high degree of confirmation on the basis of the relevant evidence available; but however well it may have been confirmed by careful tests, the possibility can never be precluded that it will have to be discarded later in the light of new and disconfirming evidence. Thus, all the theories and hypotheses of empirical science share this provisional character of being established and accepted "until further notice," whereas a mathematical theorem, once proved, is established once and for all; it holds with that particular certainty which no subsequent empirical discoveries, however unexpected and extraordinary, can ever affect to the slightest extent. It is the purpose of this paper to examine the nature of that proverbial "mathematical certainty" with special reference to geometry, in an attempt to shed some light on the question as to the validity of geometrical theories, and their significance for our knowledge of the structure of physical space.

The nature of mathematical truth can be understood through an analysis of the method by means of which it is established. On this point I can be very brief: it is the method of mathematical demonstration, which consists in the logical deduction of the proposition to be proved from other propositions, previously established. Clearly, this procedure would involve an infinite regress unless some propositions were accepted without proof; such propositions are indeed found in every mathematical discipline which is rigorously developed; they are the *axioms* or *postulates* (we shall use these terms interchangeably) of the theory. Geometry provides the historically first example of the axiomatic presentation of a mathematical discipline. The classical set of postulates, however, on which Euclid based his system, has proved insufficient for the deduction of the well-known theorems of so-called euclidean geometry; it has therefore been revised and supplemented in modern times, and at present various adequate systems of postulates for euclidean geometry are available; the one most closely related to Euclid's system is probably that of Hilbert.

2. The inadequacy of Euclid's postulates. The inadequacy of Euclid's own set of postulates illustrates a point which is crucial for the axiomatic method in modern mathematics: Once the postulates for a theory have been laid down, every further proposition of the theory must be proved exclusively by logical deduction from the postulates; any appeal, explicit or implicit, to a feeling of self-evidence, or to the characteristics of geometrical figures, or to our experiences concerning the behavior of rigid bodies in physical space, or the like, is

strictly prohibited; such devices may have a heuristic value in guiding our efforts to find a strict proof for a theorem, but the proof itself must contain absolutely no reference to such aids. This is particularly important in geometry, where our so-called intuition of geometrical relationships, supported by reference to figures or to previous physical experiences, may induce us tacitly to make use of assumptions which are neither formulated in our postulates nor provable by means of them. Consider, for example, the theorem that in a triangle the three medians bisecting the sides intersect in one point which divides each of them in the ratio of 1:2. To prove this theorem, one shows first that in any triangle ABC (see figure) the line segment MN which connects the centers of AB and AC is parallel to BC and therefore half as long as the latter side. Then the lines BN and CM are drawn, and an examination of the triangles MON and BOC leads to the proof of the theorem. In this procedure, it is usually taken for granted that BN and CM intersect in a point O which lies between B and N as well as between C



and M. This assumption is based on geometrical intuition, and indeed, it cannot be deduced from Euclid's postulates; to make it strictly demonstrable and independent of any reference to intuition, a special group of postulates has been added to those of Euclid; they are the postulates of order. One of these—to give an example—asserts that if A, B, C are points on a straight line l, and if B lies between A and C, then B also lies between C and A.—Not even as "trivial" an assumption as this may be taken for granted; the system of postulates has to be made so complete that all the required propositions can be deduced from it by purely logical means.

Another illustration of the point under consideration is provided by the proposition that triangles which agree in two sides and the enclosed angle, are congruent. In Euclid's Elements, this proposition is presented as a theorem; the alleged proof, however, makes use of the ideas of motion and superimposition of figures and thus involves tacit assumptions which are based on our geometric intuition and on experiences with rigid bodies, but which are definitely not warranted by—*i.e.* deducible from—Euclid's postulates. In Hilbert's system, therefore, this proposition (more precisely: part of it) is explicitly included among the postulates.

3. Mathematical certainty. It is this purely deductive character of mathematical proof which forms the basis of mathematical certainty: What the rigorous proof of a theorem—say the proposition about the sum of the angles in a

triangle—establishes is not the truth of the proposition in question but rather a conditional insight to the effect that that proposition is certainly true *provided that* the postulates are true; in other words, the proof of a mathematical proposition establishes the fact that the latter is logically implied by the postulates of the theory in question. Thus, each mathematical theorem can be cast into the form

$$(P_1 \cdot P_2 \cdot P_3 \cdot \cdots \cdot P_N) \rightarrow T$$

where the expression on the left is the conjunction (joint assertion) of all the postulates, the symbol on the right represents the theorem in its customary formulation, and the arrow expresses the relation of logical implication or entailment. Precisely this character of mathematical theorems is the reason for their peculiar certainty and necessity, as I shall now attempt to show.

It is typical of any purely logical deduction that the conclusion to which it leads simply re-asserts (a proper or improper) part of what has already been stated in the premises. Thus, to illustrate this point by a very elementary example, from the premise, "This figure is a right triangle," we can deduce the conclusion, "This figure is a triangle"; but this conclusion clearly reiterates part of the information already contained in the premise. Again, from the premises, "All primes different from 2 are odd" and "n is a prime different from 2," we can infer logically that n is odd; but this consequence merely repeats part (indeed a relatively small part) of the information contained in the premises. The same situation prevails in all other cases of logical deduction; and we may, therefore, say that logical deduction—which is the one and only method of mathematical proof—is a technique of conceptual analysis: it discloses what assertions are concealed in a given set of premises, and it makes us realize to what we committed ourselves in accepting those premises; but none of the results obtained by this technique ever goes by one iota beyond the information already contained in the initial assumptions.

Since all mathematical proofs rest exclusively on logical deductions from certain postulates, it follows that a mathematical theorem, such as the Pythagorean theorem in geometry, asserts nothing that is *objectively* or *theoretically new* as compared with the postulates from which it is derived, although its content may well be *psychologically new* in the sense that we were not aware of its being implicitly contained in the postulates.

The nature of the peculiar certainty of mathematics is now clear: A mathematical theorem is certain *relatively* to the set of postulates from which it is derived; *i.e.* it is necessarily true *if* those postulates are true; and this is so because the theorem, if rigorously proved, simply re-asserts part of what has been stipulated in the postulates. A truth of this conditional type obviously implies no assertions about matters of empirical fact and can, therefore, never get into conflict with any empirical findings, even of the most unexpected kind; consequently, unlike the hypotheses and theories of empirical science, it can never suffer the fate of being disconfirmed by new evidence: A mathematical truth is

irrefutably certain just because it is devoid of factual, or empirical content. Any theorem of geometry, therefore, when cast into the conditional form described earlier, is analytic in the technical sense of logic, and thus true *a priori*; *i.e.* its truth can be established by means of the formal machinery of logic alone, without any reference to empirical data.

4. Postulates and truth. Now it might be felt that our analysis of geometrical truth so far tells only half of the relevant story. For while a geometrical proof no doubt enables us to assert a proposition conditionally—namely on condition that the postulates are accepted—, is it not correct to add that geometry also unconditionally asserts the truth of its postulates and thus, by virtue of the deductive relationship between postulates and theorems, enables us unconditionally to assert the truth of its theorems? Is it not an unconditional assertion of geometry that two points determine one and only one straight line that connects them, or that in any triangle, the sum of the angles equals two right angles? That this is definitely not the case, is evidenced by two important aspects of the axiomatic treatment of geometry which will now be briefly considered.

The first of these features is the well-known fact that in the more recent development of mathematics, several systems of geometry have been constructed which are incompatible with euclidean geometry, and in which, for example, the two propositions just mentioned do not necessarily hold. Let us briefly recollect some of the basic facts concerning these *non-euclidean geometries*. The postulates on which euclidean geometry rests include the famous postulate of the parallels, which, in the case of plane geometry, asserts in effect that through every point P not on a given line l there exists exactly one parallel to l, *i.e.*, one straight line which does not meet l. As this postulate is considerably less simple than the others, and as it was also felt to be intuitively less plausible than the latter, many efforts were made in the history of geometry to prove that this proposition need not be accepted as an axiom, but that it can be deduced as a theorem from the remaining body of postulates. All attempts in this direction failed, however; and finally it was conclusively demonstrated that a proof of the parallel principle on the basis of the other postulates of euclidean geometry (even in its modern, completed form) is impossible. This was shown by proving that a perfectly self-consistent geometrical theory is obtained if the postulate of the parallels is replaced by the assumption that through any point P not on a given straight line l there exist at least two parallels to l. This postulate obviously contradicts the euclidean postulate of the parallels, and if the latter were actually a consequence of the other postulates of euclidean geometry, then the new set of postulates would clearly involve a contradiction, which can be shown not to be the case. This first non-euclidean type of geometry, which is called hyperbolic geometry, was discovered in the early 20's of the last century almost simultaneously, but independently by the Russian N. I. Lobatschefskij, and by the Hungarian J. Bolyai. Later, Riemann developed an alternative geometry, known as elliptical geometry, in which the axiom of the parallels is replaced by the postulate that no line has any parallels. (The acceptance of this postulate,

however, in contradistinction to that of hyperbolic geometry, requires the modification of some further axioms of euclidean geometry, if a consistent new theory is to result.) As is to be expected, many of the theorems of these non-euclidean geometries are at variance with those of euclidean theory; thus, *e.g.*, in the hyperbolic geometry of two dimensions, there exist, for each straight line l, through any point P not on l, infinitely many straight lines which do not meet l; also, the sum of the angles in any triangle is less than two right angles. In elliptic geometry, this angle sum is always greater than two right angles; no two straight lines are parallel; and while two different points usually determine exactly one straight line connecting them (as they always do in euclidean geometry), there are certain pairs of points which are connected by infinitely many different straight lines. An illustration of this latter type of geometry is provided by the geometrical structure of that curved two-dimensional space which is represented by the surface of a sphere, when the concept of straight line is interpreted by that of great circle on the sphere. In this space, there are no parallel lines since any two great circles intersect; the endpoints of any diameter of the sphere are points connected by infinitely many different "straight lines," and the sum of the angles in a triangle is always in excess of two right angles. Also, in this space, the ratio between the circumference and the diameter of a circle (not necessarily a great circle) is always less than 2π.

Elliptic and hyperbolic geometry are not the only types of non-euclidean geometry; various other types have been developed; we shall later have occasion to refer to a much more general form of non-euclidean geometry which was likewise devised by Riemann.

The fact that these different types of geometry have been developed in modern mathematics shows clearly that mathematics cannot be said to assert the truth of any particular set of geometrical postulates; all that pure mathematics is interested in, and all that it can establish, is the deductive consequences of given sets of postulates and thus the necessary truth of the ensuing theorems relatively to the postulates under consideration.

A second observation which likewise shows that mathematics does not assert the truth of any particular set of postulates refers to *the status of the concepts in geometry*. There exists, in every axiomatized theory, a close parallelism between the treatment of the propositions and that of the concepts of the system. As we have seen, the propositions fall into two classes: the postulates, for which no proof is given, and the theorems, each of which has to be derived from the postulates. Analogously, the concepts fall into two classes: the primitive or basic concepts, for which no definition is given, and the others, each of which has to be precisely defined in terms of the primitives. (The admission of some undefined concepts is clearly necessary if an infinite regress in definition is to be avoided.) The analogy goes farther: Just as there exists an infinity of theoretically suitable axiom systems for one and the same theory—say, euclidean geometry—, so there also exists an infinity of theoretically possible choices for the primitive terms of that theory; very often—but not always—different axiomatizations of the same theory involve not only different postulates, but also differ-

ent sets of primitives. Hilbert's axiomatization of plane geometry contains six primitives: point, straight line, incidence (of a point on a line), betweenness (as a relation of three points on a straight line), congruence for line segments, and congruence for angles. (Solid geometry, in Hilbert's axiomatization, requires two further primitives, that of plane and that of incidence of a point on a plane.) All other concepts of geometry, such as those of angle, triangle, circle, *etc.*, are defined in terms of these basic concepts.

But if the primitives are not defined within geometrical theory, what meaning are we to assign to them? The answer is that it is entirely unnecessary to connect any particular meaning with them. True, the words "point," "straight line," *etc.*, carry definite connotations with them which relate to the familiar geometrical figures, but the validity of the propositions is completely independent of these connotations. Indeed, suppose that in axiomatized euclidean geometry, we replace the over-suggestive terms "point," "straight line," "incidence," "betweenness," *etc.*, by the neutral terms "object of kind 1," " object of kind 2," "relation No. 1," "relation No. 2," *etc.*, and suppose that we present this modified wording of geometry to a competent mathematician or logician who, however, knows nothing of the customary connotations of the primitive terms. For this logician, all proofs would clearly remain valid, for as we saw before, a rigorous proof in geometry rests on deduction from the axioms alone without any reference to the customary interpretation of the various geometrical concepts used. We see therefore that indeed no specific meaning has to be attached to the primitive terms of an axiomatized theory; and in a precise logical presentation of axiomatized geometry the primitive concepts are accordingly treated as so-called logical variables.

As a consequence, geometry cannot be said to assert the truth of its postulates, since the latter are formulated in terms of concepts without any specific meaning; indeed, for this very reason, the postulates themselves do not make any specific assertion which could possibly be called true or false! In the terminology of modern logic, the postulates are not sentences, but sentential functions with the primitive concepts as variable arguments.—This point also shows that the postulates of geometry cannot be considered as "self-evident truths," because where no assertion is made, no self-evidence can be claimed.

5. Pure and physical geometry. Geometry thus construed is a purely formal discipline; we shall refer to it also as *pure geometry*. A pure geometry, then,—no matter whether it is of the euclidean or of a non-euclidean variety—deals with no specific subject-matter; in particular, it asserts nothing about physical space. All its theorems are analytic and thus true with certainty precisely because they are devoid of factual content. Thus, to characterize the import of pure geometry, we might use the standard form of a movie-disclaimer: No portrayal of the characteristics of geometrical figures or of the spatial properties or relationships of actual physical bodies is intended, and any similarities between the primitive concepts and their customary geometrical connotations are purely coincidental.

But just as in the case of some motion pictures, so in the case at least of euclidean geometry, the disclaimer does not sound quite convincing: Historically speaking, at least, euclidean geometry has its origin in the generalization and systematization of certain empirical discoveries which were made in connection with the measurement of areas and volumes, the practice of surveying, and the development of astronomy. Thus understood, geometry has factual import; it is an empirical science which might be called, in very general terms, the theory of the structure of physical space, or briefly, *physical geometry*. What is the relation between pure and physical geometry?

When the physicist uses the concepts of point, straight line, incidence, *etc.*, in statements about physical objects, he obviously connects with each of them a more or less definite physical meaning. Thus, the term "point" serves to designate physical points, *i.e.*, objects of the kind illustrated by pin-points, cross hairs, *etc.* Similarly, the term "straight line" refers to straight lines in the sense of physics, such as illustrated by taut strings or by the path of light rays in a homogeneous medium. Analogously, each of the other geometrical concepts has a concrete physical meaning in the statements of physical geometry. In view of this situation, we can say that physical geometry is obtained by what is called, in contemporary logic, a semantical interpretation of pure geometry, Generally speaking, a semantical interpretation of a pure mathematical theory, whose primitives are not assigned any specific meaning, consists in giving each primitive (and thus, indirectly, each defined term) a specific meaning or designatum. In the case of physical geometry, this meaning is physical in the sense just illustrated; it is possible, however, to assign a purely arithmetical meaning to each concept of geometry; the possibility of such an arithmetical interpretation of geometry is of great importance in the study of the consistency and other logical characteristics of geometry, but it falls outside the scope of the present discussion.

By virtue of the physical interpretation of the originally uninterpreted primitives of a geometrical theory, physical meaning is indirectly assigned also to every defined concept of the theory; and if every geometrical term is now taken in its physical interpretation, then every postulate and every theorem of the theory under consideration turns into a statement of physics, with respect to which the question as to truth or falsity may meaningfully be raised—a circumstance which clearly contradistinguishes the propositions of physical geometry from those of the corresponding uninterpreted pure theory.—Consider, for example, the following postulate of pure euclidean geometry: For any two objects x, y of kind 1, there exists exactly one object l of kind 2 such that both x and y stand in relation No. 1 to l. As long as the three primitives occurring in this postulate are uninterpreted, it is obviously meaningless to ask whether the postulate is true. But by virtue of the above physical interpretation, the postulate turns into the following statement: For any two physical points x, y there exists exactly one physical straight line l such that both x and y lie on l. But this is a physical hypothesis, and we may now meaningfully ask whether it is true or

false. Similarly, the theorem about the sum of the angles in a triangle turns into the assertion that the sum of the angles (in the physical sense) of a figure bounded by the paths of three light rays equals two right angles.

Thus, the physical interpretation transforms a given pure geometrical theory—euclidean or non-euclidean—into a system of physical hypotheses which, if true, might be said to constitute a theory of the structure of physical space. But the question whether a given geometrical theory in physical interpretation is factually correct represents a problem not of pure mathematics but of empirical science; it has to be settled on the basis of suitable experiments or systematic observations. The only assertion the mathematician can make in this context is this: If all the postulates of a given geometry, in their physical interpretation, are true, then all the theorems of that geometry, in their physical interpretation, are necessarily true, too, since they are logically deducible from the postulates. It might seem, therefore, that in order to decide whether physical space is euclidean or non-euclidean in structure, all that we have to do is to test the respective postulates in their physical interpretation. However, this is not directly feasible; here, as in the case of any other physical theory, the basic hypotheses are largely incapable of a direct experimental test; in geometry, this is particularly obvious for such postulates as the parallel axiom or Cantor's axiom of continuity in Hilbert's system of euclidean geometry, which makes an assertion about certain infinite sets of points on a straight line. Thus, the empirical test of a physical geometry no less than that of any other scientific theory has to proceed indirectly; namely, by deducing from the basic hypotheses of the theory certain consequences, or predictions, which are amenable to an experimental test. If a test bears out a prediction, then it constitutes confirming evidence (though, of course, no conclusive proof) for the theory; otherwise, it disconfirms the theory. If an adequate amount of confirming evidence for a theory has been established, and if no disconfirming evidence has been found, then the theory may be accepted by the scientist "until further notice."

It is in the context of this indirect procedure that pure mathematics and logic acquire their inestimable importance for empirical science: While formal logic and pure mathematics do not in themselves establish any assertions about matters of empirical fact, they provide an efficient and entirely indispensable machinery for deducing, from abstract theoretical assumptions, such as the laws of Newtonian mechanics or the postulates of euclidean geometry in physical interpretation, consequences concrete and specific enough to be accessible to direct experimental test. Thus, *e.g.*, pure euclidean geometry shows that from its postulates there may be deduced the theorem about the sum of the angles in a triangle, and that this deduction is possible no matter how the basic concepts of geometry are interpreted; hence also in the case of the physical interpretation of euclidean geometry. This theorem, in its physical interpretation, is accessible to experimental test; and since the postulates of elliptic and of hyperbolic geometry imply values different from two right angles for the angle sum of a triangle, this particular proposition seems to afford a good opportunity for a crucial experi-

ment. And no less a mathematician than Gauss did indeed perform this test; by means of optical methods—and thus using the interpretation of physical straight lines as paths of light rays—he ascertained the angle sum of a large triangle determined by three mountain tops. Within the limits of experimental error, he found it equal to two right angles.

6. On Poincaré's conventionalism concerning geometry. But suppose that Gauss had found a noticeable deviation from this value; would that have meant a refutation of euclidean geometry in its physical interpretation, or, in other words, of the hypothesis that physical space is euclidean in structure? Not necessarily; for the deviation might have been accounted for by a hypothesis to the effects that the paths of the light rays involved in the sighting process were bent by some disturbing force and thus were not actually straight lines. The same kind of reference to deforming forces could also be used if, say, the euclidean theorems of congruence for plane figures were tested in their physical interpretation by means of experiments involving rigid bodies, and if any violations of the theorems were found. This point is by no means trivial; Henri Poincaré, the great French mathematician and theoretical physicist, based on considerations of this type his famous *conventionalism concerning geometry.* It was his opinion that no empirical test, whatever its outcome, can conclusively invalidate the euclidean conception of physical space; in other words, the validity of euclidean geometry in physical science can always be preserved—if necessary, by suitable changes in the theories of physics, such as the introduction of new hypotheses concerning deforming or deflecting forces. Thus, the question as to whether physical space has a euclidean or a non-euclidean structure would become a matter of convention, and the decision to preserve euclidean geometry at all costs would recommend itself, according to Poincaré, by the greater simplicity of euclidean as compared with non-euclidean geometrical theory.

It appears, however, that Poincaré's account is an oversimplification. It rightly calls attention to the fact that the test of a physical geometry G always presupposes a certain body P of non-geometrical physical hypotheses (including the physical theory of the instruments of measurement and observation used in the test), and that the so-called test of G actually bears on the combined theoretical system $G \cdot P$ rather than on G alone. Now, if predictions derived from $G \cdot P$ are contradicted by experimental findings, then a change in the theoretical structure becomes necessary. In classical physics, G always was euclidean geometry in its physical interpretation, GE; and when experimental evidence required a modification of the theory, it was P rather than GE which was changed. But Poincaré's assertion that this procedure would always be distinguished by its greater simplicity is not entirely correct; for what has to be taken into consideration is the simplicity of the total system $G \cdot P$, and not just that of its geometrical part. And here it is clearly conceivable that a simpler total theory in accordance with all the relevant empirical evidence is obtainable by going over to a non-euclidean form of geometry rather than by preserving the euclidean structure of physical space and making adjustments only in part P.

And indeed, just this situation has arisen in physics in connection with the development of the general theory of relativity: If the primitive terms of geometry are given physical interpretations along the lines indicated before, then certain findings in astronomy represent good evidence in favor of a total physical theory with a non-euclidean geometry as part G. According to this theory, the physical universe at large is a three-dimensional curved space of a very complex geometrical structure; it is finite in volume and yet unbounded in all directions. However, in comparatively small areas, such as those involved in Gauss' experiment, euclidean geometry can serve as a good approximative account of the geometrical structure of space. The kind of structure ascribed to physical space in this theory may be illustrated by an analogue in two dimensions; namely, the surface of a sphere. The geometrical structure of the latter, as was pointed out before, can be described by means of elliptic geometry, if the primitive term "straight line" is interpreted as meaning "great circle," and if the other primitives are given analogous interpretations. In this sense, the surface of a sphere is a two-dimensional curved space of non-euclidean structure, whereas the plane is a two-dimensional space of euclidean structure. While the plane is unbounded in all directions, and infinite in size, the spherical surface is finite in size and yet unbounded in all directions: a two-dimensional physicist, travelling along "straight lines" of that space would never encounter any boundaries of his space; instead, he would finally return to his point of departure, provided that his life span and his technical facilities were sufficient for such a trip in consideration of the size of his "universe." It is interesting to note that the physicists of that world, even if they lacked any intuition of a three-dimensional space, could empirically ascertain the fact that their two-dimensional space was curved. This might be done by means of the method of traveling along straight lines; another, simpler test would consist in determining the angle sum in a triangle; again another in determining, by means of measuring tapes, the ratio of the circumference of a circle (not necessarily a great circle) to its diameter; this ratio would turn out to be less than π.

The geometrical structure which relativity physics ascribes to physical space is a three-dimensional analogue to that of the surface of a sphere, or, to be more exact, to that of the closed and finite surface of a potato, whose curvature varies from point to point. In our physical universe, the curvature of space at a given point is determined by the distribution of masses in its neighborhood; near large masses such as the sun, space is strongly curved, while in regions of low mass-density, the structure of the universe is approximately euclidean. The hypothesis stating the connection between the mass distribution and the curvature of space at a point has been approximately confirmed by astronomical observations concerning the paths of light rays in the gravitational field of the sun.

The geometrical theory which is used to describe the structure of the physical universe is of a type that may be characterized as a generalization of elliptic geometry. It was originally constructed by Riemann as a purely mathematical theory, without any concrete possibility of practical application at hand. When

Einstein, in developing his general theory of relativity, looked for an appropriate mathematical theory to deal with the structure of physical space, he found in Riemann's abstract system the conceptual tool he needed. This fact throws an interesting sidelight on the importance for scientific progress of that type of investigation which the "practical-minded" man in the street tends to dismiss as useless, abstract mathematical speculation.

Of course, a geometrical theory in physical interpretation can never be validated with mathematical certainty, no matter how extensive the experimental tests to which it is subjected; like any other theory of empirical science, it can acquire only a more or less high degree of confirmation. Indeed, the considerations presented in this article show that the demand for mathematical certainty in empirical matters is misguided and unreasonable; for, as we saw, mathematical certainty of knowledge can be attained only at the price of analyticity and thus of complete lack of factual content. Let me summarize this insight in Einstein's words:

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

FUNCTIONS OF SEVERAL COMPLEX VARIABLES*

W. T. MARTIN, Syracuse University

1. Definition of an analytic function. Consider a domain D in the $2n$-dimensional euclidean space of n complex variables $z_1, \cdots, z_n$. A function $f(z_1, \cdots, z_n)$ is said to be analytic in D if in some neighborhood of every point $(z_1, \cdots, z_n)$ of D it can be represented as the sum of an (absolutely) convergent multiple power-series

$$\sum_{j_1, \cdots, j_n=0} a_{j_1, \cdots, j_n}(z_1 - z_1^0)^{j_1} \cdots (z_n - z_n^0)^{j_n}.$$

An alternative definition is that f is analytic in D if it has derivatives of all orders, mixed and iterated, at every point of D. These two definitions are very easily seen to be equivalent.

An important result due to Osgood [1] in 1899 states that *if $f(z_1, \cdots, z_n)$ is bounded in D and if the n partial derivatives $\partial f/\partial z_j$, $j=1, \cdots, n$ all exist at every point of D, then f is analytic in D.*

In 1899 and again in 1900 Osgood [1, 2] raised the question as to whether the boundedness restriction could be removed, and in 1906 Hartogs [1] showed that it actually could be removed. This means that if a function is analytic in each variable separately, it is analytic as a function of all n variables. This is a key result in the theory.

* This paper is an amplification of an invited address delivered at the annual meeting of the Mathematical Association of American in Wellesley, Massachusetts, on August 12, 1944.

2. Some general remarks. Roughly the results of analytic functions of several complex variables can be divided into two main groups, those results which are generalizations of the results of one complex variable, and those results which are peculiar to several variables, either in that they have no analogues in one variable or in that their analogues in one variable are false. It is necessary and important that both types of results be investigated. Often it is extremely difficult to carry a result over from one to several variables, and completely new ideas are needed. Sometimes these new ideas even clarify ideas in one variable. In this particular talk I want to limit myself to a few examples of the second type, that is to a few results which are more or less peculiar to several variables. One such result is the key theorem of Hartogs mentioned above—this theorem has no meaning in one variable and hence is truly a result on several variables. As a matter of fact it is a result peculiar to several *complex* variables since it is false when the variables are real, as one sees by considering the example

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & 0 < x^2 + y^2, \\ 0, & x = y = 0, \end{cases}$$

in a neighborhood of the origin (*cf.* Osgood [3, p. 142]). In proving the theorem Hartogs was led to the introduction of subharmonic functions, a topic which has proved very fruitful in other parts of analysis.

I will not dwell further here on this theorem but will pass to a second example which also bears the name of Hartogs. It is perhaps interesting to remark that Hartogs' key theorem given above involves mainly the work of two men, Osgood and Hartogs, and was proved in 1906; while the theorem to be considered next has been contributed to by many mathematicians over the period of the past four decades.

3. Hartogs' theorem on analytic continuation from the boundary of a domain. An analytic function of one complex variable in a domain D has isolated zeros in D and its reciprocal has isolated singularities (poles) in D. This situation no longer prevails in several variables—an analytic function of several complex variables cannot have isolated zeros or singularities (except for trivial removable singularities). For example, the function $1/z_1$ has as singularities all points of the $(2n-2)$-dimensional set $z_1=0$; $z_2, \cdots, z_n$ arbitrary. This property, as well as other considerations, led mathematicians early to state roughly that "the singularities of an analytic function of several complex variables must penetrate the boundary." This idea is rigorously embodied in the following theorem:

HARTOGS' THEOREM. *Let D be a bounded domain in $(z_1, \cdots, z_n)$-space $(n>1)$ with a connected boundary C, and let $f(z_1, \cdots, z_n)$ be a single-valued function which is analytic in some domain containing C. Then f has a unique analytic continuation into all of D.*

The hypothesis $n>1$ of the theorem is necessary—every domain in the plane is the (exact) existence domain of an analytic function of one complex variable.

Hartogs [2] proved the theorem in 1906 for the case in which D is a cylindrical domain, that is, the topological product $[z_1 \in D_1, \cdots, z_n \in D_n]$, where D_j is a domain in the z_j-plane. He did this by repeated use of Cauchy's integral formula for one variable. Since 1906 the theorem has attracted various mathematicians.

There are three main methods of attack on the theorem. The first method uses Cauchy's integral formula for one variable to yield an important lemma concerning continuation from the boundary of a hypersphere (Lemma 1 below), and then bases the proof upon this lemma. The second method is closely related; it again uses Cauchy's integral formula for one variable, treating the other variables as parameters. The third method differs importantly from the other two in that it uses a recent form of Cauchy's integral formula for n variables taken over the $(2n-1)$-dimensional boundary of the domain. We shall amplify these remarks a little.

3a. The first method. The following important lemma is due essentially to Levi [1] and Osgood [3] although it is based upon Hartogs' work.

LEMMA 1. *Let P be a boundary point of a hypersphere in E_{2n} $(n>1)$ and let $f(z_1, \cdots, z_n)$ be a function which is defined and analytic in the portion of a neighborhood of P which lies outside the (closed) hypersphere. Then f has a unique analytic continuation into a ($2n$-dimensional) neighborhood containing P.*

The lemma is proved very readily by use of Cauchy's integral formula for one variable; the fact that n is greater than unity is of importance in allowing one to shift the contour. The idea of the proof will become evident as we consider an example in Section 3b.

Osgood [3, 4] used this lemma in attacking Hartogs' general theorem. Because of certain geometric difficulties which apparently were not fully met in Osgood's proof, Brown [1] in 1936 gave a detailed proof of the theorem, again using the above lemma. Gaps in at least one other proof by this method have been observed; suffice it to say that a proof of Hartogs' theorem by this method calls for very careful and painstaking analysis.

3b. The second method. By use of Cauchy's integral formula for one complex variable, Severi [1] in 1932 proved the analogue of Hartogs' theorem for a wide class of domains for analytic functions $f(z, u)$ of one complex and one real variable. In the paper mentioned earlier, Brown [1] extended Severi's result for functions $f(z_1, \cdots, z_n, u)$, $(n>0)$, with $(z_1, \cdots, z_n)$ complex, u real, and for general bounded domains D in E_{2n+1} with connected boundaries. A function $f(z_1, \cdots, z_n, u_1, \cdots, u_m)$ of n complex variables and m real variables is said to be an analytic function of these $n+m$ variables in a domain D of E_{2n+m} if in some neighborhood of every point of D it can be developed in an (absolutely) convergent multiple power-series.

We will illustrate Severi's method by the following example. For the sake of simplicity, we will work with one real variable.

Example. Let $f(z, u)$ *be analytic in the spherical shell*

$$(R - \epsilon)^2 < |z|^2 + u^2 < (R + \epsilon)^2, \qquad (0 < \epsilon < R). \tag{1}$$

Then f has an analytic continuation into the sphere

$$|z|^2 + u^2 < (R + \epsilon)^2. \tag{2}$$

Proof. For each fixed value of u in

$$-R < u < R, \tag{3}$$

define

$$\phi(z, u) = \frac{1}{2\pi i}\int_{C(u)} \frac{f(t, u)}{t - z}\, dt, \tag{4}$$

where $C(u)$ is the circle

$$|t| = [R^2 - u^2]^{1/2}. \tag{5}$$

Since $f(t, u)$ is a continuous (even analytic) function of t for t on the contour $C(u)$ it follows that $\phi(z, u)$ is an analytic function of z for z in the interior of the circle $C(u)$,

$$|z| < [R^2 - u^2]^{1/2}. \tag{6}$$

We next observe that since $f(t, u)$ is analytic in the shell (1) we may shift the contour in (4) to any simple closed contour contained within the annulus

$$[(R - \epsilon)^2 - u^2]^{1/2} < |t| < [(R + \epsilon)^2 - u^2]^{1/2} \tag{7}$$

(u fixed in (3)), and not change the value of the resulting function $\phi(z, u)$. We will use this fact a little later.

Now let u_0 be a fixed value of u in (3),

$$-R < u_0 < R, \tag{8}$$

and let us form the integral

$$\frac{1}{2\pi i}\int_{C(u_0)} \frac{f(t, u)}{t - z}\, dt, \tag{9}$$

where the contour $C(u_0)$ is fixed but where we now let u vary over an interval

$$u_0 - \epsilon < u < u_0 + \epsilon. \tag{10}$$

Since $f(z, u)$ is given to be analytic in (z, u) in the shell (1), and since the point set

$$[t \in C(u_0),\ u_0 - \epsilon < u < u_0 + \epsilon] \tag{11}$$

lies in the shell (1), it follows easily that the integral (9) defines a function which is analytic in (z, u) in the domain

$$[|z|^2 < R^2 - u_0^2, u_0 - \epsilon < u < u_0 + \epsilon]. \tag{12}$$

We now make two simple observations. First, the function (9) and the function $\phi(z, u)$ are identical for $u = u_0$, as one sees by comparing their definitions. Secondly, by the remark made in the preceding paragraph, the function (9) and the function $\phi(z, u)$ actually agree for all values of u in (10) since for such values of u the contour $C(u_0)$ serves as an admissible contour. Hence $\phi(z, u)$ is analytic in (z, u) throughout the entire sphere $|z|^2 + u^2 < R^2$.

It remains to identify ϕ with f. This is very easily done. Consider for example a value of u in, say,

$$R - \epsilon < u < R - \frac{\epsilon}{2}.$$

For such a value of u the function $f(t, u)$ is analytic, not just in a neighborhood of the contour $C(u)$ but even throughout its interior. Hence for such values of u we have $\phi(z, u) \equiv f(z, u)$, and hence $\phi(z, u)$ furnishes the desired analytic continuation of $f(z, u)$ into $|z|^2 + u^2 < R^2$. This yields the desired result.

This example illustrates Severi's approach; by such methods he proved Hartogs' theorem for a wide class of domains for functions of one complex and one real variable. In addition to the work of Brown mentioned earlier, further work in this direction is found in a manuscript by Bochner and the speaker [1].

3c. The third method. As mentioned earlier this method is rather recent and uses an integral formula for analytic functions taken over the $(2n-1)$-dimensional boundary of a domain. The corresponding theorem for analytic functions of quaternions was proved by Fueter [1] in 1939 by this method; this yielded Hartogs' theorem in its general form for $n=2$. Bochner [1, 2], Fueter [2], Martinelli [1] and May [1] have proved the result for general domains and for general values of $n \geqq 2$ by this method. The papers of Fueter and Martinelli are closely related and independent of the work of Bochner and his student, May. The Fueter approach is from the standpoint of quaternions and other Cayley number systems, while Bochner and May approach the problem directly from the standpoint of analytic functions of several complex variables. We shall now give an outline of Bochner's approach.

Given a bounded domain D in E_{2n} $(n>1)$ with connected boundary C, and given a function $f(z_1, \cdots, z_n)$ defined and analytic in a neighborhood N of C, define a function

$$\phi(z_1, \cdots, z_n) = (-1)^n \frac{(n-1)!}{(2\pi i)^n} \int_C \frac{f(t_1, \cdots, t_n)}{[\sum_{j=1}^n |t_j - z_j|^2]^n} \sum_{j=1}^n \overline{t_j - z_j} [dt]^{(\bar{j})},$$

where $[dt]^{(\bar{j})}$ is a $(2n-1)$-dimensional element on C which can be defined as follows in terms of the Jacobian:

$$[dt]^{(\bar{j})} = \frac{\partial(t_1, \bar{t}_1, \cdots, t_{j-1}, \bar{t}_{j-1}, t_j, t_{j+1}, \bar{t}_{j+1}, \cdots, t_n, \bar{t}_n)}{\partial(\sigma_1, \sigma_2, \cdots, \sigma_{2n-1})} d\sigma_1 \cdots d\sigma_{2n-1},$$

where $\sigma_1, \cdots, \sigma_{2n-1}$ form a system of (real) coordinates on C. (In case the boundary C of the domain D is not sufficiently smooth, it may be necessary to shift the path of integration slightly. Bochner and May have shown that this can always be done in such a way as to remain in any preassigned neighborhood N of C, *cf.* May [1].) With $\phi(z_1, \cdots, z_n)$ so defined one first shows that ϕ is analytic throughout the interior of D and one next identifies ϕ with f by considering points near the boundary C. The facts that n is greater than 1 and that D is bounded both enter very much into the proof. We shall omit the details.

This method of attack avoids the gaps which were mentioned earlier as having arisen in certain of the proofs by the first method.

3d. In concluding section 3 we merely mention that Levi [1] has carried Hartogs' theorem over to the case of meromorphic functions. The paper by Brown also includes the case of meromorphic functions.

4. The problems of Cousin and Poincaré. Among the earliest and most difficult problems are those of trying to carry over to several complex variables the results of Mittag-Leffler and Weierstrass on the existence of a meromorphic (analytic) function having given poles (zeros) and the related question on the quotient representation of a meromorphic function. Especial difficulties arise in several variables due to the fact that the zeros and poles of an analytic function of several variables are no longer isolated but are $(2n-2)$-dimensional manifolds.

Cousin's first problem is an attempt to carry over to several variables Mittag-Leffler's theorem on the existence of a meromorphic function having prescribed poles. His second problem is to carry over Weierstrass's theorem on the existence of an analytic function with given zeros. Poincaré's problem is to show that a meromorphic function in a domain D can be represented as a quotient of two functions throughout D, not just locally.

Cousin's First Problem. To every point P of a domain D let there be a neighborhood $N(P)$ and a function f_p meromorphic in $N(P)$. In the intersection $N(P)\cap N(Q)$ of the neighborhoods of two points P and Q of D let f_p-f_q be regular. Then does there always exist a single-valued meromorphic function F in D so that $F-f_p$ is regular in $N(P)$?

Cousin's Second Problem. To every point P of a domain D let there be a neighborhood $N(P)$ and a function f_p regular in $N(P)$. In the intersection $N(P)\cap N(Q)$ of the neighborhoods of two points P and Q of D let f_p/f_q be regular and different from zero. Then does there always exist a single-valued analytic function F in D so that F/f_p is regular and different from zero in $N(P)$?

Poincaré's Problem. If a function f is single-valued and meromorphic in a domain D, do there always exist two functions g and h, analytic in D and prime to each other, so that the relation

$$h\cdot f \equiv g$$

holds throughout D?

(Two analytic functions g and h in a domain D are said to be prime to each other if to every common zero P_0 in D, there is no triple of functions d, g_1, h_1

analytic in a neighborhood of P_0, all vanishing at P_0 and such that $g \equiv d \cdot g_1$, $h = d \cdot h_1$ in $N(P_0)$.)

If for a given domain D, Cousin's first problem always can be answered in the affirmative then we say that Cousin's first assertion is valid for D. We do similarly for Cousin's second assertion and Poincaré's assertion.

Now all three assertions are valid for every *schlicht* domain in one complex variable, but the situation is entirely different for several variables. We shall see in what ways it differs as we proceed.

In 1883 Poincaré [1] for $n=2$ proved that his assertion is valid for the entire space. Later, in 1895, Cousin [1] proved that all three assertions are valid for all simply connected cylindrical domains D: $[z_1 \in D_1, \cdots, z_n \in D_n]$. Cousin claimed to prove that all three assertions are valid for all cylindrical domains, whether simply connected or not, but Gronwall [1] in 1917 found a gap in Cousin's work and showed by a counterexample that Cousin's second assertion is in general false if D is not simply connected; the same prevails for Poincaré's assertion. As far as cylindrical domains go, Cousin's first assertion is valid for all cylindrical domains while Cousin's second assertion and Poincare's assertion hold for all cylindrical domains at most one of whose components is multiply connected.

In 1934 Cartan [1] proved that for $n=2$, Cousin's first assertion can hold at most in domains which are (exact) existence domains for some analytic function. (The proof of this is given in Behnke and Stein [1].) As far as I have been able to determine it is still an open question as to whether this result of Cartan is true for $n>2$.

In 1936, 1937, Oka [1, 2] proved a result related to that of Cartan; he showed for $n \geqq 2$ that Cousin's first assertion is valid in all finite *schlicht* domains D which are existence domains for some analytic function.

Two other important results are the following:

Result 1. The validity of Cousin's second assertion for a domain implies the validity of Poincaré's assertion for that domain.

Result 2. If Cousin's first assertion and Poincaré's assertion are both valid for a domain D then Cousin's second assertion is also valid for D (Behnke-Stein [1], 1937).

Result 1 may be seen as follows (compare Behnke-Thullen [1, p. 67]). If F is meromorphic in D and P is a point of D, then by definition there exists a pair of functions of f_p and h_p, regular at P and prime to each other at P, so that

$$F = \frac{h_p}{f_p}$$

in a neighborhood of P; if F is regular at P, then $f_p \equiv 1$. These functions f_p obviously satisfy the conditions of Cousin's second problem for suitably selected neighborhoods $N(P)$. If therefore D is a domain for which Cousin's second assumption is valid then there exists a function G, regular in D, which is such that

G/f_p is regular and non-vanishing at P. Now consider the function $F\cdot G$; in the neighborhood of any point P of D it is representable in the form

$$F\cdot G = h_p \cdot \frac{G}{f_p}$$

and each of the two factors h_p and (G/f_p) is analytic in a neighborhood of P. Hence $F\cdot G$ is analytic throughout D; that is, there is a function H, analytic in D, such that

$$F\cdot G = H$$

holds throughout D; also G and H are prime to each other in any common zero.

This yields Result 1. We will not give the proof of Result 2.

Behnke and Stein [1] have made a very interesting analysis of the relationship of the three problems under consideration. We shall next give a brief résumé of their analysis and in the next section shall discuss briefly two other important contributions related to the work of this section.

If the three problems were entirely independent of each other there would be $2^3 = 8$ possibilities for a given domain. Actually they are not entirely independent since we have seen (Results 1 and 2 of this section) that Cousin's second assertion for a given domain implies Poincaré's, while Cousin's first assertion with Poincaré's assertion implies Cousin's second assertion. Therefore there are at most five possibilities for a given domain; Behnke and Stein have shown these five possibilities all actually do occur. They have given the following table:

	Cousin I	*Cousin II*	*Poincaré*	*Possible?*
1	+	+	+	+
2	+	+	−	−
3	+	−	+	−
4	+	−	−	+
5	−	+	+	+
6	−	+	−	−
7	−	−	+	+
8	−	−	−	+

As just mentioned, combinations 2, 3, and 6 cannot occur. Behnke and Stein have given examples showing domains in which each of the other five combinations actually do occur. Thus five of the eight conceivable combinations occur for $n > 1$ while for $n = 1$ the first combination occurs for all (*schlicht*) domains.

We shall close this section by remarking that Stein [1] and Behnke and Stein [2] have investigated these problems on Riemann surfaces while Oka [3] and Stein [2] have obtained topological criteria for the solvability of Cousin's and Poincaré's problems for domains which are exact existence domains of some analytic function. The theory of existence domains (regularity domains) and regularity envelopes is closely related to the three problems, as various writers have observed.

5. Certain results related to Poincaré's and Cousin's problems. Various other important results related to the three problems of the previous section have been obtained. Two of the most interesting of these are due respectively to Bergman and to Bochner.

We shall first present an early result of Bergman. In presenting this result we shall follow the presentation of Behnke and Thullen [1]. In 1932 Bergman [1] first asked the following question: *given a set of functions g_m, $m=1, 2, \cdots$, analytic in a domain D, what conditions must be imposed upon the functions in order that there shall exist a function f, analytic in D, which has these functions g_m exactly as its zero-functions*? (A function g analytic in a domain D is called a *zero-function* of an analytic function F in D in case F vanishes at all points of D in which $g=0$.) Bergman proved the following result for E_4:

THEOREM. *Denote by $A(g)$ the lower limit of the volume integral*

$$\int_D |1 - v(w, z)g(w, z)|^2 dV, \qquad (dV = dxdydudv),$$

where $v(w, z)$ runs over the totality of all non-vanishing functions analytic in D. If then $\{g_m(w, z)\}$, $m=1, 2, \cdots$, is an infinite sequence of functions analytic and of integrable square in D and if furthermore there exists a $p>0$ so that

$$\sum_{m=1}^{\infty} [A(g_m)]^p$$

converges, then there exists a function $f(w, z)$ analytic in D and having exactly the g_m as its zero-functions.

Bergman has followed up this theorem with a series of interesting results on existence of analytic functions with prescribed zeros, and also similar results for zeros and singularities of meromorphic functions or harmonic functions. Some of these results are given in Bergman [2, 3, 4]. Certain applications are given in Bergman and Martin [1].

We now pass to a very brief discussion of the next result. As stated in the previous section, Poincaré proved his assertion valid for the entire space; he applied methods of potential theory to obtain this result. In extending Poincaré's work to the more general case of cylindrical domains, Cousin used the method of iterating Cauchy's integral in one complex variable. Recently Bochner [2] used a simplified approach to Poincaré's original method and connected Poincaré's theorem to Hartogs' theorem on analytic continuation from the boundary. By making use of Green's formula for general harmonic functions, he not only connected the two results mentioned but he gave new proofs of them and also obtained various new results in very general versions. One of the new results is an extension of Poincaré's theorem to functions on the torus.

6. Concluding remarks. In a talk of this nature it is necessary to limit very drastically the number of topics which can be discussed. In no way do I want

to imply that the results mentioned here are the most important or even the most typical. If there were sufficient time I would like to discuss the theory of analytic mappings, the problem of carrying Runge's theorem over to several variables, the problem of the determination of an analytic function from its boundary values on characteristic surfaces, the theory of regularity domains and regularity envelopes and the related work on various types of convexity, Levi's condition on the boundary of a domain which is the existence domain of an analytic function (a regularity domain), Fourier transforms, Hurwitz's theorem on rational functions, and many other topics. There are many important unsolved problems in the theory. Discussion of these as well as discussion of many of the topics just mentioned will be found in the excellent books of Bergman [5], Osgood [3, 4], and Behnke and Thullen [1].

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METHODS OF PRESENTING e AND π*

KARL MENGER, University of Dame Notre

1. Introduction. Initiating a student into calculus is about what sailing through the straits of Messina used to be: On one side the Charybdis dragging the boat into her whirlpool, on the other side the Scylla waiting for the vessel to shatter on her rock. The whirlpool engulfing so many teachers consists of the false statements concerning infinitely small quantities, the rock on which the beginner goes to pieces is the solid foundation of analysis. The proper initiation into calculus must painstakingly avoid all senseless statements and at the same time avoid unduly rigorous reasoning. If we add that even a first introduction should aim at conveying to the student the understanding of calculus rather than a mere mechanical ability to handle formulae, then we have about described the difficulties confronting the teacher.

The solution is to present only statements and arguments which the student can easily visualize and which are capable of rigorous proofs, but to present them without any attempt at rigorously proving them beyond what may come up in answering the questions of intelligent students. As an example of a presentation in this spirit, in what follows I outline methods of introducing e and π which for years I have found useful in teaching.

2. Concerning e. We start by plotting the exponential curves E_a given by the equations $y=a^x$ for particular values of $a(>1)$, especially for the bases $a=2$ and $a=4$. Then we compare the curve E_a with an auxiliary line L given by the equation $y=x+1$. Clearly, each of the curves E_a has the point (0, 1) in common with L. The curve $E_2(y=2^x)$ has a second point with positive abscissa in common with L, *viz.*, the point (1, 2), the curve E_4 a second point with negative abscissa, *viz.*, the point $(-1/2, 1/2)$.

The student easily realizes that if we let a increase beyond 2, the abscissa of the second point of intersection of E_a and L comes closer to 0 from the right side, and that if we let a decrease below 4, the abscissa of this point comes closer to 0 from the left. In fact, if we preassign the abscissa $\xi(>-1)$ of the second point of intersection, it is easy to find exactly one value of a for which the curve E_a intersects L at the point $(\xi, \xi+1)$. We have to satisfy the equation $a^\xi=\xi+1$, which yields $a=(1+\xi)^{1/\xi}$. The student actually computes the values of a corresponding to some small positive and negative values of ξ (for very small values of ξ by means of decadic or natural logarithmic tables), *e.g.*, for $\xi=1/2, 1/10, \cdots; -1/4, -1/10, \cdots$. On the basis of this experience he will readily believe that as ξ approaches 0, the value of a approaches a number whose first five decimals have been computed as 2.71828. He is told that this number has been given the name e while the curve E_e is called the natural exponential curve, and e^x the natural exponential function.

The student will further admit:

* An address to the teachers of calculus in the V-12 program at the University of Notre Dame.

that for $a>e$ the curve E_a intersects L at a second point, with a negative abscissa, which is the closer to 0 the closer a is to e; that each secant joining the point (0, 1) to a close neighboring point on E_a is appreciably steeper than L, and that consequently the tangent to E_a at (0, 1) has a slope >1;

that for $a<e$ the curve E_a intersects L at a second point, with a positive abscissa, which is the closer to 0 the closer a is to e; that each secant joining the point (0, 1) to a close neighboring point on E_a is appreciably flatter than L, and that consequently the tangent to E_a at (0, 1) has a slope <1;

that for $a=e$ the curve E_e with the equation $y=e^x$ does not intersect L in any second point, that the secants joining the point (0, 1) to neighboring points on E_e come the closer to L the closer the neighboring point is to (0, 1), and that consequently L is the tangent to E_e at (0, 1); in other words, that the tangent to E_e at (0, 1) has the slope 1.

Thus, in addition to introducing e, our method has made plausible this last statement, which we shall reformulate after the introduction of the concept of the derivative, by saying that the derivative of the function e^x at $x=0$ is $=1$, a fact of fundamental importance for the entire calculus. If a rigorous proof of the fact that $(e^h-1)/h$ approaches 1 as h approaches 0 is desired, it can easily be supplemented for the case that h assumes the values 1, 1/2, 1/3, $\cdots$.

Clearly, from $a=b+(a-b)$ and $a>b>1$ it follows that $a^m>b^m+mb^{m-1}(b-a)$ $>b^m+m(b-a)$. Thus we obtain the following

LEMMA

$$|\alpha-\beta|>m|\alpha^{1/m}-\beta^{1/m}| \quad \text{if} \quad \alpha>\beta>1.$$

In order to show that with increasing m the numbers $(e^{1/m}-1)/(1/m)$ approach 1 we form the auxiliary numbers $e_m=(1+1/m)^m$ for which $e_m^{1/m}=1+1/m$. By our lemma,

$$|e^{1/m}-e_m^{1/m}|<|e-e_m|/m,$$

that is,

$$|e^{1/m}-1-1/m|<|e-e_m|/m.$$

Dividing this inequality by $1/m$ we obtain

$$\left|\frac{e^{1/m}-1}{1/m}-1\right|<|e-e_m|,$$

from which our contention follows since, by the definition of e, the numbers e_m approach e, and thus the numbers $|e-e_m|$ approach 0.

The reason for the importance of the fact that $(e^h-1)/h$ approaches 1, is that in conjunction with the functional equation $e^{x+y}=e^x\cdot e^y$ this fact enables us to differentiate e^x:

$$\frac{e^{x+h}-e^x}{h}=e^x\cdot\frac{e^h-1}{h}\to e^x \quad \text{as} \quad h\to 0;$$

and that from the formula $(e^x)' = e^x$ the entire differential calculus of elementary functions, excluding the trigonometric functions, can be derived by virtue of the two rules concerning the differentiation of a sum of two functions, and the differentiation of a function of a function. (For a detailed development of this "Algebra of Derivation" the reader is referred to the author's booklet *Algebra of Analysis*, Notre Dame Mathematical Lectures No. 3, 1944.)

3. Concerning π. In introducing the beginner into the differentiation of the trigonometric functions, the teacher will, of course, use the concept of π familiar to the student from elementary geometry, and the trigonometric functions as they are known from trigonometry, including the concept of radian measure. Again the essential fact which he has to learn, is that for $x=0$ the derivative of the sine, or still better of the tangent, is equal to 1. In view of the fact that $\tan 0=0$, this amounts to the statement that $\tan h/h$ approaches 1 as h approaches 0.

Again, the reason for the importance of the fact that $\tan h/h$ approaches 1, is that in conjunction with the functional equation,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y},$$

it enables us to differentiate $\tan x$:

$$\frac{\tan(x+h) - \tan x}{h} = \frac{1}{h}\left[\frac{\tan x + \tan h}{1 - \tan h \cdot \tan x} - \tan x\right]$$

$$= \frac{1}{1 - \tan h \cdot \tan x} \cdot \frac{\tan h}{h}(1 + \tan^2 x) \to (1 + \tan^2 x) = \sec^2 x;$$

and that by means of the formula $(\tan x)' = \sec^2 x$ all the trigonometric and arc functions can be differentiated. Thus to complete the differentiation of all elementary functions we merely have to use the formulae

$$\sin x = \frac{\tan(x/2)}{1 + \tan^2(x/2)} \quad \text{and} \quad \cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)},$$

which obviously are important also for other reasons, and

$$\tan(\arctan x) = x, \qquad \sin(\arcsin x) = x.$$

(Again for details, the reader is referred to the author's *Algebra of Analysis*.)

In order to prove that $\tan h/h \to 1$ as $h \to 0$, the following idea is useful. It applies to the case that h assumes the values $\pi/3, \pi/4, \cdots, \pi/n, \cdots$ but can easily be adapted to the general case, which, in the opinion of the author, should not be presented to the beginner. If we circumscribe to the circle of radius 1 a regular polygon of n sides, then from elementary trigonometry it is clear that each side has the length $2\tan(\pi/n)$ where π is the radian measure of two right

angles. Hence the length of the polygon is $2n \tan(\pi/n)$. As n gets larger, the lengths of the polygons approach that of the circle (elementary geometry!), that is, 2π. Dividing the formula $2n \tan(\pi/n) \to 2\pi$ by 2π, we obtain

$$\frac{\tan(\pi/n)}{\pi/n} \to 1.$$

The same reasoning applied to inscribed polygons would, of course, yield

$$\frac{\sin(\pi/n)}{\pi/n} \to 1.$$

While this procedure is quite satisfactory as a first introduction, the teacher should realize that the use of radian measure in developing the differentiation of the trigonometric functions is objectionable. The radian measure is based on the concept of the length of a circle which is the limit of the lengths of inscribed and circumscribed polygons. Whenever we consider $\tan x$ where x is measured in radians, we really presuppose a process of the same logical order as the formation of e^x. Now we certainly would not start the exposition of exponential functions with the study of e^x. We start with 2^x, 4^x, *etc.*, then proceed to the idea of a one-parameter family of functions a^x. In fact, we begin with rational values of x and then, on the basis of considerations of continuity, extend a^x to all values of x. Subsequently a procedure described in the first part of this paper singles out the natural exponential curve E_e from the curves E_a (and e from the numbers a) in such a way that for $x=0$ the derivative of the natural exponential function is $=1$.

4. Analogous methods of approach. It may thus be of interest that in a completely analogous way we may start with a one-parameter family of trigonometric functions T_p, among which those corresponding with rational values, like 2 and 4, can be as easily handled as 2^x and 4^x. Then a procedure can be developed which is completely analogous to that which we developed for exponential functions and which singles out a "natural" trigonometric function T_π from the T_p (and π from the numbers p) in such a way that for $x=0$ the derivative of the natural trigonometric function is $=1$.

We shall describe this development for the tangential functions. We call tangential curve of period p, and denote by T_p, the curve with the equation

$$y = \tan(2Rx/p).$$

The ordinate of T_p at the abscissa x is the trigonometric tangent of an angle which is $2x/p$ times a right angle. Clearly, this definition is independent of the concept of length. In order to construct T_p for a given number p, we declare that on the X-axis the point with the abscissa $p/2$ will symbolize the right angle, and the points with other abscissae will stand for proportional multiples of the right angle, *e.g.*, the point $p/4$ for one-half of a right angle, $p/6$ for one-third of a right angle, the point x for $2x/p$ times a right angle. As ordinates at these

abscissae we lay off the tangents of $R/2$, $R/3$, $2Rx/p$, respectively. For instance, we can construct T_4 exactly as we constructed E_4. We start with rational abscissae and then extend the curve T_4 by virtue of continuity considerations to all values of x.

Next we might compare the curves T_p with the auxiliary line L_0 given by the equation $y=x$. Clearly, each curve T_p has the origin in common with L_0. The curve T_4 intersects L_0 in another point with an abscissa between 0 and $p/2$ (and only this interval will be considered in what follows). The curve T_3 has only the origin in common with L_0. One readily sees that if we let p decrease below 4, the abscissa of the second point of intersection of T_p and L_0 approaches 0. In fact, if we preassign the abscissa ξ of the second point of intersection, it is easy to compute by means of trigonometric tables (in degrees or in radians) a value of p for which T_p and L_0 intersect at the abscissa ξ. We have to satisfy the equation $\tan(2R\xi/p)=\xi$, which yields

$$p = \frac{2R\xi}{\arctan \xi},$$

so that T_p has the equation

$$y = \tan \frac{\arctan \xi}{\xi} x.$$

Unfortunately, while in the theory of exponential functions the expression of the base a in terms of ξ leads to the ordinary definition of e when ξ approaches 0, for tangential functions the expression of the period p in terms of ξ does not give us a clear insight into the nature of the limit which p approaches as $\xi \to 0$. We thus shall define as "natural" tangential curve the curve T_p which has L_0 as tangent at the origin. This procedure has the double advantage of guaranteeing the existence and unicity of the natural tangent, and of implying that the natural tangential curve is T_π where π denotes the length of a semicircle of radius 1.

In fact, since every curve T_p can be obtained from any one of them by a proper change of the unit on the X-axis, thus every tangential function $t(x)$ can be obtained from any one of them, say, from $t_0(x)$, by a transformation $t(x)=t_0(cx)$ for a proper constant c, it is clear that the tangents to the various lines T_p at the origin form the pencil of straight lines through this point with the exception of the two coördinate axes; in other words, that each finite slope $\neq 0$ is associated with exactly one number p. There is exactly one curve, let us call it T_q, with the slope 1 at 0. It is easily seen that for $p<q$ the line T_p has no point other than the origin in common with L_p while in this case the sine curve S_p with the equation

$$y = \sin \frac{2R}{p} x$$

does intersect L_0 in another point, and that for $p>q$ the line T_p has another point in common with L_0.

In order to prove that $q=\pi$, we repeat our former reasoning about the regular polygons with n sides in slightly changed form. We remark that with increasing n the lengths $2n \tan (2R/n)$ approach the length of the circle. If we call this latter length 2π, then

$$\frac{\tan (2R/n)}{\pi/n} \to 1 \quad \text{as} \quad n \to \infty.$$

It clearly follows that

$$\frac{\tan \frac{\pi}{\pi} \frac{2R}{n}}{\pi/n} \to 1 \quad \text{as} \quad n \to \infty.$$

Now only for $p=q$ do we have

$$\frac{(\tan 2Rx/p)}{x} \to 1 \quad \text{as} \quad x \to 0.$$

Setting $x=\pi/n$ we see: only for $p=q$ do we have

$$\frac{\tan \frac{\pi}{p} \frac{2R}{n}}{\pi/n} \to 1 \quad \text{as} \quad n \to \infty.$$

We have seen that the above formula does hold for $p=\pi$. Hence $q=\pi$.

A PROBLEM OF REGIONS

H. A. ROBINSON, U. S. Military Academy

If a pie is cut twice, each slice intersecting the other *once* within the pie, four pieces are formed. A third cut, not passing through the point of intersection of the first two, intersects the existing slices in two points (0-cells) which form on the third cut three 1-cells, the partial boundaries of the three new partitions of the pie. Hence three cuts yield $4+3=7$ pieces, one of which is *interior*, *i.e.* does not touch the boundary or crust of the pie.*

This note is concerned with a generalization of this problem. In d-space, let $S(n, d, k)$ be the maximum number of d-cells formed by the partitioning of a simply connected euclidean region R_d by n hyperplanes of $d-1$ dimensional subspaces, each hyperplane intersecting each other in k $d-2$-cells within R_d, and no $d+1$ or more hyperplanes passing through a $d-2$-cell; and let $C(n, d, k)$

* The author, this MONTHLY, vol. 33, pp. 466–469. Recent work on allied subjects will be found in this MONTHLY as follows: W. B. Carver, The polygonal regions into which a plane is divided by n straight lines, vol. 48, pp. 667–675; R. C. Buck, Partition of space, vol. 50, pp. 541–544; Solution to problem E554, vol. 50, pp. 564–565.

be the number of interior regions formed. For the example given above, $S(3, 2, 1) = 7$ and $C(3, 2, 1) = 1$. Suppose we consider any plane region R_2 and $k = 3$. The cutting elements are isomorphic to cubical parabolas; each will intersect the other three times within R_2 and will cut the boundary twice.

By reasoning similar to that used in the first paragraph, in general, the number of new regions created by an nth cut are the same as the number of partitions $S(n-1, d-1, k)$, hence

$$S(n, d, k) = S(n-1, d, k) + S(n-1, d-1, k),$$

and a similar equation for $C(n, d, k)$. Integrating the difference equations and adjusting the constants, we have

$$S(n, d, k) = \sum_{i=0}^{d} \binom{n}{i} + (k-1)\binom{n}{d},$$

and

$$C(n, d, k) = \binom{n-1}{d} + (k-1)\binom{n}{d} \quad \text{where} \quad n \geqq 2, n \geqq d,$$

and

$$C(n, n, k) = k - 1.$$

Consider the partitions of a plane by curves isomorphic to circles, *i.e.* each curve intersects each other in two points $(k=2)$, and no circle intersects the boundary R_2. If the circles are opened into straight lines, the number of partitions lost are the same as the number of interior regions $C(n, 2, 1)$. Let prime letters be used in this case. Hence

$$S'(n, 2, 2) = S(n, 2, 1) + c(n, 2, 1).$$

For the extension of this case, which occurs only when k is even, we obtain

$$S'(n, d, k) = \sum_{i=0}^{d} \binom{n}{i} + \binom{n-1}{d} + (k-2)\binom{n}{d},$$

and

$$C'(n, d, k) = \binom{n-1}{d-2} + k\binom{n}{d}, \quad \text{where} \quad n > 0,\ d > 1,$$

and

$$S'(0, d, k) = C'(0, 1, k) = 1, \qquad C'(n, 1, k) = nk.$$

An interesting result occurs when $n > d$,

$$S(n, d, k) = S'(n, d, k) = 2^n.$$

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

FEUERBACH'S THEOREM BY "MEAN POSITION"

HOWARD EVES, Syracuse University

The "Theory of Mean Position" often furnishes an elegant method for finding distances between notable points, and in this capacity may be successfully employed in proving the famous Feuerbach Theorem (that the nine-point circle of a triangle is tangent to the inscribed and the three escribed circles). Although I devised the following proof several years ago it is still, so far as I am aware, new, and might therefore be worth adding to the host of already existing solutions of the Feuerbach Theorem.

FUNDAMENTAL THEOREM. *If O is the mean center of a system of points $A, \cdots$ for the system of multiples $a, \cdots$, and if P is any point, then*

$$\sum (a\overline{AP}^2) = \sum (a\overline{AO}^2) + \sum (a)\overline{OP}^2. \tag{1}$$

This is a well known and basic theorem in the "Theory of Mean Position." See, *e.g.*, art. 55 in M'Clelland's *The Geometry of the Circle*.

LEMMA. *If O is the incenter of triangle ABC with opposite sides, a, b, c, P any point, and $2s=a+b+c$, then*

$$\sum (a\overline{AP}^2) = abc + 2s\overline{OP}^2. \tag{2}$$

The incenter of a triangle is the mean center of the vertices for multiples proportional to the opposite sides. If, then, in (1) we let P coincide with A, B, C in succession we get

$$\begin{aligned} bc^2 + cb^2 &= \sum (a\overline{AO}^2) + \sum (a)\overline{AO}^2, \\ ca^2 + ac^2 &= \sum (a\overline{AO}^2) + \sum (a)\overline{BO}^2, \\ ab^2 + ba^2 &= \sum (a\overline{AO}^2) + \sum (a)\overline{CO}^2. \end{aligned}$$

Multiplying the first by a, the second by b, the third by c, and adding, we get

$$2abc(a + b + c) = (a + b + c)\sum (a\overline{AO}^2) + (a + b + c)\sum (a\overline{AO}^2),$$

whence

$$\sum (a\overline{AO}^2) = abc.$$

Hence the relation (1), where A, B, C are the vertices of a triangle, a, b, c the opposite sides, O the incenter, and P any point, becomes the relation (2).

FEUERBACH'S THEOREM. *The nine-point circle of a triangle is tangent to the inscribed and the three escribed circles.*

Let N be the center of the nine-point circle. Then, for $P \equiv N$, (2) becomes

$$\sum (a\overline{AN}^2) = abc + 2s\overline{ON}^2. \tag{3}$$

Now, if R is the circumradius,

$$\overline{AN}^2 + \overline{BN}^2 = \tfrac{1}{2}c^2 + \tfrac{1}{2}R^2, \tag{4}$$

$$\overline{CN}^2 + \overline{AN}^2 = \tfrac{1}{2}b^2 + \tfrac{1}{2}R^2, \tag{5}$$

$$\overline{BN}^2 + \overline{CN}^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}R^2. \tag{6}$$

(The sum of the squares of two sides of a triangle is equal to one half the square of the third side plus twice the square of the median on that side.)

Adding (4) and (5) and subtracting (6), and then multiplying by $\frac{1}{2}a$, we find

$$a\overline{AN}^2 = \tfrac{1}{4}a(R^2 + b^2 + c^2 - a^2).$$

Similarly

$$b\overline{BN}^2 = \tfrac{1}{4}b(R^2 + c^2 + a^2 - b^2),$$

$$c\overline{CN}^2 = \tfrac{1}{4}c(R^2 + a^2 + b^2 - c^2).$$

Therefore

$$\begin{aligned}\sum (a\overline{AN}^2) &= \tfrac{1}{4}R^2(a + b + c) + \tfrac{1}{4}(ab^2 + ac^2 + bc^2 + ba^2 + ca^2 \\ &\quad + cb^2 - a^3 - b^3 - c^3) \\ &= \tfrac{1}{2}R^2s + \tfrac{1}{4}[8(s - a)(s - b)(s - c) + 2abc] \\ &= \tfrac{1}{2}R^2s + \tfrac{1}{4}[8r\Delta + 8R\Delta],\end{aligned}$$

where Δ is the area of the triangle and r is the inradius. Substituting the above in (3) we find

$$\tfrac{1}{2}R^2s + 2r\Delta + 2R\Delta = 4R\Delta + 2s\overline{ON}^2,$$

or

$$\overline{ON}^2 = \tfrac{1}{4}R^2 - Rr + r^2 = (\tfrac{1}{2}R - r)^2.$$

That is, the difference between the radii of the nine-point circle and the incircle equals the distance between the centers of these circles, and consequently the circles touch internally.

Similarly, if O_a is the excenter opposite A, we may show that

$$-a\overline{AN}^2 + b\overline{BN}^2 + c\overline{CN}^2 = -abc + 2(s - a)\overline{O_aN}^2,$$

a relation corresponding to (3). Proceeding as above we finally find

$$O_aN = \tfrac{1}{2}R + r_a,$$

whence the nine-point circle and the escribed circle opposite A touch externally.

Hence the theorem.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent to American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the editors or officers of the Association.

Exact Values of the first 200 Factorials. By H. S. Uhler. New Haven, H. S. Uhler, 1944. 4+18 pages. $0.80.

This booklet consists of an introduction and a table of the factorials of the integers from 1 to 200. Spacing is used to separate groups of ten digits in expressing the factorials of numbers greater than 13. Thus 24! appears as 6204 4840173323 9434360000. Use is made of the device of writing the number of terminal zeros (enclosed in parentheses) at the right of the other digits though the number of digits so indicated is always a multiple of 5. Thus 25! appears as 1 5511210043 3309859840 (5)—a number with six terminal zeros. The table gives exact values of the first 200 factorials and a value of the reciprocal of 155! with an error of less than one unit in the 578th decimal place.

In the brief introduction the author discusses his purpose in computing these tables, the methods of computation and checking, and the consistency with previously published results. It appears that the results here published are new for $120 < n \leqq 200$ and that considerable diligence has been exercised in seeing that the published results are free from error.

P. S. DWYER

Mathematical and Physical Principles of Engineering Analysis. By W. C. Johnson. New York, McGraw-Hill Book Company, Inc., 1944. 10+346 pages. $3.00.

This book is intended as a text for engineering students during an upperclass year and presents the mathematical methods appropriate to the solution of problems in various branches of engineering, together with the physical background. The treatment is eminently practical and the explanations are clear, concise and interesting. It represents an original piece of work, possible only by an author who has had much experience in solving engineering problems and who possesses the skill to pass on his experience to his pupils. Mathematical rigor would be out of place in such a book and is disclaimed.

To analyze an engineering problem, it is explained in the first chapter, one must first determine what the problem is, what data are needed, what methods of attack are available and which is the most promising, and what simplifying assumptions should be made. Then the physical conditions are put in mathematical form, the solution is carried out analytically and numerically, being checked by appropriate methods, and finally the results are summarized in useful form.

Chapter 2 reviews "some basic physical principles," properly emphasizing the concept of energy and relating mainly to mechanics and electricity. (The following statement on p. 23 requires modification: "It has been found experimentally that a change in flux linkage of 10^8 maxwell-turns per second will induce an emf of 1 volt." This is really a matter of definition, both as to the general physical relation and as to the units—although, of course, there are experimental facts behind it.)

Linear differential equations and their initial conditions are briefly introduced in Chapter 3 on "transient and steady-state conditions" and are more fully discussed in Chapter 4 on "setting-up equations." The "graphical and numerical solution of differential equations" in the next chapter first takes up quadratures and then step-by-step solutions, the particular methods emphasized being commendable. The long Chapter 6 on "ordinary differential equations" discusses analytical solutions, the greater part being devoted to linear equations with constant coefficients. Although operational notation and ideas are used freely, complete operational solutions unfortunately are not given. Such operational solutions are easier and can be presented more briefly than the methods of the text. Moreover all cases can be embraced in a single procedure; and only a single evaluating formula is required. At the end of this chapter appears a discussion of Bessel's equation and Bessel functions, together with the gamma function.

Chapter 7 takes up "vector representation of sinusoids," through complex numbers, as long used for alternating-current circuits, and applies it also to mechanical vibrations. "The checking of equations" in the next chapter includes dimensional checking and checking by taking simple limiting cases. (On p. 207 appears the statement that "nearly all physical equations are dimensionally homogeneous"; it would be more satisfying if the word "nearly" were omitted and there were added the clause, *if all dimensional constants are written.*)

An unusually clear presentation of "dimensional analysis" in Chapter 9 includes the changing of units and the derivation of general formulas. (The matter of "dimensional constants" is rather briefly dismissed at the end; it might better have been brought out that the method of dimensional analysis really fails in certain cases, in the sense that the dimensions of such a constant are not known until the desired law has been formulated: thus a student would be disappointed if he tried to find the adiabatic or the isothermal $p-v$ law of an ideal gas.)

Fourier series are treated in Chapter 10, with numerical methods of harmonic analysis and a brief introduction to the Fourier integral. (The third example of a "periodic function" on p. 239 is, of course, non-periodic—as is explained later, on p. 249.) The final chapter, on "systems with distributed constants," introduces partial differential equations. Various problems are worked out in terms of the elementary functions, either in closed form or as Fourier series. Graphical field plotting is applied to two-dimensional problems and, more briefly, to three-dimensional problems having an axis of symmetry.

Besides the problems worked out in the text, there are given a large number

of practice problems, some formal and many of a practical nature. It is to be regretted that no answers are given.

The question arises, by whom can the subject matter of this book best be taught? A teacher having an engineering point of view is called for; and he will most often be selected from an engineering department. But a mathematics department would be the stronger if it possessed such a man. In any case, the book will be of great value to teachers of mathematics (as well as to teachers of physics), to show the purest of them "how the other half lives."

ALAN HAZELTINE

Sistemas de Ecuaciones Analiticas en Terminos Finitos, Diferenciales y en Derivadas Parciales. (Monografias publicadas por la Facultad de Ciencias Matematicas, Fisico-Quimicas y Naturales, applicadas a la Industria, Universidad Nacional del Litoral, No. 1.) By Beppo Levi. Rosario, Argentina, 1944. 218 pages. $8.00 m/n.

This is volume one of a series of monographs to be published in Spanish by the Faculty of Mathematical, Physical-chemical and Natural Sciences of the National Coastal University (Universidad Nacional del Litoral), Rosario, Argentina. According to the fly-leaf, volumes two, three and four have also been published, but titles are not given; presumably these do not deal with mathematical subjects. The author of the present volume is director of the Mathematical Institute of the above-mentioned university.

As announced by the title, Professor Levi has presented a theory of analytic equations in several complex variables, embracing finite, ordinary differential and partial differential equations. (Reviewer's note: For the sake of English sentence structure we have substituted the phrase "finite equation" in place of "equation in finite terms.") To paraphrase the opening words of the introduction, "The problems related to the resolution of equations, whether finite or differential, assume fundamentally different aspects according to whether we postulate only the existence of such properties of continuity and differentiability of the given functions and the unknowns as are strictly necessary if a particular problem is to have meaning, or admit additional hypotheses relative to the existence of a certain number of successive derivatives, or finally grant at once the existence of all derivatives." Although it seems clear that the author is most intrigued by the first of these three alternatives, he evidently feels that the last has greatest importance for physical applications, and promises that it will be adopted in the monograph. Nevertheless there are to be found some twenty-five notes in small type scattered throughout the book, some of which will be of aid to a student wishing to relax the hypotheses. In addition there are several pages of bibliographical notes at the end of the volume.

The monograph is not to be regarded as a textbook in the American sense; it contains a number of examples but no problems. Chapter I begins with some brief but illuminating remarks on complex numbers and passes on to the study

of complex power series in several complex variables. By means of analytic continuation the functions studied are defined throughout their natural domain of existence. Derivatives are introduced directly from the power series, and an analogue is developed of the Cauchy Integral Theorem. The problem of solving a holomorphic equation $f(x_1, x_2, \cdots, x_n)=0$ for one of the variables is attacked in a more or less algebraic fashion via the Weierstrass Preparation Theorem and leads to the notion of an algebroid function. Little is said however about the difficult problem of classifying types of singularity of an analytic function of several variables. Chapter II introduces the Jacobian in connection with the solution of finite equations, and also considers the solution of systems of first-order ordinary differential equations.

In Chapter III the author turns aside from the main train of thought in order to prove two elementary algebraic lemmas, useful for the sequel. As evidence of the careful scholarship which has gone into this book we should like to quote (in paraphrase) the second of these: "Lemma II. Suppose given a finite system M of monomials in k variables $x_1, x_2, \cdots, x_k$, such that no monomial of the system is a multiple of any other. We shall show that *there can be determined (often in various ways) another system N of monomials, and, at the same time, certain groups of variables* (multiplier variables) *in such a manner that all monomials which are not multiples of any monomial of M may be obtained in one and only one way by multiplication of a monomial of N by a monomial formed from the corresponding multiplier variables.*"

Chapter IV gives an extended treatment of finite systems of analytic partial differential equations of finite order in a finite number of variables. The notion of a complete system is fundamental. An orderly method is presented whereby a given system may either be shown to be "impossible" or be reduced to an equivalent system comprising a finite number of complete systems. Most of the examples of the book appear in this chapter, and there are many references to the literature. The latter part of the chapter is devoted to the study of initial conditions and to the formal calculation of series solutions. Chapter V offers a necessary complement to the preceding with convergence proofs for the solutions of certain normal systems. In the notation of the author: "*A system of partial differential equations always has an analytic solution satisfying given initial conditions analytic with respect to a normal ordering of the derivatives.*" There are also given certain extensions of this result. The tract concludes with a few remarks on Cauchy conditions and characteristic varieties.

The reviewer did not make a careful analysis of the proofs throughout the whole of the book; however a detailed examination of Chapter I sufficed to give him complete confidence in the author. There appear to be few misprints and only one minor misstatement was noted. In conclusion we should like to remark upon the elegance of mathematical expression which is possible in Spanish and to suggest that with the present volume the new series of monographs has made a most satisfactory beginning.

R. H. BRUCK

Spherical Trigonometry. By Aaron Freilich, Henry Shanholt, and Joseph Seidlin. New York, Silver Burdett Co., 1943. 4+140 pages. $1.28.

Here is a text-book which is designed to meet the needs of students who have had a course in plane trigonometry and who need some knowledge of spherical trigonometry and its application to problems in navigation. Inasmuch as many, if not the majority, of such students will not have the background of solid geometry, it is necessary to include some sort of an introduction to those theorems which form the foundation for the development of spherical trigonometry.

As the authors note, this introduction has been made on a rather informal, intuitive basis, omitting, with a casual "it can be proved" proofs which could have been supplied with a few additional lines. The inclusion of such proofs, while not seriously increasing the size of the book, would render its study more intellectually satisfying.

The development of the formulas for solution of spherical triangles is essentially the same as that to be found in any standard text on the subject. Exercises for students of varying ability and needs are provided with completely worked illustrative examples, which should help the student in developing an adequate problem solving technique.

The usefulness of the text might have been improved by providing haversine tables, since the haversine formulas are introduced. Incidentally this might bring the work more in line with current navigation practice.

The conversational style adopted in much of the treatment has its drawbacks, since many of the conclusions drawn are dependent upon the student's producing the correct answers to the questions posed. In actual classroom practice, the teacher would have to see that the correct answers are produced, together with the reasoning which leads to the stated conclusion. These gaps in the reasoning would detract materially from the value of the book for self instruction.

The presence of occasional historical notes should stimulate the interest of the student. Cumulative reviews offer an excellent means for testing progress in acquiring the desired knowledge.

In the hands of a capable teacher, the text should prove to be very useful.

H. P. Pettit

NEW BOOKS RECEIVED

Methods of Advanced Calculus. By Philip Franklin. New York and London, McGraw-Hill Book Co., Inc., 1944. 12+486 pages. $4.50.

Introduction to the Theory of Divergent Series. By Otto Szász. (Lectures by Otto Szász, written by Joshua Barlaz.) Cincinnati, University of Cincinnati, 1944. 6+72 pages, lithoprinted. $1.25.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 651. *Proposed by E. D. Schell, Arlington, Virginia*

You have eight similar coins and a beam balance. At most one coin is counterfeit and hence underweight. How can you determine whether there is an underweight coin, and if so, which one, using the balance only twice?

E 652. *Proposed by V. Thébault, San Sebastián, Spain*

In which scales of notation can a four-digit number $aabc$ be the square of a two-digit number mn, if $c=b+1$ and $n=m+1$?

E 653. *Proposed by J. H. Butchart, Grinnell College*

The ends of a chord UV of the circle $r=a$ have the parametric angles ϕ and $k\phi$, where k is a constant greater than 1. Show that the locus of the midpoint of UV is a prolate epitrochoid whose polar equation is

$$r = a \cos \frac{k-1}{k+1} \theta.$$

Show also that the envelope of UV is an epicycloid, the point of contact dividing UV in the ratio $1:k$.

E 654. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $n!_1$ denote the coefficient of $x^n/n!$ in the expansion of $e^x/(1-x)$. Show that, for a prime p and any nonnegative integer k,

$$\sum_{n=k}^{k+p-1} n!_1 \equiv -1 \pmod{p}.$$

(*Cf.* E 488 [1942, 478].)

E 655. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

The first term of a geometrical progression is a three-digit number, and the ratio is 9. Each term is divisible by the sum of its digits. Find the sequence having the largest number of terms. (The one beginning with 100 has thirteen terms.)

SOLUTIONS

A Class of Convergent Series

E 615 [1944, 162]. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be real numbers such that $0<g_p<1$ and such that the series

$$g_1 + (1 - g_1)g_2 + (1 - g_2)g_3 + \cdots$$

converges to a sum not exceeding unity. Establish the convergence of

$$1 + \sum_{p=1}^{\infty} \frac{g_1g_2 \cdots g_p}{(1 - g_1)(1 - g_2) \cdots (1 - g_p)}.$$

Solution by M. F. Smiley, U. S. Naval Academy. Rewrite the sum of the first p terms of the first series as $S_p = T_p + g_p$, where

$$T_p = g_1(1 - g_2) + g_2(1 - g_3) + \cdots + g_{p-1}(1 - g_p).$$

By hypothesis $\{T_p\}$ is strictly increasing and bounded by unity. Consequently this sequence has a limit $t>0$. Setting $s = \lim S_p$, we have

$$g = \lim g_p = s - t \leqq 1 - t < 1.$$

Since the limit of the pth term of the first series is zero, we deduce that $g=0$. The ratio test then ensures the convergence of the second series.

Also solved by Murray Barbour, Henry Helson, W. J. Thron, and the proposer. Thron followed the proposer in seeing a connection with continued fractions and citing Perron, *Die Lehre von den Kettenbrüchen* (second edition), p. 259.

A Variant of Gregory's Series for π

E 618 [1944, 231]. *Proposed by Thorold Gasset, Cambridge, England*

Prove that, for any value of θ between 0 and π,

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots = \frac{\pi}{4}.$$

Solution by Irving Kaplansky, New York, N. Y. This is a standard result in Fourier series (see, *e.g.*, Goursat, *Mathematical Analysis*, vol. 1, p. 421). However, the following formal derivation (in the spirit of eighteenth century mathematics) may be of interest. Let

$$f(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

Then

$$f(e^{i\theta}) = \tfrac{1}{2}\log\frac{1+e^{i\theta}}{1-e^{i\theta}} = \tfrac{1}{2}\log\,(i\cot\tfrac{1}{2}\theta)$$

$$= \begin{cases} \tfrac{1}{2}\log\cot\tfrac{1}{2}\theta + i\pi/4 & (0<\theta<\pi), \\ \tfrac{1}{2}\log\,(-\cot\tfrac{1}{2}\theta) - i\pi/4 & (\pi<\theta<2\pi). \end{cases}$$

Hence

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots = \begin{cases} \pi/4 & (0 < \theta < \pi), \\ -\pi/4 & (\pi < \theta < 2\pi), \end{cases}$$

and also

$$\cos\theta + \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} + \cdots = \frac{1}{2}\log\left|\cot\frac{\theta}{2}\right|.$$

Also solved by D. H. Browne, Howard Eves, Edward Fleisher, Bernard Greenspan, R. W. Hamming, H. D. Lipsich, E. B. Roessler, and P. D. Thomas. The following literature was cited: Bromwich, *Theory of Infinite Series*, p. 356; Jackson, *Fourier Series and Orthogonal Polynomials* (Carus Monograph No. 6), p. 209; Knopp, *Theory and Application of Infinite Series* (1928), p. 374; Loney, *Plane Trigonometry* (1925), p. 122; Woods, *Advanced Calculus*, p. 298.

The Quadrangle of Circumcenters

E 619 [1944, 231]. *Proposed by W. B. Clarke, San José*

Prove that the four triangles of the complete quadrangle formed by the circumcenters of the four triangles of any complete quadrilateral are similar to those triangles.

Solution by Howard Eves, Syracuse University. Let $a_1a_2a_3a_4$ be the complete quadrilateral and $A_1A_2A_3A_4$ the complete quadrangle, so that A_1 is the circumcenter of triangle $a_2a_3a_4$, and so on. Now (see art. 196 in Johnson's *Modern Geometry*) the circumcircles (A_1), (A_2), (A_3), (A_4) are concurrent at a point P. Let c_{34} and c_{41} denote the lines joining P to the points of intersection (a_3, a_4) and (a_4, a_1). Since these lines are perpendicular respectively to A_1A_2 and A_2A_3, we have

$$\sphericalangle c_{34}c_{41} = \sphericalangle A_1A_2A_3.$$

But $\sphericalangle c_{34}c_{41} = \sphericalangle a_3a_1$, these being inscribed in the same arc of circle (A_2). Hence $\sphericalangle A_1A_2A_3 = \sphericalangle a_3a_1$. Similarly, for all suitable i, j, k, we have

$$\sphericalangle A_iA_jA_k = \sphericalangle a_ka_i.$$

This proves the theorem.

Triangles with a Given Ratio for Two Angles

E 620 [1944, 231]. *Proposed by Alan Wayne, Rhodes School, New York*

Find integral sides for a triangle in which one angle is six times another.

Solution by R. C. Buck, Cambridge, Mass. Consider a triangle ABC with angle $B = nA$ and side $a = 1$. We will find all such triangles with *rational* sides. (Then, multiplying by the common denominator of b and c, we shall have all integral solutions for one angle n times another.) Applying the law of sines to ABC (with $a = 1$), we obtain

$$b = R_n, \qquad c = R_{n+1}, \tag{1}$$

where $R_n = (\sin nA)/(\sin A)$. In terms of $r = 2\cos A$, a well known formula gives

$$(2) \qquad R_{n+1} = r^n - \binom{n-1}{1} r^{n-2} + \binom{n-2}{2} r^{n-4} - \binom{n-3}{3} r^{n-6} + \cdots;$$

e.g.,

$$R_6 = r(r^2 - 3)(r^2 - 1), \qquad R_7 = r^2(r^2 - 3)(r^2 - 2) - 1.$$

If r is rational, so is R_k, and hence so are b and c; conversely, if b and c are rational, so is r, for $r = (b^2 + c^2 - 1)/bc$. In order that a value of r may give a valid triangle, we must have

$$0 < A + B < \pi;$$

in terms of r, this means that

$$(3) \qquad 2\cos\frac{\pi}{n+1} < r < 2.$$

This completes the general solution. For any rational r satisfying (3), we can find b and c by (1) and (2), and these will be rational. Conversely, every rational solution is given by some rational r.

For the case when $n = 6$, we choose any rational p/q such that

$$1.80194 \leqq p/q < 2.$$

Then multiplication by q^6 yields a triangle with integral sides

$$a = q^6,$$
$$b = pq(p^2 - 3q^2)(p^2 - q^2),$$
$$c = p^2(p^2 - 3q^2)(p^2 - 2q^2) - q^6.$$

The smallest solution is given by $p/q = 11/6$:

$$a = 46656, \qquad b = 72930, \qquad c = 30421.$$

Here are the solutions for smaller values of n.

$n = 2$: $a = q^2$, $b = pq$, $c = p^2 - q^2$; *e.g.* (4, 6, 5).

$n = 3$: $a = q^3$, $b = q(p^2 - q^2)$, $c = p(p^2 - 2q^2)$; *e.g.* (8, 10, 3).

$n = 4$: $a = q^4$, $b = pq(p^2 - 2q^2)$, $c = p^4 - 3p^2q^2 + q^4$; *e.g.* (81, 105, 31).

$n = 5$: $a = q^5$, $b = q(p^4 - 3p^2q^2 + q^4)$, $c = p(p^2 - 3q^2)(p^2 - q^2)$; *e.g.* (1024, 1220, 231).

Also solved by Murray Barbour, H. N. Carleton, R. W. Hamming, Frank Hawthorne, E. P. Starke, and the proposer. The proposer points out that, if (a_n, b_n, c_n) is the general primitive solution with $B = nA$, the expressions can be quickly written down by means of the recurrence formulas

$$a_{n+1} = qa_n, \qquad b_{n+1} = qc_n, \qquad c_{n+1} = pc_n - qb_n.$$

In particular, we always have $a_n = q^n$.

The proposer remarks, further, that repeated application of the identity

$$\sin(n+1)A = r \sin nA - \sin(n-1)A,$$

where $r = 2\cos A$, yields the expansion

$$\frac{b_n}{c_n} = \frac{R_n}{R_{n+1}} = \frac{\sin nA}{\sin(n+1)A} = \frac{1}{r} \, \frac{}{-}\, \frac{1}{r} \, \frac{}{-} \cdots \frac{1}{r}$$

(to n components). Setting $r = p/q$ (or $\cos A = p/2q$), we deduce that b_n/c_n is the nth convergent of the continued fraction

$$\frac{q}{p} \, \frac{}{-}\, \frac{q^2}{p} \, \frac{}{-}\, \frac{q^2}{p} \, \frac{}{-} \cdots ;$$

e.g., with $p/q = 11/6$,

$$\frac{b_6}{c_6} = \frac{6}{11} \, \frac{}{-}\, \frac{36}{11} \, \frac{}{-}\, \frac{36}{11} \, \frac{}{-}\, \frac{36}{11} \, \frac{}{-}\, \frac{36}{11} \, \frac{}{-}\, \frac{36}{11} = \frac{72930}{30421}.$$

A Periodic Continued Fraction

E 621 [1944, 285]. *Proposed by C. D. Olds, Purdue University*

Show directly (*i.e.*, without substituting θ into the value given by the standard algebraic proofs) that

$$\frac{1}{a} \, \frac{}{+}\, \frac{1}{a} \, \frac{}{+} \cdots = e^{-\theta},$$

where a is a positive number and $\theta = \arg\sinh(a/2)$.

I. *Solution by E. D. Schell, Arlington, Virginia.* Assuming the continued fraction to be convergent, there is no loss of generality in calling its value $e^{-\theta}$. Then

$$e^{\theta} = a + \frac{1}{a} \, \frac{}{+} \cdots = a + e^{-\theta},$$

$$2\sinh\theta = a,$$

and

$$\theta = \arg\sinh \tfrac{1}{2}a.$$

II. *Solution by the proposer.* In terms of $\theta = \arg\sinh\frac{1}{2}a$, the successive convergents are

$$\frac{\cosh\theta}{\sinh 2\theta}, \quad \frac{\sinh 2\theta}{\cosh 3\theta}, \quad \frac{\cosh 3\theta}{\sinh 4\theta}, \quad \frac{\sinh 4\theta}{\cosh 5\theta}, \cdots.$$

The limit of this sequence is clearly $e^{-\theta}$.

Also solved by Murray Barbour.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4145. *Proposed by Howard Eves, Syracuse University*

Find the positions of three non-overlapping circles in a triangle which have a maximum combined area.

4146. *Proposed by G. B. Lang, Emory University*

Suppose that n men check their hats, and then each man takes a hat at random. Let $g(n)$ denote the probability that no one gets his own hat. Find the value of $g(n)$ and its limit for $n\rightarrow\infty$.

4147. *Proposed by V. Thébault, San Sebastián, Spain*

Two circles varying in magnitude and position roll on two fixed circles. Find the loci of their centers of similitude. If the straight line of their centers has a constant direction, the midpoint of the segment of their centers describes a straight line.

SOLUTIONS

The Game of Fourteen

4093 [1943, 516]. *Proposed by B. M. Stewart, Michigan State College*

A store conducted the following game in which the player paid 10¢, and if he won received \$1 in merchandise. Out of five throws, each time throwing ten dice, a win was declared if the player had *fourteen* or more appearances of the numbered face named by him before playing; the player had the additional advantage that on the *first* throw of ten dice he could count the face occurring the greatest number of times as if it were the special face he had selected. The problem is to compare the theoretical probability with the odds offered by the store.

Solution by A. G. Clark, Colorado State College. The only complicating factor is the privilege of choosing the face which occurs the greatest number of times on the first throw. If $i=2, 3, \cdots, 10$ denotes the greatest number of times any face whatever occurs on the first throw, then the probability of such occurrence is given by

$$\sum_i \frac{6!10!6^{-10}}{\sum_{j=1}^{i}(j!)^{n_j}n_j!} \qquad \text{or} \qquad \sum_i \text{ for brevity,}$$

where the n_j are non-negative integers such that

$$\sum_{j=1}^{i} jn_j = 10$$

and the summation for a given value of i is over all combinations of the n_j which satisfy this condition.

The probability that i occurrences of some face will be followed by a total of $14-i$ or more occurrences of that face on the next four throws of the ten dice will evidently be

$$6^{-40} \sum_{r=14-i}^{40} \binom{40}{r} 5^{40-r}$$

which for purposes of computation can be expressed in terms of beta functions, *i.e.*,

$$I_{1/6}(14 - i, 27 + i) = 1 - I_{5/6}(27 + i, 14 - i).$$

The probability of winning the game will then be

$$\sum_{i=2}^{10} I_{1/6}(14 - i, 27 + i) \sum_i .$$

A tabular arrangement of the computation follows. Interpolations for $I_{1/6}(14-i, 27+i)$ use only first differences.

i	$I_{1/6}(14-i, 27+i)$	$\sum_i$	Product
2	.026	.0675	.00176
3	.058	.5293	.03070
4	.116	.3105	.03602
5	.211	.0781	.01648
6	.347	.0130	.00451
7	.509	.0015	.00076
8	.677	.0001	.00007
9	—	—	—
10	—	—	—
	The probability of winning is		.09030.

Thus, the player's expectation is about 9¢.

Note: A more interesting and much more commonly played variant of this game is for the player to let lie those dice which turn up the face he has chosen and to recast the remaining dice each time. The player would win if at the conclusion of five casts the total number of favorable dice amounts to say seven or more.

Solved also by E. P. Starke and the proposer. The other two solutions are similar except in the modifications of the final formula for computation. Starke's result is .09108 and the proposer's result lies between .091066 and .091067.

An Alternant Type of Determinant

4098 [1943, 569]. *Proposed by P. R Halmos, Urbana, Illinois*

Evaluate the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & & x_n^2 \\ x_1^4 & x_2^4 & x_3^4 & \cdots & x_n^4 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ x_1^{2^{n-1}} & x_2^{2^{n-1}} & x_3^{2^{n-1}} & \cdots & x_n^{2^{n-1}} \end{vmatrix}$$

Solution by Howard Eves, Syracuse University In 1841 Jacobi established the following factorization of the general simple alternant:

$$(1) \qquad \left| x_1^{\alpha_1}, x_1^{\alpha_2}, \cdots, x_1^{\alpha_n} \right| = \zeta^{1/2}(x_1x_2\cdots x_n) \left| H_{\alpha_1-n+1}, H_{\alpha_1-n+2}, \cdots, H_{\alpha_1} \right|,$$

where $\zeta^{1/2}(x_1x_2\cdots x_n)$ is Sylvester's notation for the continued product

$$(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)(x_2-x_3)\cdots(x_2-x_n)\cdots(x_{n-1}-x_n)$$

and where H_j is the complete homogeneous polynomial with unit coefficients of degree j in $x_1, x_2, \cdots, x_n$.

Using this theorem the determinant of the problem can be factored as

$$\left| x_1, x_1^2, x_1^4, \cdots, x_1^{2^{n-1}} \right| = (x_1x_2\cdots x_n)\zeta^{1/2}(x_1x_2\cdots x_n)D,$$

where

$$D = \begin{vmatrix} H_{-n+1} & H_{-n+2} & \cdot H_{-3}H_{-2}H_{-1}H_0 \\ H_{-n+2} & H_{-n+3} & \cdot H_{-2}H_{-1}H_0\ H_1 \\ H_{-n+4} & H_{-n+5} & \cdot H_0\ H_1\ H_2\ H_3 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ H_{-n+2^{n-1}} & H_{-n+2^{n-1}+1}\cdot & \cdots\ H_{2^{n-1}-1} \end{vmatrix}$$

Since $H_{-j}=0$ and $H_0=1$, D reduces to

$$(2) \qquad -\begin{vmatrix} H_{-n+4} & H_{-n+5} & \cdot H_0 & H_1 \\ \cdot & \cdot & \cdot & \cdot \\ H_{-n+2^{n-1}} & H_{-n+2^{n-1}+1}\cdot & H_{2^{n-1}-4} & H_{2^{n-1}-3} \end{vmatrix}$$

Denoting the determinant of the problem by Δ_n, and the determinant in (2) by H, we then have

$$\Delta_n = -(x_1x_2\cdots x_n)\zeta^{1/2}(x_1x_2\cdots x_n)H,$$

where H is a determinant of order $n-2$. It seems difficult to improve much on this partial expansion of Δ_n.

For the special cases $n=3$ and $n=4$ we find

$$\Delta_3 = - x_1x_2x_3(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3)$$

and

$$\Delta_4 = - x_1x_2x_3x_4(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)(H_5 - H_1H_4).$$

Jacobi's relation (1) was established in an improved manner by N. Trudi in 1864. Others who have published the relation are Clebsch and Gordan (1868), Naegelsbach (1871), Malet (1874), Kostka (1875), Cayley (in a note in Salmon's *Modern Higher Algebra*, 3rd ed. 1876), Garbieri (1878), and Mansion (1879). See vol. iii, topic *Alternants*, of Muir's *History of Determinants*, or Chapter XI of *Theory of Determinants* by Muir and Metzler.

Pedal Curves

4099 [1943, 569]. *Proposed by J. H. Butchart, Grinnell College*

If P is any point of a curve and Q is the corresponding point of the pedal with respect to the point O, then OQ makes the same angle with the pedal that OP makes with the curve.

Solution by A. Sisk, Maryville, Tenn. Suppose the given curve c is well behaved, with simple tangents near P. Let P' be near P on c, and suppose the tangents to c at P and P' intersect in T. Take Q' as the image of P'. Let angles OTQ' and OQQ' be denoted by θ and ϕ. Since the points O, T, Q, Q' are cyclic, $\theta=\phi$. Hence the limiting values of these angles are equal as P' approaches P along c, causing T to approach P along PQ and Q' to approach Q along the pedal.

For this problem and others kin to it, see Hilton's *Plane Algebraic Curves*, p. 166 · · · .

Solved also by H. Demir, H. Eves, S. Hughart, H. Siller, and P. D. Thomas.

Editorial Note. The solution by Eves is similar to the above; he gave the above reference and also Williamson's *Differential Calculus*, 1927, p. 227, and remarked that this is an old and well known result usually established in that manner. Siller remarked that the problem is solved in the second reference. Hughart uses polar coordinates and derivatives. Thomas uses rectangular coordinates and derivatives.

Demir proved the theorem for the case where the angle PQO is an arbitrarily given angle by considering the instantaneous center I of rotation of the rigid system of two distinct straight lines QP and QO, the first tangent at P to the curve $\{P\}$ and the second passing through the fixed point O. For any given position of P the point I is the point of concurrence of the normals PI, QI to the curves $\{P\}$, $\{Q\}$, and the perpendicular to QO at O. If QT and PQ are corresponding directions of the tangents for this position, then it easily follows that angles OQT, OPQ, QIO are equal. For the theory of the point I he referred to Mannheim, *Principes et développements de géométrie cinématique.* However, this theory is quite simple in this case and probably is given in many books on kinematics and some on geometry.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

At New York University Associate Professor F. W. John has been promoted to a professorship and H. E. Wahlert to an assistant professorship.

In the department of mathematics and mechanics of the University of Minnesota, Assistant Professors H. A. Doeringsfeld and F. E. Miller have been promoted to associate professorships; Assistant Professors H. L. Turrittin and N. R. Amundson have been granted leaves of absence, the former to serve as a mathematician with the Operations Analysis Division of the Army Air Force and the latter as Fellow in the Applied Mechanics Program at Brown University.

Associate Professor L. E. Babcock of Newberry College, South Carolina, has been appointed to an assistant professorship at the University of Richmond.

Ralph Mansfield of Chicago Teachers' College has been appointed research engineer with Jos. Weidenhoff, Inc., Chicago, Illinois.

Helen K. Milleson of Carleton College has been appointed statistician in the Bureau of Census, Washington, D. C.

Assistant Professor Edwin Nilson of Mount Holyoke College has been given leave of absence to serve in the United States Navy.

Professor C. F. Thomas of Case School of Applied Science has been appointed head of the mathematics department.

Assistant Professor C. J. Thorne of Louisiana State University has been granted leave of absence to serve as development engineer in the Curtiss Wright Corporation at Bloomfield, New Jersey.

The following appointments to instructorships are announced:
Coe College (Iowa): Aletha C. Gaddis
Sunflower Junior College, Moorhead, Mississippi: Virginia Felder
University of Buffalo: R. C. Luippold
University of New Mexico: E. Marie Hove

Professor J. S. Miller of Emory and Henry College, Emory, Virginia, died March, 16, 1944.

Ruth Newlin of Creston Junior College and High School, Creston, Iowa, died October 20, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

AID TO LIBRARIES IN WAR AREAS

Up to the end of 1943, $160,000 had been spent for subscriptions to 325 scholarly and scientific journals, to be stored in this country, for distribution after the war to libraries in war areas. The money is provided by a grant from the Rockefeller Foundation, which has allotted from $50,000 to $70,000 annually for this purpose since 1941. The fund is administered by the Committee on Aid to Libraries in War Areas of the American Library Association, and inquiries should be addressed to Miss Edith A. Wright, Committee on Aid to Libraries in War Areas, Library of Congress Annex, Study 251, Washington, D. C.

Originally the committee was formed to study ways in which libraries in this country could help similar institutions abroad in their struggle with wartime conditions, and especially to find some means of preserving for later distribution U. S. publications not available because of the war. The Rockefeller grant has enabled the committee to implement its findings.

The American Library Association and the constituent societies of the American Council of Learned Societies have prepared lists of outstanding publications in their various fields of interest which were published in this country during 1939–44. They will be available to individuals and groups interested in the selection of books for foreign libraries.

In the case of periodicals, several factors influence orders placed for each title, namely, the journal's foreign circulation before the war, the ratio of foreign circulation to total circulation, the number of free or exchange subscriptions, the number of copies available through gifts from subscribers in this country, and the relative importance of the journal to research. In no case have purchases equaled more than half the number of paid institutional subscriptions in Europe and Asia discontinued because of the war. For this reason, and because no funds are available for purchasing periodicals from individuals nor for the acquisition of volumes issued before 1939, the committee has been gratified to note the efforts of governments-in-exile and other groups to maintain periodical subscriptions for important libraries in various countries.

The committee's periodical purchases are supplemented by gifts. In this way current issues and back files of important periodicals have been secured for use in restocking libraries that have been damaged or destroyed. Free storage space is difficult to obtain, however, and whenever possible donors are urged to store materials until central storage space is available. Transportation costs are paid,

but material accepted for storage is limited to what seems of first importance. Gifts are being stored in a number of libraries in all parts of the country, and offers of additional storage space are always welcome. Prospective donors are asked to first report titles and dates of the journals available to the office of the committee. Shipping instructions will then be issued, indicating where and how shipment should be made. The committee has prepared three lists of desirable periodicals, which are available on request: a general list, a list of medical journals, and a list of technical periodicals.

No definite decisions as to disposal of books and periodicals can be made until more is known of library conditions in war areas. Present efforts of the committee are therefore confined to collecting material, obtaining all available information about the needs of libraries, and recording what is being done in the same field by other organizations.

The mathematical journals especially desired as gifts are the following: American Journal of Mathematics, American Mathematical Monthly, Bulletin of the American Mathematical Society, Transactions of the American Mathematical Society, Journal of the American Statistical Association, Annals of Mathematics, Annals of Mathematical Statistics, Duke Mathematical Journal, Journal of Mathematics and Physics, Mathematical Reviews, Quarterly of Applied Mathematics, Review of Economic Statistics, School Science and Mathematics, and Scripta Mathematica. The committee is now purchasing some copies of all these journals with the funds made available by the Rockefeller Foundation. However, it is certain that additional copies will be needed. There is a serious demand, also, for long runs of the various journals published before the war to replace those that have been destroyed. In addition to the titles listed in mathematics, there is a need for similar journals in other fields and some of the general scientific magazines, such as Science and The Scientific Monthly.

EDUCATION OF STUDENTS FROM WAR-TORN COUNTRIES

A conference on providing educational opportunities for students from the war-torn countries was held in Philadelphia, February 28–29, 1944; the consultation included representatives of twenty-four national educational associations and representatives of six agencies of the Federal Government. The Institute of International Education and the Association of American Colleges joined with The Edward W. Hazen Foundation in sponsoring the conference. A general statement of principles adopted by the group follows.

"It is evident that the war-torn countries will be desperately in need of assistance in educational and cultural reconstruction after the war. Our people very properly will wish the Government of the United States to cooperate in this program. In doing so, the first step may be to negotiate bilateral agreements for the reception and exchange of students and teachers; and to do this in collaboration with the institutions and agencies representing both publicly supported and privately supported education in the United States. We believe that education of students from other countries is primarily the function of our existing

educational institutions. We anticipate, however, that the magnitude of the task will require supplemental support from our Government, in addition to that which will be supplied by other Governments, the institutions themselves, and by other agencies. In cooperation with the U. S. Office of Education, other Government departments, and with educational institutions, and other agencies, the Department of State should formulate the general policies governing this program, based upon the determined needs. It should also negotiate agreements with foreign governments and obtain, in collaboration with other agencies, the necessary data on which the program is to be founded. It should survey existing and proposed legislation and immigration procedures, with the view to facilitating the entrance and residence of students from other countries by removing unnecessary restrictions upon such students and also with the view of extending privileges enjoyed by such students to trainees from other countries who come for training in industrial, business or government agencies. The educational institutions and agencies should provide information as to their resources and facilities. They should undertake to adapt their regular curricula to the students coming to the United States or organize other programs to meet the particular needs. Where possible, they should provide some assistance to the students. The sponsors of this conference are authorized to organize a small provisional continuing committee, consisting of representatives of educational associations of the country. This continuation committee shall cooperate with officials appointed by the Government and will assist in formulating policies regarding an appropriate program. It should plan procedures for enlisting the resources of educational institutions and agencies making use of available experience and facilities. It should enlist public interest and support."

The continuation committee of the conference has further investigated the problems involved. This led to recognition of a need for some cooperative agency in the United States to give continuing leadership in the field. A proposal for such an agency has been drafted. This confidential document has been submitted to the Division of Cultural Cooperation of the Department of State for study and appropriate action. It is to be expected that when a decision regarding next steps has been reached, colleges, universities, and educational associations will be informed promptly. In the meantime, information concerning educational facilities that may be available for students from war-torn countries is being assembled.

ACADEMIC STANDARDS FOR V-12 OFFICER CANDIDATES

Previous to the start of the Navy V-12 term upon November 1, 1944, commanding officers of the various units were requested to scrutinize the academic qualifications of men completing their V-12 studies upon that date. The exact instruction follows.

"Any student who is not in good standing academically upon completion of his allowed number of terms in the Navy V-12 Program, regardless of the fact that he may have successfully completed the *minimum* course requirements for the type of candidacy to which he has been assigned, shall not be considered

academically qualified for further training as an officer candidate, nor shall a student who fails to obtain a satisfactory grade with respect to officer aptitude be continued in the program. A mark of less than 2.5 (on the Navy 4.0 basis) is regarded as unsatisfactory."

Presumably, the requirements explained in this communication will be maintained for the duration of the V-12 Program. A mark of 2.5 is approximately equivalent to an average grade of D upon the usual scale A, B, C, D, and F.

CIVIL SERVICE APPOINTMENTS FOR MATHEMATICIANS

On October 9, 1944, the U. S. Civil Service Commission announced that applications would be accepted for war service appointment as "mathematician." Although many mathematicians have been employed in war research by agencies of the Federal Government, there has been limited recognition of mathematics as a profession until recent months. Thus, the announcement of the Civil Service Commission is of considerable interest.

The duties of the mathematician are defined as follows:

A. To plan, direct, conduct or assist in the planning or conducting of mathematical research.

B. To make specialized computations.

C. To make investigations in the field of applied mathematics involving (1) the analyses of practical and theoretical problems requiring the use of advanced mathematical theory and principles and (2) the interpretation of mathematical results in terms of physical concepts so that the results may be used in engineering practice and scientific work.

D. To develop and introduce new mathematical procedures and check current mathematical practices.

The qualifications of the mathematician are described below.

Grade P-1. At least 3 years of progressive technical experience in mathematics requiring an intimate knowledge of the theory and the applications of the principles of all the following: algebra, trigonometry, analytical geometry, and differential and integral calculus. This experience must be of such scope and character as to demonstrate clearly that the applicant's training and ability are adequate for the conduct of the mathematical duties indicated above. Salary: \$2,000+overtime pay of \$433.

In addition to the minimum requirement specified for the P-1 grade, applicants for grades P-2 and above must show additional experience of a progressively higher level as follows:

Grade P-2. At least 2 years of professional work in mathematics, involving the use of the principles of theoretical or applied mathematics in the solution of scientific and related problems. Salary: \$2,600+overtime pay of \$563.

Grade P-3. At least 3 years of progressive professional experience in mathematics of such character as to demonstrate capacity for original mathematical research or the ability to plan and develop mathematical projects and to prepare full reports of work accomplished. Salary: \$3,200+overtime pay of \$628.

Grade P-4. At least 5 years of responsible progressive professional experience in mathematics of such character as to demonstrate marked capacity for original research, outstanding professional attainments, or the ability to plan, administer, or perform productive research in mathematics and to develop mathematical projects. Credit will be given for all valuable experience of the type required, regardless of whether compensation was received or whether the experience was gained in a part time or full time occupation. Such experience will be credited on the basis of time actually spent in appropriate activities. Salary: \$3,800+overtime pay of \$628.

Nonqualifying Experience. Experience involving routine computations or statistical compilations, work in pure statistics, or high school teaching of mathematics will not be accepted as qualifying professional experience in mathematics.

Substitution of Education for Experience. Successful completion of pertinent undergraduate study in a college or university of recognized standing may be substituted for the required experience on the basis of 1 year of academic study for each 9 months of experience, provided that for each year of academic study substituted, the applicant must show at least 5 semester hours in algebra, trigonometry, analytical geometry, differential and integral calculus, the solution of differential equations, or other courses in higher mathematics.

Graduate study in mathematics successfully completed in a college or university of recognized standing may be substituted year for year for the experience prescribed for these positions up to a maximum of 3 years of experience. In order to substitute the full 3 years of graduate study, applicants must have completed all the requirements for the Ph.D. degree, including the thesis.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE MAY MEETING OF THE WISCONSIN SECTION

The twelfth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Milwaukee Downer College on Saturday, May 13, 1944. Sessions were held in the morning and in the afternoon, with Professor May M. Beenken, Chairman of the Section, presiding.

There were twenty-six in attendance, including the following thirteen members of the Association: R. H. Bardell, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, R. C. Huffer, Lionel London, C. C. MacDuffee, Sister Mary Felice, A. C. Moeller, R. E. Norris, H. P. Pettit, J. I. Vass, Louise A. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, Morris Marden, University of Wisconsin at Milwaukee; Program Committee, Sister Mary Felice, Mount Mary College, B. R. Ullsvik, Eau Claire State Teachers College. It was voted that the next meeting be held in

May, 1945, at Milwaukee State Teachers College, the exact date to be set by that institution.

The following papers were presented:

1. *The Hagge circle of a point in the plane of a triangle*, by Sister Mary Felice, Mount Mary College.

This paper was a summary of an article entitled *Der Fuhrmannsche Kreis und der Brocardsche Kreis als Sonderfälle eines allgemeineren Kreises* which was published by K. Hagge in 1836. The Hagge circle of the incenter of a triangle is the Furmann circle of that triangle, while that of the centroid is the Brocard circle of the triangle formed by the lines drawn through the vertices of the triangle parallel to the opposite sides. The Hagge circle of a point on the circumference of the circumcircle of the triangle provides a proof for Steiner's theorem on the collinearity of the orthocenters of the triangles of a complete quadrilateral. Finally, proofs by the methods of pure geometry were given for some theorems enunciated by A. M. Peiser in his paper *The Hagge circle of a triangle* which was published in this MONTHLY, vol. 49, 1942, pp. 524–527. Peiser had proved the theorems by the method of conjugate coördinates.

2. *Groups, quasigroups, and Cayley squares*, by Dr. R. H. Bruck, University of Wisconsin, introduced by Professor Kenney.

Dr. Bruck discussed the interrelations of the theories of groups, loops, quasigroups, and Cayley, or latin, squares. He outlined some of the fundamental theorems common to these subjects, and touched upon certain applications to algebra, geometry, and biometry. He pointed out that Cayley squares may be used to give concrete meaning to abstract concepts such as those of homomorphism and isotopy.

3. *Determination of latitude in an emergency*, by Joel Brenner, Lawrence College, introduced by Professor Marden.

Mr. Brenner's paper was published in this MONTHLY, vol. 51, 1944, p. 343.

4. *The rise of matrices and normal coördinates for the solution of torsional vibration problems*, by K. E. Bisshopp, Fairbanks-Morse Company, introduced by Mr. Battig.

Matrix methods are convenient for obtaining numerical solutions of problems involving the calculation of normal modes and frequencies of torsional vibration in systems with many degrees of freedom. In general, the types of systems considered have no fixed points, so that it is necessary to remove the rigid body mode of vibration corresponding to the degenerate frequency $p=0$ before inverting the stiffness matrix. The contribution of this paper is the presentation of a general method for computing the inverse matrix of a free system comprising an arbitrary number of masses connected by elastic shafts without mass. The usual iterative process based upon Sylvester's theorem then can be used to evaluate the frequencies successively in increasing order of magnitude

with the aid of the orthogonality relations between the normal modes. The elements of the modal columns obtained by iteration are proportional to the factors required for the reduction of the original system to normal coördinates. The practical advantage of the normal coördinate method is that a partial fraction expansion for the forced vibration amplitudes can be constructed directly from the matrix results, thereby avoiding a tedious torque summation calculation.

5. *Functions and services of the Armed Forces Institute*, by Lt. Col. C. W. Hansen, AGD, Commandant, United States Armed Forces Institute, introduced by Miss Wolf.

The United States Armed Forces Institute provides instructional services whereby military personnel may continue their education while in military service. Eight branches have been established overseas. Three types of instructional service are provided, namely: (1) institute correspondence courses, which are short courses ranging from high school to junior college level, and from academic to vocational and technical subjects; (2) university extension correspondence courses, which are regular university extension courses offered through the institute at reduced cost; (3) self-teaching texts and special editions in a wide variety of subjects, which may be studied by individuals on their own initiative or in study groups.

Service men and women may if they desire have the Institute prepare reports of their training and experience to be submitted to civilian schools for credit evaluation, or, at the time of discharge or later, to employers for use in occupational placement. The Institute reports facts only, leaving all evaluation up to civilian schools. Tests designed to measure educational development are administered as part of the service.

P. L. TRUMP, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA, Cedar Rapids, April 14, 1945
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, Brooklyn, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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AN ISOPERIMETRIC PROBLEM WITH AN INEQUALITY

CLAIRE FISHER ADLER, New York University

The problem of this paper is a generalization of the classical isoperimetric problem: "Of all curves inclosing a domain of given area, to find one of minimum length." It was suggested by Dr. R. Courant in connection with new types of variational problems, some of which can be illustrated by soap film experiments.*

1. Formulation of the problem. Let M denote a closed, continuous, sectionally smooth curve in a bounded, closed, plane region R. That is, every such curve consists of a finite number of arcs each of which has a continuously turning tangent at each of its points up to and including its end points. Such a curve can therefore have at most a finite number of corners or cusps. Also, each such curve can be represented parametrically by equations of the form: $x=x(t)$, $y=y(t)$, $t_0\leqq t\leqq t_1$, the curve being described just once as the parameter t describes the interval t_0, t_1. The correspondence between points $P(t)$ of curve M and values of t need not however be one-to-one. That is, to an arc as P_1P_2 (Fig. 1) may correspond more than one sub interval of t_0, t_1. Such an arc will be said

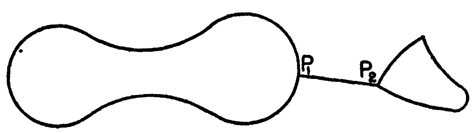


FIG. 1

to be *described more than once* as $P(t)$ describes the curve once. From this fact it follows that M is not necessarily a Jordan curve. In order therefore to speak of a domain inclosed by such a curve it will be necessary to extend the ordinary concept of exterior, interior points of a (Jordan) curve. To do so, join a point $P(t)$ of curve M to any point Q not on the curve and denote by $\theta(t)$ the angle which the vector QP makes with the positive x axis. Then, as the parameter t describes the interval t_0, t_1 the vector QP generates an angle $\theta(t_1)-\theta(t_0)=2\pi n$ where the number n is independent not only of the parametric representation of the curve but also of the determination of $\theta(t)$. The number n is called the *index* of *the point* Q *with respect to curve* M. As is well known, when curve M is a Jordan curve, exterior, interior points are points of index 0, 1 respectively. We shall consider therefore only those curves M for which *every point not on the curve is of index* 1 *or* 0. *By definition, exterior, interior points of curve* M *are points of index* 0, 1 *respectively.*

Points of index 1 form an open set D consisting of a finite number of domains

* See "What Is Mathematics" by R. Courant and H. Robbins, pp. 385–397, and a paper by R. Courant to be published after the war.

inclosed by the curve. *The area of the set D is said to be the area inclosed by the curve. If D is void, M is said to inclose zero area.*

If a point P of M is a boundary point of a domain inclosed by curve M, the point P is called a *boundary point of the curve*; otherwise, point P is a *non-boundary point.*

The *length of curve M* is the sum total of the lengths of the arcs of the curve, *each arc being counted as many times as it is described.*

γ denotes the class of all such curves M.

Admissible curves are those curves of the class γ which inclose a domain or domains 1) of a prescribed fixed area $S \geqq 0$ and 2) these domains contain a given set A of fixed points in their interior or on their boundaries. Points of the set A are denoted $A_1, A_2 \cdots A_n$, $n > 2$. The problem to be solved is then:

PROBLEM. *In the class of admissible curves find one whose length is a minimum.*

The existence of a solution is assumed in Part I of the paper and proved in Part II. C always denotes a solution and $L(S)$ its length. Part I contains 1) properties of a solution C showing its form or structure and 2) continuity considerations showing the dependence of the length $L(S)$ of curve C on the area S. In addition to existence proofs, Part II contains an illustrative example showing the changing form of a solution as the area increases.

From the classical theory, it is known that, C is a circle, when the condition 2 of admissibility, called an *inequality*, is *unessential*, *i.e.*, when the problem *with* the inequality has the same solution as the problem *without* the inequality. Only the *essential* inequality is considered here.

Part I

2. Three preliminary properties of a solution C.

PROPERTY 1. *Every non-boundary point of C is on a straight line segment and every boundary point on an arc of a circle (which may be a straight line).*

a) Every non-boundary point of C is on a straight line segment since such a point must be connected to a boundary point by a curve of shortest length: *i.e.*, a *straight line segment.*

b) Every boundary point is on an arc of a circle:

Proof. From the definition of an admissible curve, C consists of a *finite* number of smooth arcs and incloses only a finite number of domains. Moreover, two domains may have a common boundary arc only if the arc is a point for otherwise removal of this common boundary arc would result in an admissible curve of smaller length than C thus contradicting the hypothesis that C is a solution. Hence *the only arcs of C which are described more than once are straight line segments.* Let G denote the finite set of boundary points consisting of:

1) *Extremities of line segments,*
2) *Points common to the boundaries of different domains,*
3) *Points of the set A which are on C.*

Then, if a boundary point P is not a point of the set G, there exists a positive quantity ϵ such that the circle whose center is P and radius ϵ meets C in only two points, R and Q (Fig. 2), contains points of one domain only and does not contain any points of the set A. From the classical theory, it is known that in this circular domain the arc of smallest length joining R and Q is an arc of a circle. Hence the property is proved for a boundary point not belonging to the set G.

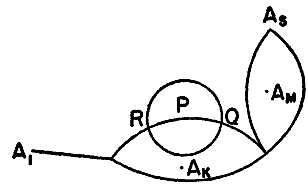


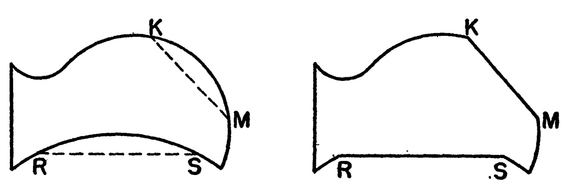
FIG. 2

However, since these arguments are valid as long as the point P is not a point of the set G, an extremity of an arc of a circle must be a point of the set G. C therefore consists of straight line segments and arcs of circles. If extremities of arcs of circles be called *end-points* it has also been shown that:

COROLLARY: *End-points of C belong to the set G.*

PROPERTY 2. *If one arc of a circle of C is convex (concave), every arc of a circle is convex (concave).*

For, otherwise, it is always possible to replace a convex and a concave arc by their respective chords KM and RS (Fig. 3a) where the arcs are so chosen that the resulting curve is still admissible. But since chords have smaller length than their arcs, the new curve (Fig. 3b) has smaller length than C and from this contradiction Property 2 follows.



(a) (b)

FIG. 3

PROPERTY 3. *Arcs of circles of C have equal radii.*

For, otherwise, there exist chords PQ of radius r_1 and $P'Q'$ of radius $r_2 \neq r_1$ such that the chords are equal and the domains inclosed by these arcs and their respective chords do not contain any points of the set G or of the set A. Then the curve which results by interchanging arc PQ with arc $P'Q'$ is still an admis-

sible curve but an end point such as P' (Fig. 4b) does not belong to the set G. Since this contradicts corollary, property 1, the property is proved.

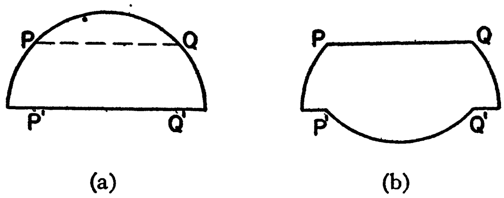


FIG. 4

3. Continuity considerations. The dependence of the length $L(S)$ of C on the area S is given by a theorem whose proof depends on the following:

LEMMA. *The length $L(S)$ of C is a) a monotonic increasing function of S if an arc of a circle of C is convex and b) a monotonic decreasing function of S if the arc is concave.*

Proof: If a convex, or a concave arc of a circle of C, inclosing with its chord a domain not containing any fixed points, be replaced by its chord, the new curve E 1) is still admissible, and 2) incloses area $S'<S$ if the arc is convex and area $S'>S$ if the arc is concave. Moreover, since a chord is less than its subtended arc, the length $E(S')$ satisfies:

$$E(S') < L(S). \tag{1}$$

But, since $L(S')$ is the length of a solution inclosing area S',

$$E(S') > L(S'). \tag{2}$$

Therefore:

$$L(S') < L(S) \tag{3}$$

where $S'<S$ if the arc is convex and $S'>S$ if the arc is concave and thus the lemma is proved.

CONTINUITY THEOREM. *The length $L(S)$ of a solution C inclosing a domain or domains of total area S varies continuously with S.*

Proof. Consider first a solution C in which an arc of a circle of C is convex. Let a domain of area h be adjoined to a domain inclosed by C by replacing an arc PQ by a circular arc PQ of different radius (Fig. 5). Such a domain can always be chosen so that the resulting curve E is still admissible when the inclosed area is $S+h$. Moreover, if h is a sufficiently small positive number there exists an $\epsilon>0$ such that if $E(S+h)$ denote the length of E, we have

$$0 < E(S+h) - L(S) < \epsilon \tag{1}$$

where $L(S)$ denotes the length of C. But,

$$E(S+h) \geqq L(S+h) \tag{2}$$

since $L(S+h)$ is the length of a solution inclosing area $S+h$. From the lemma just proved:

$$L(S + h) > L(S). \tag{3}$$

Therefore

$$0 < L(S + h) - L(S) < \epsilon \tag{4}$$

and the theorem is proved when an arc of C is convex. If an arc of C is concave, the domain of area h may be chosen 1) as a sub-domain of a domain inclosed by C and 2) so that the domain contains no points of the set A. Then E incloses area $S-h$; and if $S-h$ replace $S+h$, the above inequalities still hold. That is, the theorem is proved also when an arc of C is concave. Since these arguments hold for any value of S however small, $L(S)$ is continuous at $S=0$.

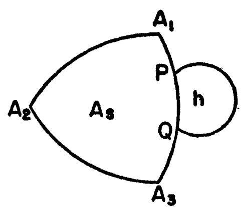


FIG. 5

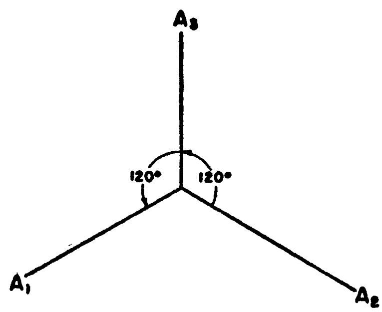


FIG. 6

4. Additional properties of C. Steiner's classic problem "To find a point O at a minimum distance from three fixed points," is the problem of this paper when $S=0$ and $A_1A_2A_3$ are the given points. Its solution in theorem form is:

THEOREM A. (STEINER). *The point O at a minimum distance from three fixed points $A_1A_2A_3$ is the point O (Fig. 6) from which each of the three sides of the triangle $A_1A_2A_3$ subtends an angle of* 120°, *if no angle of the triangle is greater than or equal to* 120°; *otherwise, it is the vertex of the largest angle of the triangle.*

PROPERTY 4. *When $S=0$, C consists of straight line segments and no more than three of them meet in a point.*

For, by definition, every point of C is a non-boundary point when $S=0$. Hence, by property 1, C consists of straight line segments. Suppose now that more than three of these segments meet in a point P; then one of the angles at P say A_1PA_2 is less than 120°, and since C is a solution this is the shortest path connecting points A_1, P, and A_2. Since this contradicts Theorem A, Property 4 follows.

Let a point of meeting of exactly 3(2) line segments of C be called a *regular* (*degenerate*) contact point, or, where no confusion will arise, simply a *contact* point.

From Theorem A it then follows immediately that:

PROPERTY 5. *a) Each obtuse angle formed by lines meeting at a regular contact point equals* 120°; *b) the obtuse angle formed at a degenerate contact point is* $\geqq 120°$.

The smoothing effect of the minimizing property is shown by:

PROPERTY 6. *A line segment of C is tangent to an arc of a circle of C at its point of meeting with the arc.*

Proof. Suppose the contrary, *i.e.*, a solution C has a corner at an end point e lying on a line segment of C. Fig. 7a. There is then a point P of C so near e that if r is the radius of arcs of circles of C, a circle of radius r and through P meets C

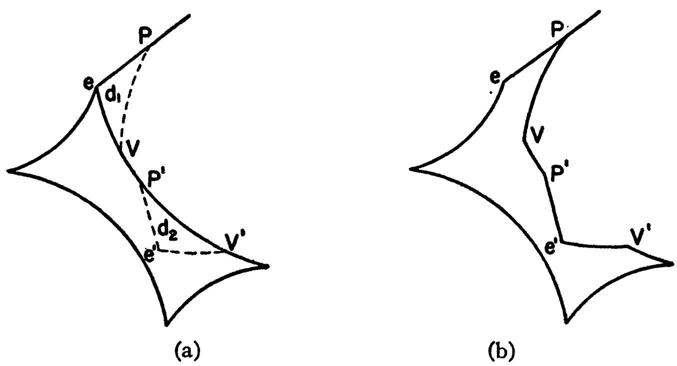


FIG. 7

in another point V and incloses with arc PeV a domain d_1 of arbitrarily small area and outside any domain inclosed by C. Also, there exists a closed domain d_2 congruent to d_1 and its boundary, and not containing any points of the set A. Now, replace arcs PeV and $P'V'$ by arcs PV and $P'e'V'$ respectively. There results an admissible curve E, Fig. 7b (by hypothesis, the arc Pe was described *twice* and removed *once*). Moreover E has the same length as C, for E is simply curve C except for 1) arc PeV which is replaced by arc PV and 2) arc $P'V'$ which is replaced by arc $P'e'V'$, and $P'e'V'$ is congruent to PeV. But E has an endpoint as P' not belonging to the set G (page 60) and hence by corollary, property 1, its length and therefore the length of C is not a minimum. From this contradiction the property follows.

Similar arguments hold if 1) e is a point common to the boundary of two different domains or 2) C contains both a contact point and incloses a domain of area $S>0$. Hence, if a contact point be called a domain of zero area:

COROLLARY 1. *If any domain inclosed by C has area greater than (equal to) zero, so has every domain.*

COROLLARY 2. *C has no corners when* $S>0$ *except possibly boundary points belonging to the set A.*

COROLLARY 3. *If C contains a line segment when $S>0$, arcs of circles of C are concave. For, otherwise, a point of meeting of a convex arc and a straight line segment would be a corner.*

LEMMA. *If S is sufficiently small, no domain contains more than one fixed point, and arcs of circles of C are concave.*

Proof. The maximum diameter of any domain is less than or equal to twice the radius r of arcs of circles of C so if d is the minimum distance between any two fixed points of the set A, and if $2r<d$, no domain contains more than one fixed point. For such a value of r, a domain must be joined to a fixed point by a line segment (of length greater than zero), and by corollary 3, property 6, arcs of circles are concave. Hence the lemma is proved.

PROPERTY 7. *C incloses at most $n-2$ domains.*

Proof. For simplicity, let a point common to the boundary of two domains be called a line segment joining the same. Similarly, if an end-point of a domain is a fixed point, the two points are said to be joined by a line segment of length zero. From corollary 1, property 6, it follows that if any domain inclosed by a solution C has area $S>0$, so has every domain. Suppose first that area $S>0$ and the conditions of the lemma are satisfied. Then each domain has at least three end-points of which, if k is the number of domains inclosed, $2(k-1)$ of them are on the $k-1$ line segments connecting these k domains. But each of the remaining $3k-2(k-1)=k+2$ end-points must be joined to a fixed point by a line segment and this line segment as well as those joining domains may contain more than one point of the set A. Thus the correspondence between these $k+2$ end-points and the n points of the set A is: 1) To each end-point e_i, $i=1, 2, \cdots, k+2$, corresponds a point A_j (or points), $j=1, 2, \cdots, n$, and 2) no two end-points correspond to the same fixed point. Therefore $k+2\leqq n$ or $k\leqq n-2$, where k is the number of domains inclosed by C, and property 7 is proved when area $S>0$.

If area $S=0$, by corollary 1, property 6, every domain d_i has zero area and d_i is a contact point. Hence there are again three line segments corresponding to each domain and the remaining arguments are the same as those just given.

If a domain with three and only three end-points be called a *triangular domain*, the next property is:

PROPERTY 8. *When* 1) *the area inclosed by C is sufficiently small and* 2) *the set A contains more than three points, C incloses more than one domain and each domain is triangular.*

Proof. Let K denote the class of admissible curves each of which incloses zero area and has more than three line segments meeting at a point. Consider the set of numbers representing the difference between the length $L(0)$ of a solution inclosing zero area and the length $M(0)$ of a curve M of class K. This set of positive numbers has a greatest lower bound $B>0$ since otherwise there would be a curve of the set whose length would be arbitrarily near $L(0)$ and by Theorem A this is impossible. Hence

$$M(0) - L(0) \geqq B. \tag{1}$$

Suppose now that C incloses an area satisfying lemma, page 65. Then in each domain d_i, $(1 \leqq i \leqq n-2)$, inclosed by C, select a point P_i (where P_i is the fixed point if d_i contains one) and join P_i to each end-point of d_i by a line segment. *Remove all arcs of circles of C.* There results a curve M of class K, if any domain of C has *more than three end-points*. Suppose such is the case. If then $\epsilon > 0$ is the area inclosed by C, there exists a positive quantity δ depending on ϵ such that the length $L(\epsilon)$ of C satisfies

$$\left| L(\epsilon) - M(0) \right| = \delta, \tag{2}$$

where if ϵ is small so is δ.

Suppose now that δ is chosen so small that

$$\delta < B. \tag{3}$$

Then from 1), 2), and 3) it follows that

$$L(\epsilon) > L(0), \tag{4}$$

which contradicts lemma, page 62, and hence property 8 is proved, and also:

COROLLARY 1. *When the area inclosed by C is sufficiently small, C incloses $n-2$ domains.*

COROLLARY 2. *A solution curve inclosing area near zero is near a solution curve inclosing zero area.*

Two additional properties, whose proofs are omitted because of their extremely elementary nature, are:

PROPERTY 9. *If each end-point of a triangular domain of C is a point of tangency of the arcs through it, the end-points form an equilateral triangle (Fig. 8a); if two and only two end-points are points of tangency, the end-points form an isosceles triangle (Fig. 8b).*

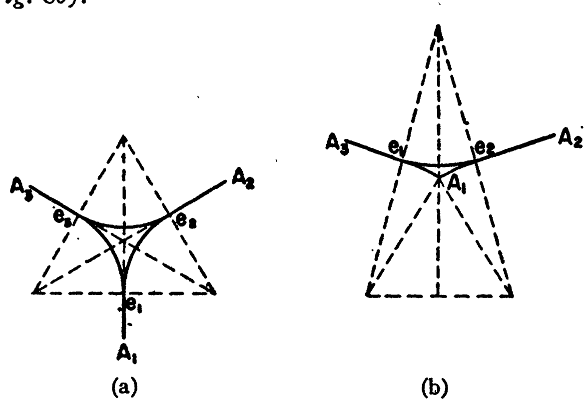


FIG. 8

PROPERTY 10. *If a domain inclosed by C has four and only four end-points each of which is a point of tangency of arcs through it, these end-points form a rectangle* (*Fig.* 9).

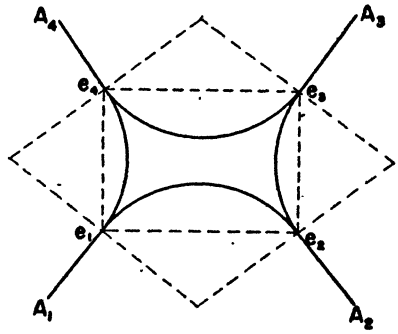


FIG. 9

This completes the description of the structure of C.

Part II

5. Existence Proof. By definition, all admissible curves inclose the prescribed constant area.

When $S>0$, denote by α the sub-class of admissible curves having properties 1, 2, 3 and 7 and

When $S=0$, denote by β the sub-class of admissible curves having properties 1, 4 and 7.

From the arguments in part I, if a solution exists in the class $\alpha(\beta)$ it is the solution of the original problem. That this solution exists is shown as follows:

EXISTENCE THEOREM. *There exists a solution in α and a solution in β.*

Proof. Let k denote the number of end-points of a curve of α or contact points of a curve of β. Then, from property 7, there exists a positive integer s such that

$$0 < k < s. \tag{1}$$

It will be shown first that a solution exists in α. Let r denote the radius of arcs of circles of a curve E of α, x_i, $y_i(i=1, 2, \cdots, k)$ the rectangular coordinates of end-points P_i of E, and

$\lambda=\text{arc tan } r$. Then if $E(S)$ denote the length of E,

$$E(S) = \Phi(x_1, \cdots, x_k, y_1, \cdots, y_k, \lambda) \tag{2}$$

where the variables $x_1, \cdots, x_k, y_1, \cdots, y_k, \lambda$, denoted simply x, y, λ, satisfy the subsidiary condition:

$$f(x, y, \lambda) = S, \tag{3}$$

and both Φ and f are continuous functions of x, y, λ. Since all of these variables

lie in a bounded, closed region T, by elementary function theory $E(S)$ has a minimum in T. Let $L_k(S)$ denote this minimum. Now from 1), k can take on only a finite set of values. The corresponding values assumed by $L_k(S)$ therefore are finite in number. In this finite set there exists a minimum $L(S)$. C is a curve whose length is $L(S)$.

If x_i, y_i, are coordinates of end-points of a curve E of β and $E(0)$ its length, then $E(0)$ is a continuous function of x and y alone and the remaining arguments are the same as those just given. Hence there exists a solution in β and the theorem is proved.

6. Example. The set A consists of four points A_i, $(i=1-4.)$ In a rectangular coordinate system in which a_i, b_i are coordinates of A_i, $a_1=b_1=0$; $a_1<a_4<a_2<a_3$; $b_2<b_1<b_3<b_4$, Fig. 10.

A solution curve C inclosing zero area is shown in Fig. 10 where C consists of the line segments containing the points A_i and meeting at contact points O_1O_2. Another solution curve C' (dotted line, Fig. 10) has its contact points $O_1'O_2'$ ly-

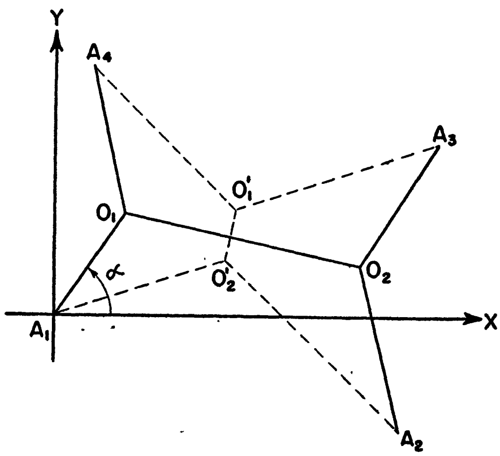


FIG. 10

ing on arcs 1) whose chords are A_4A_3 and A_1A_2 and 2) in which a 120° angle may be inscribed. If α is the inclination of A_1O_1, then

$$\tan\alpha = \frac{b_3 - (a_4 - a_2)\sin 60° + (b_4 - b_2)\cos 60°}{a_3 + (a_4 - a_2)\cos 60° + (b_4 - b_2)\sin 60°}.$$

The changing form of a solution as the area S increases is shown graphically in Fig. 11 and Fig. 12.

Fig. 11 shows a solution curve inclosing area S near zero. The number of domains inclosed is two (property 8, corollary 1), and the end-points of each domain form an equilateral triangle (property 9). Boundaries of domains are concave arcs of circles of equal radii (properties 2, 3, 6, corollary 3).

Domains are connected by line segments and fixed points are joined to domains by line segments (property 1).

As S increases, the line segment joining domains shrinks and finally disappears. A solution curve C then incloses only one domain, Fig. 12, and its end-points form a rectangle (property 10) as long as every end-point is a point of tangency of the arcs through it.

For increasing values of S, line segments through the fixed points become shorter and finally disappear and when arcs of circles of a solution curve C are straight lines (*i.e.*, circles of infinite radius) C is the quadrilateral $A_1A_2A_3A_4$, and S is its area (dotted line, Fig. 12).

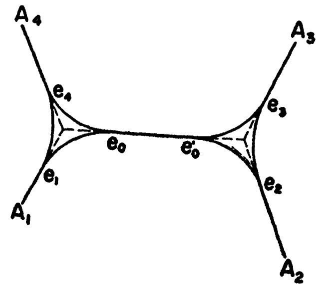


FIG. 11

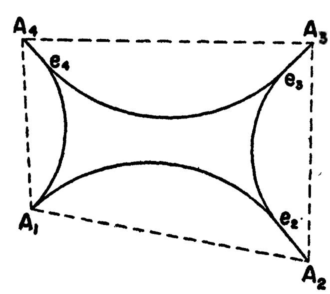


FIG. 12

For all larger values of S, curve C consists of 4 convex arcs of circles, and every end-point is a fixed point as long as S is not too large. But when the area S is so large that arcs through the vertex of the second largest angle of the quadrilateral have a common tangent, a solution C is a circle having the vertex of the largest angle of the quadrilateral on its interior. This point then represents an *unessential* inequality.

7. Generalizations. Only a few additional arguments are needed to solve the problem when any fixed point is replaced by an admissible curve.

The theory cannot be extended to a space of three dimensions, however, if by a three-dimensional space generalization is meant one in which the area is replaced by volume and the length by surface. This is seen most easily by considering the surface of minimum area inclosing a given volume, *i.e.*, a sphere. When a sphere is joined to the fixed points by spine-like projections, the area of the resulting surface can be made to approach the area of the surface of the sphere to any degree of accuracy, but in no case are the two areas equal. The problem leads directly to the theory of minimal surfaces, however, if the surface is required to contain not only the given points but also the line segments joining them. It is known that a surface inclosing a given volume has least area when its mean curvature is constant, but a general treatment of the problem along the lines of this paper has not yet been given.

CONVERGENT MONOTONE SERIES

R. W. HAMMING, University of Louisville

1. Introduction. We shall use the words "monotone series" to mean a series $\sum a_n$ with positive monotone decreasing terms, $a_1 \geqq a_2 \geqq \cdots \geqq 0$. The following theorems form a background for the present paper and their proofs are so well known that they need not be repeated here.*

THEOREM 1. *If $\sum a_n$ is a convergent series with positive terms, then $a_n \to 0$.*

This is a "best possible" theorem in the following sense.

THEOREM 1*a*. *For every sequence (p_n) approaching infinity, no matter how slowly, there exists a convergent series $\sum a_n$ with positive terms, such that $p_n a_n$ does not approach zero.*

If the hypotheses of theorem 1 are increased by supposing that the terms are both positive and monotone decreasing, then a stronger conclusion can be drawn.

THEOREM 2. *If $\sum a_n$ is a convergent monotone series, then $na_n \to 0$.*

THEOREM 2*a*. *For every sequence (p_n) approaching infinity, no matter how slowly, there exists a convergent monotone series $\sum a_n$ such that $np_n a_n$ does not approach zero.*

This paper extends the above sequence of theorems and for this the Cauchy Condensation Test is needed.

THEOREM 3. (*Cauchy Condensation Test*) *If $\sum a_n$ is a convergent (divergent) monotone series, then the series $\sum 2^n a_{2^n}$ is convergent (divergent).*

2. Cauchy derived series. The Cauchy Condensation Test replaces a monotone series $\sum a_n$ by a derived series $\sum 2^n a_{2^n}$, which we shall call *the first Cauchy derived series*. The derived series of the first Cauchy derived series will be called the *second Cauchy derived series, etc.* As shown in the following example, the first Cauchy derived series need not be monotone. Let $\sum a_n$ be any convergent monotone series. From it obtain a series $\sum b_n$ according to the following rule: for n satisfying $2^{2k} < n \leqq 2^{2k+2}$ set $b_n = a_{2^{2k+2}}$. This makes $\sum b_n$ both monotone and convergent. However its first Cauchy derived series is not monotone since

$$2^{2k+1} b_{2^{2k+1}} < 2^{2k+2} b_{2^{2k+2}}.$$

THEOREM 4. *If $\sum a_n$ is a convergent monotone series and the first Cauchy derived series is also monotone, then n* log *$na_n \to 0$.*

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* See, for example, any edition of Knopp, K. Theorie und Anwendung der Unendlichen Reihen.

Proof. The first Cauchy derived series satisfies theorem 2 so that

$$n2^n a_{2^n} \to 0,$$

$$(\log 2) n2^n a_{2^n} \to 0,$$

$$2^n \log 2^n a_{2^n} \to 0. \tag{2.1}$$

Thus a sub-sequence of $(n \log na_n)$ approaches zero. Suppose, contrary to what we wish to prove, that $(n \log na_n)$ does not approach zero. Then there exists an $\epsilon > 0$ and infinitely many m_i such that

$$m_i \log m_i a_{m_i} \geqq \epsilon.$$

However, by (2.1) there exists an n_0 such that for all $n \geqq n_0$

$$2^n \log 2^n a_{2^n} \leqq \epsilon/8.$$

Thus for $m_i \geqq 2^{n_0}$ and satisfying $2^n \leqq m_i < 2^{n+1}$

$$\begin{aligned} \epsilon \leqq m_i \log m_i a_{m_i} &< 2^{n+1} \log 2^{n+1} a_{2^n} \\ &< 2\left(\frac{n+1}{n}\right) 2^n \log 2^n a_{2^n} \leqq 4 \cdot \epsilon/8 = \epsilon/2 \end{aligned}$$

which is a contradiction. Thus $(n \log na_n)$ approaches zero.

It is often thought, contrary to theorem 2*a*, that if $\sum a_n$ is merely monotone and convergent then necessarily $n \log na_n$ approaches zero.* It is true that *if* $\sum a_n$ *is a convergent monotone series and if* n log na_n *approaches a limit, then this limit is zero,*† but the following elementary example‡ shows that $n \log na_n$ need not approach a limit. Define the terms of $\sum a_n$ as $a_n = 1/(k^{k^2} \log k^{k^2})$ whenever n satisfies $(k-1)^{(k-1)^2} < n \leqq k^{k^2}$. Clearly $\sum a_n$ is monotone. The proof of its convergence runs as follows: for every N there exists a k such that $N < k^{k^2}$, and hence

$$\begin{aligned} S_N = \sum_{n=2}^{N} a_n \leqq \sum_{n=2}^{k^{k^2}} a_n &= \sum_{m=2}^{k} \frac{m^{m^2} - (m-1)^{(m-1)^2}}{m^{m^2} \log m^{m^2}} \\ \leqq \sum_{m=2}^{\infty} \frac{1}{\log m^{m^2}} &= \sum_{m=2}^{\infty} \frac{1}{m^2 \log m} = C. \end{aligned}$$

Thus S_N is bounded and must approach a limit. However, it is clear that for $n = k^{k^2}$

$$n \log na_n = 1.$$

THEOREM 4*a*. *For every sequence* (p_n) *approaching infinity, no matter how slowly, there exists a convergent monotone series* $\sum a_n$ *with a monotone first Cauchy derived series such that* n log $np_n a_n$ *does not approach zero.*

* For references see Pringsheim, A. Allgemeine Theorie der Divergenz und Convergenz von Reihen mit positiven Gliedern, Math. Annalen, V. 35, 1890, pp. 297–394.

† The proof follows readily from the contrary assumption.

‡ "Elementary" with respect to those in the literature.

Proof. Applying theorem 2a to the sequence (p_{2^n}) there exists a convergent monotone series $\sum b_n$ such that $nb_np_{2^n}$ does not approach zero. Set $b_n = 2^na_{2^n}$ and multiply by log 2:

$$n2^na_{2^n}p_{2^n}\log 2 = 2^n \log 2^n p_{2^n}a_{2^n}.$$

Define a_m for m satisfying $2^{n-1} < m \leqq 2^n$ as equal to a_{2^n}. Thus $\sum a_n$ is monotone with a monotone first Cauchy derived series. The proof of the convergence runs as follows:

$$\sum a_n = \sum (2^n - 2^{n-1})a_{2^n} = \tfrac{1}{2}\sum 2^na_{2^n} = \tfrac{1}{2}\sum b_n.$$

Since a sub-sequence of $(n \log np_na_n)$ does not approach zero the sequence itself cannot approach zero, and we have proved the theorem.

COROLLARY TO THEOREM 4. *If $\sum a_n$ is a convergent monotone series and the first k Cauchy derived series are monotone, then $n \log n \log_2 n \log_3 n \cdots \log_k na_n \to 0$, where $\log_2 n \equiv \log\log n$, $\log_3 n \equiv \log\log\log n$, etc.*

Proof. We have actually to prove an infinite sequence of theorems, one for each integer k. To prove these we apply mathematical induction. Theorem 4 provides a basis for the induction with $k=1$.* Thus assume that the corollary is true for $k-1$. Label the m-th Cauchy derived series as $\sum a_n^{(m)}$. We have by the hypotheses of the corollary that the $k-1$th Cauchy derived series of $\sum a_n^{(1)}$ is monotone, so that by the hypotheses of the induction,

$$n \log n \log_2 n \cdots \log_{k-1} na_n^{(1)} \to 0.$$

But $a_n^{(1)} = 2^na_{2^n}$, and the above becomes,

$$2^nn \log n \log_2 n \cdots \log_{k-1} na_{2^n} \to 0.$$

Multiply by the constant $\log 2 \log_2 2 \cdots \log_k 2$:

$$2^n \log 2^n \log_2 2^n \cdots \log_k 2^na_{2^n} \to 0.$$

Thus a sub-sequence of $(n \log n \log_2 n \cdots \log_k na_n)$ approaches zero. The rest of the proof follows that of theorem 4.

COROLLARY TO THEOREM 4*a*. *For every sequence (p_n) approaching infinity, no matter how slowly, there exists a convergent monotone series $\sum a_n$ with the first k Cauchy derived series also monotone such that $n \log n \log_2 n \cdots \log_k np_na_n$ does not approach zero.*

The proof is left to the reader.

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* Actually we could have used theorem 2 as a basis and performed the induction then and not developed theorem 4.

MAXIMUM ANGULAR VARIATION UNDER SMALL DISPLACEMENTS

T. Y. THOMAS, Indiana University

1. Angular variation under small displacements. Let x^α and $\bar{x}^\alpha(\alpha=1, 2, 3)$ denote coordinates of points relative to the same rectangular system. A displacement of space or of a material body referred to this system can be represented by $\bar{x}^\alpha=x^\alpha+u^\alpha(x)$ where the $u^\alpha(x)$ are the components of a vector. We assume the u^α to be continuous and to have continuous first partial derivatives. To this requirement we add the further condition that the u's and their derivatives are small in the sense that the squares and higher powers of these quantities can be neglected. This is the usual assumption in the theory of elastic displacements.

If ν_1 and ν_2 are unit vectors at a point P, and θ is the angle determined by these vectors, then the change $\delta\theta$ produced in the angle θ as a result of the above displacement is given by*

$$\sin\theta\delta\theta = (e_1 + e_2) \cos\theta - 2e_{\alpha\beta}\nu_1^\alpha\nu_2^\beta, \tag{1}$$

where

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$$

and

$$e_1 = e_{\alpha\beta}\nu_1^\alpha\nu_1^\beta, \qquad e_2 = e_{\alpha\beta}\nu_2^\alpha\nu_2^\beta.$$

Observe that in the formula for $e_{\alpha\beta}$ we have written u_α instead of u^α for the "sake of appearances" and this is legitimate since we shall limit ourselves to rectangular coordinate systems. The "comma" denotes partial differentiation, and of course the summation convention is employed. If $\nu_1=\nu_2$ or $\nu_1=-\nu_2$ we have $\theta=0$ or $\theta=\pi$ and for each of these cases $\delta\theta=0$. We exclude these two cases and thus suppose $0<\theta<\pi$ in the following discussion.

2. An existence proof. The actual proof of the existence of a maximum or minimum value of $\delta\theta$ is immediate. In fact each of the directions ν_1 and ν_2 can be interpreted as a point on a unit sphere and the product of these spheres is a bicompact topological space. Since $\delta\theta$ is obviously a continuous function over this space, it follows that $\delta\theta$ assumes maximum and minimum values at points of the space.† With the existence of directions giving maximum and minimum values of $\delta\theta$ thus assured, we proceed to the actual determination of these directions and to the calculation of the maximum and minimum values of $\delta\theta$ involved.

3. Stationary values of the angular variation. A first insight into the problem of finding the above directions ν_1 and ν_2 is given by the following fact. *Stationary values of $\delta\theta$ can occur only for $\theta=\pi/2$ unless $\delta\theta=0$ for arbitrary directions at P.* To prove this we choose rectangular axes with origin at P and such that the vectors ν_1 and ν_2 lie in the x^1, x^2 plane. Let ν_1 and ν_2 make angles θ_1 and θ_2 with the x^1 axis and suppose furthermore that the axes are so taken that $\theta=\theta_2-\theta_1>0$. Then

* *Cf.* Love, The Mathematical Theory of Elasticity, Cambridge, 4th ed., 1927, p. 62. Also S. Timoshenko, Theory of Elasticity, McGraw-Hill, 1934, p. 192.

† See P. Alexandroff and H. Hopf, Topologie, Berlin, J. Springer, 1935, p. 96.

$$\nu_1^1 = \cos\theta_1, \qquad \nu_1^2 = \sin\theta_1, \qquad \nu_1^3 = 0,$$
$$\nu_2^1 = \cos\theta_2, \qquad \nu_2^2 = \sin\theta_2, \qquad \nu_2^3 = 0.$$

Substituting these values of the ν's and θ into formula (1) for $\delta\theta$ we have

$$\begin{aligned} \sin(\theta_2 - \theta_1)\delta\theta = {} & [e_{11}\cos^2\theta_1 + 2e_{12}\sin\theta_1\cos\theta_2 + e_{22}\sin^2\theta_1 + e_{11}\cos^2\theta_2 \\ & + 2e_{12}\sin\theta_2\cos\theta_2 + e_{22}\sin^2\theta_2]\cos(\theta_2 - \theta_1) \\ & - 2[e_{11}\cos\theta_1\cos\theta_2 + e_{12}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ & + e_{22}\sin\theta_1\sin\theta_2]. \end{aligned} \tag{2}$$

If the vectors ν_1 and ν_2 yield a stationary value of $\delta\theta$ we must have $\delta\partial\theta/\partial\theta_1 = 0$. After performing the differentiation involved in this condition, let us set $\theta_1 = 0$ whereupon $\theta_2 = \theta$. This is permissible since we can suppose that the coordinate system has been chosen to fulfill this condition (in addition to the above requirements). We thus obtain, after some trigonometric simplification:

$$\sin^2\theta \frac{\partial\delta\theta}{\partial\theta_1} = e_{11} - e_{22}.$$

Hence the above condition for stationary $\delta\theta$ gives $e_{11} = e_{22}$. Substitution of this relation in (2) gives

$$\delta\theta = -2e_{12}\sin^2\theta. \tag{3}$$

Similarly the condition $\partial\delta\theta/\partial\theta_2 = 0$ in conjunction with the relation $e_{11} = e_{22}$ yields, after some calculation,

$$\frac{\partial\delta\theta}{\partial\theta_2} = -2e_{12}\sin\theta\cos\theta = 0. \tag{4}$$

Now if $e_{12} = 0$ (in the special coordinate system selected) then from (3) we have $\delta\theta = 0$. Since our consideration applies in particular to directions ν_1 and ν_2 giving an absolute maximum or minimum variation $\delta\theta$ this result shows that there is no variation in angle for arbitrary directions at P. If, however, $e_{12} \neq 0$ it follows from (4) that $\cos\theta = 0$ and hence $\theta = \pi/2$ (the case $\sin\theta = 0$ is excluded by the assumption $0 < \theta < \pi/2$). This completes the proof of the italicized statement.

4. Formal simplification of the main problem. The result just proved suggests that we replace (1) by the simplified equation which is obtained by taking $\theta = \pi/2$. This gives

$$\delta\theta = -2e_{\alpha\beta}\nu_1^\alpha\nu_2^\beta, \tag{5}$$

where

$$\sum \nu_1^\alpha\nu_1^\alpha = 1, \qquad \sum \nu_2^\alpha\nu_2^\alpha = 1, \qquad \sum \nu_1^\alpha\nu_2^\alpha = 0.$$

The first two of these latter equations express the fact that ν_1 and ν_2 are unit vectors, which is inherent in (1), while the last equation contains the require-

ment that the vectors are perpendicular. We now seek stationary values of $\delta\theta$ in the set of all values of $\delta\theta$ determined by perpendicular vectors ν_1 and ν_2. For the determination of the perpendicular directions ν_1 and ν_2 which give these stationary values of $\delta\theta$ we take a rectangular coordinate system with origin at P relative to which $e_{\alpha\beta} = \epsilon_\alpha \delta_{\alpha\beta}$. We suppose furthermore that the axes of this system are so enumerated that $\epsilon_1 \geqq \epsilon_2 \geqq \epsilon_3$. The above equation (5) now becomes

$$\delta\theta = -2\sum \epsilon_\alpha \nu_1^\alpha \nu_2^\alpha, \tag{6}$$

and the perpendicular directions ν_1 and ν_2 are those which satisfy the condition

$$\sum \epsilon_\alpha(\nu_2^\alpha \delta\nu_1^\alpha + \nu_1^\alpha \delta\nu_2^\alpha) = 0, \tag{7}$$

in which the variations $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$ are subject to the conditions

$$\sum \nu_1^\alpha \delta\nu_1^\alpha = 0, \qquad \sum \nu_2^\alpha \delta\nu_2^\alpha = 0, \qquad \sum (\nu_2^\alpha \delta\nu_1^\alpha + \nu_1^\alpha \delta\nu_2^\alpha) = 0. \tag{8}$$

To find the vectors ν_1 and ν_2 satisfying (7) we multiply the three equations (8) by constants k_1, k_2 and k_3 and add the resulting equations to (7) in accordance with the usual procedure. This gives

$$\sum [k_1\nu_1^\alpha + (\epsilon_\alpha + k_3)\nu_2^\alpha]\delta\nu_1^\alpha + \sum [(\epsilon_\alpha + k_3)\nu_1^\alpha + k_2\nu_2^\alpha]\delta\nu_2^\alpha = 0.$$

Equating the coefficients of $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$ to zero we now have

$$k_1\nu_1^\alpha + (\epsilon_\alpha + k_3)\nu_2^\alpha = 0, \tag{9}$$

$$(\epsilon_\alpha + k_3)\nu_1^\alpha + k_2\nu_2^\alpha = 0, \tag{10}$$

and the solutions ν_1 and ν_2 of these equations (for suitable selections of the constants k_1, k_2 and k_3) give the required directions. Now deduce from (9) and (10) those equations which result by elimination of ν_1^α and ν_2^α in turn. They are

$$[(\epsilon_\alpha + k_3)^2 - k_1k_2]\nu_1^\alpha = 0, \qquad [(\epsilon_\alpha + k_3)^2 - k_1k_2]\nu_2^\alpha = 0 \tag{11}$$

in which there is of course no summation on the index α. If none of the bracket expressions in (11) vanish, the only solution of these equations is given by $\nu_1 = \nu_2 = 0$. If only one of the bracket expressions is equal to zero, for example that corresponding to $\alpha = 1$, then the vectors ν_1 and ν_2 will lie along the x^1 axis. This gives $\theta = 0$ or $\theta = \pi$ according as ν_1 and ν_2 have the same or the opposite direction on the x^1 axis and this case has been excluded from the discussion. Hence *at least two of the bracket expressions in* (11) *must be equal to zero*. A slight simplification in (9), (10) and (11) is obtained as follows. Multiplying (9) and (10) respectively by ν_1^α and ν_2^α and summing on the repeated index α we see that $k_1 = k_2 = k$. Hence these equations become

$$k\nu_1^\alpha + (\epsilon_\alpha + k_3)\nu_2^\alpha = 0, \qquad (\epsilon_\alpha + k_3)\nu_1^\alpha + k\nu_2^\alpha = 0, \tag{12}$$

$$[(\epsilon_\alpha + k_3)^2 - k^2]\nu_1^\alpha = 0, \qquad [(\epsilon_\alpha + k_3)^2 - k^2]\nu_2^\alpha = 0. \tag{13}$$

We have shown that all pairs of perpendicular directions ν_1 and ν_2 which

yield stationary values of $\delta\theta$ in the set of values of $\delta\theta$ determined by perpendicular directions are given as solutions of (12). We now show that any pair of perpendicular directions ν_1 and ν_2 satisfying (12) will likewise give a stationary value of $\delta\theta$ without the restriction that variations in direction satisfy the condition of perpendicularity, *i.e.*, the third equation (8). Removing this latter condition we must now determine the variation $\delta(\delta\theta)$ of $\delta\theta$ directly from (1) in which $\cos\theta = \Sigma\nu_1^\alpha\nu_2^\alpha$ in terms of the unit vectors ν_1 and ν_2. Making the required variations and then imposing the condition that the vectors ν_1 and ν_2 are perpendicular initially, we are led to the condition

$$\delta(\delta\theta) = [(e_1 + e_2)\delta_{\alpha\beta} - 2e_{\alpha\beta}](\nu_2^\beta\delta\nu_1^\alpha + \nu_1^\alpha\delta\nu_2^\beta) = 0.$$

Or, making the substitution $e_{\alpha\beta} = \epsilon_\alpha\delta_{\alpha\beta}$ we have

$$\sum \tau_\alpha\nu_2^\alpha\delta\nu_1^\alpha + \sum \tau_\alpha\nu_1^\alpha\delta\nu_2 = 0, \qquad \left[\tau_\alpha = \epsilon_\alpha - \tfrac{1}{2}(e_1 + e_2)\right], \tag{14}$$

with $\Sigma\nu_1^\alpha\delta\nu_1^\alpha = 0$ and $\Sigma\nu_2^\alpha\delta\nu_2^\alpha = 0$. Then multiplying these latter conditions by constants k_1 and k_2 and combining with (14) we deduce finally as conditions on the directions ν_1 and ν_2 for stationary $\delta\theta$ the following

$$k_1\nu_1^\alpha + \tau_\alpha\nu_2^\alpha = 0, \qquad \tau_\alpha\nu_1^\alpha + k_2\nu_2^\alpha = 0. \tag{15}$$

But multiplying the two sets of equations in (12) by ν_2^α and ν_1^α respectively and summing on the repeated index α we find $e_2+k_3=0$ and $e_1+k_3=0$. Hence $k_3 = -e_1 = -e_3$ and $k_3 = -(e_1+e_2)/2$. But when we make this latter substitution into (12) these equations assume the form (15). Hence (12) is the condition for a pair of perpendicular vectors ν_1 and ν_2 to give a stationary value of $\delta\theta$ relative to *arbitrary* variations in direction.

5. Directions giving a maximum or minimum value of the angular variation. We now consider in detail the problem of finding directions ν_1 and ν_2 for which $\delta\theta$ has a maximum or minimum value. For this purpose we divide the discussion into separate cases accordingly as the ϵ's are all distinct, the values of two of the ϵ's are equal, or all ϵ's have the same value.

Case I. The ϵ's have distinct values, i.e., $\epsilon_1 > \epsilon_2 > \epsilon_3$. Suppose that the bracket expressions in (13) vanish for $\alpha = 1, 2$. Then $(\epsilon_1+k_3)^2 = k^2$ and $(\epsilon_2+k_3)^2 = k^2$ or $\epsilon_1+k_3 = \pm k$ and $\epsilon_2+k_3 = \pm k$. Both plus or both minus signs in these relations cannot be taken since we then have $\epsilon_1 = \epsilon_2$ contrary to hypothesis. Hence we must have (a) $\epsilon_1+k_3 = k$, $\epsilon_2+k_3 = -k$ or (b) $\epsilon_1+k_3 = -k$, $\epsilon_2+k_3 = k$. Taking first the equations (a) we find on solving that $k_3 = -(\epsilon_1+\epsilon_2)/2$ and $k = (\epsilon_1-\epsilon_2)/2$. Substituting these values of k_3 and k into the remaining bracket expression in (13), *i.e.*, the expression for which $\alpha = 3$, we have

$$(\epsilon_3 + k_3)^2 - k^2 = (\epsilon_3 + k_3 + k)(\epsilon_3 + k_3 - k) = (\epsilon_3 - \epsilon_2)(\epsilon_3 - \epsilon_1) > 0.$$

It follows from (13) that $\nu_1^3 = \nu_2^3 = 0$ so that the vectors ν_1 and ν_2 lie in the x^1, x^2 plane. From (12) with $\alpha = 1, 2$ we now find

$$k\overset{1}{\nu_1} + (\epsilon_1 + k_3)\overset{1}{\nu_2} = 0 \rightarrow k\overset{1}{\nu_1} + k\overset{1}{\nu_2} = 0, \quad \text{or} \quad \overset{1}{\nu_1} = -\overset{1}{\nu_2},$$

$$k\overset{2}{\nu_1} + (\epsilon_2 + k_3)\overset{2}{\nu_2} = 0 \rightarrow k\overset{2}{\nu_1} - k\overset{2}{\nu_2} = 0, \quad \text{or} \quad \overset{2}{\nu_1} = \overset{2}{\nu_2},$$

since $k \neq 0$. Had we used the alternative equations (b) we would have found that (12) reduces to $\nu_1^1 = \nu_2^1$, $\nu_1^2 = -\nu_2^2$ and $\nu_1^3 = \nu_2^3 = 0$. Now imposing the condition that ν_1 and ν_2 are perpendicular, namely $\nu_1^1\nu_2^1 + \nu_1^2\nu_2^2 = 0$ we obtain $-(\nu_1^1)^2 + (\nu_1^2)^2 = 0$ and hence $\nu_1^1 = \pm\nu_1^2$. Then from the relation $(\nu_1^1)^2 + (\nu_1^2)^2 = 1$ expressing the condition that ν_1 is a unit vector, we obtain $2(\nu_1^1)^2 = 1$ or $\nu_1^1 = \pm 1/\sqrt{2}$. Hence $\nu_2^1 = \pm 1/\sqrt{2}$. From the condition $(\nu_2^1)^2 + (\nu_2^2)^2 = 1$ we can now obtain $1/2 + (\nu_2^2)^2 = 1$ and hence $\nu_2^2 = \pm 1/\sqrt{2}$. Finally $\nu_1^2 = \pm 1/\sqrt{2}$. Collecting these results we have

$$\overset{1}{\nu_1} = \pm \frac{1}{\sqrt{2}}, \qquad \overset{1}{\nu_2} = \pm \frac{1}{\sqrt{2}}, \qquad \overset{2}{\nu_1} = \pm \frac{1}{\sqrt{2}}, \qquad \overset{2}{\nu_2} = \pm \frac{1}{\sqrt{2}}$$

where any choice of algebraic signs can be made which is consistent with the requirement that the vectors ν_1 and ν_2 be perpendicular. We thus see that the possible vectors ν_1 and ν_2 can be described by saying that they bisect adjacent quadrants in the x^1, x^2 plane. If ν_1 and ν_2 bisect the first and second quadrants respectively, or if these vectors bisect the third and fourth quadrants of the x^1, x^2 plane, we find on making the appropriate substitutions of the ν's in (6) that $\delta\theta = \epsilon_1 - \epsilon_2 > 0$ in each case. When the vectors bisect the first and fourth quadrants and also when they bisect the second and third quadrants the corresponding value of $\delta\theta = -(\epsilon_1 - \epsilon_2)$.

There are two other combinations of bracket expressions in (13) which can be equated to zero and which lead to results analogous to the above. Thus stationary values of $\delta\theta$ are given by the pairs of perpendicular vectors which bisect the quadrants of the x^1, x^3 plane and the corresponding value of $\delta\theta$ is $\pm(\epsilon_1 - \epsilon_3)$. Similarly we obtain a stationary $\delta\theta$ for perpendicular vectors bisecting the quadrants of the x^2, x^3 plane, the values of $\delta\theta$ in this case being given by $\delta\theta = \pm(\epsilon_2 - \epsilon_3)$. Since there thus exist non-zero values of $\delta\theta$ it follows from the result at the beginning of this paper that the above-mentioned pairs of perpendicular vectors ν_1 and ν_2 constitute all the directions giving stationary values of $\delta\theta$.

The question now arises as to whether the above pairs of perpendicular vectors ν_1 and ν_2 give relative maximum or minimum values of $\delta\theta$. Our results show that the greatest value of $\delta\theta$ is $\epsilon_1 - \epsilon_3$ and this is determined by the pair of perpendicular vectors ν_1 and ν_2 which bisect the first and second quadrants or the third and fourth quadrants of the x^1, x^3 plane. Similarly the least value of $\delta\theta$ is $-(\epsilon_1 - \epsilon_3)$ and is determined by the vectors bisecting the first and fourth or the second and third quadrants of the x^1, x^3 plane. These are the absolute maximum and minimum values of $\delta\theta$ the existence of which has been assured by the previous discussion. There remains, however, the question whether the other stationary values of $\delta\theta$ constitute relative maxima or minima of this quantity.

This latter question is to be answered in the negative. But to show this we must calculate, to within terms of higher order, the change $\Delta\delta\theta$ in $\delta\theta$ determined

by the variations $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$. Take the case for which the vectors ν_1 and ν_2 bisect the first and second quadrants of the x^1, x^2 plane and for which $\delta\theta=\epsilon_1-\epsilon_2$ as shown above. The components of these vectors are then given by $\nu_1^1=\nu_1^2=1/\sqrt{2}$, $\nu_1^3=0$ and $-\nu_2^1=\nu_2^2=1/\sqrt{2}$, $\nu_2^3=0$. Values of ν_1^α and ν_2^α in the neighborhood of these values can be represented by the spherical coordinates of points on the unit sphere with center at the origin. Thus

$$\overset{1}{\nu}_1 = \sin\phi_1\cos\phi_2, \qquad \overset{2}{\nu}_1 = \sin\phi_1\sin\phi_2, \qquad \overset{3}{\nu}_1 = \cos\phi_1,$$
$$\overset{1}{\nu}_2 = \sin\phi_3\cos\phi_4, \qquad \overset{2}{\nu}_2 = \sin\phi_3\sin\phi_4, \qquad \overset{3}{\nu}_2 = \cos\phi_3.$$

To find the variations $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$ in terms of the arbitrary variations of the ϕ's we expand the trigonometric functions in the above equations about the following initial values: $\phi_1=\pi_2$, $\phi_2=\pi/4$, $\phi_3=\pi/_2$, and $\phi_4=\pi/2+\pi/4$. We thus find

$$\begin{aligned}
\sin\phi_1 &= 1-\frac{1}{2}(\delta\phi_1)^2+\cdots\\
\cos\phi_2 &= \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\delta\phi_2-\frac{1}{2\sqrt{2}}(\delta\phi_2)^2+\cdots\\
\sin\phi_2 &= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\delta\phi_2-\frac{1}{2\sqrt{2}}(\delta\phi_2)^2+\cdots\\
\cos\phi_1 &= -\delta\phi_1+\cdots\\
\sin\phi_3 &= 1-\frac{1}{2}(\delta\phi_3)^2+\cdots\\
\cos\phi_4 &= -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\delta\phi_4+\frac{1}{2\sqrt{2}}(\delta\phi_4)^2+\cdots\\
\sin\phi_4 &= \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\delta\phi_4-\frac{1}{2\sqrt{2}}(\delta\phi_4)^2+\cdots\\
\cos\phi_3 &= -\delta\phi_3+\cdots.
\end{aligned}$$

Now $\cos\theta=\sum\nu_1^\alpha\nu_2^\alpha$. The right member of this equation can be expressed in terms of the $\delta\phi$'s by the above substitutions. In the left member put $\theta=\pi/2+\Delta\theta$ and expand about $\pi/2$. This leads to the relations

$$\cos\theta = -\Delta\theta = \delta\phi_2-\delta\phi_4+\delta\phi_1\delta\phi_3,$$

to within the order of terms which we need to consider. Turning now to the equation (1) we replace $\delta\theta$ by $\delta\theta+\Delta\delta\theta$ and express $\sin\theta$ in terms of $\Delta\theta$ by expansion about $\pi/2$. To within terms of the second order in the variations $\delta\phi$ the left member of (1) then becomes

$$\Delta\delta\theta+(\epsilon_1-\epsilon_2)-\tfrac{1}{2}(\epsilon_1-\epsilon_2)\left[(\delta\phi_2)^2-2\delta\phi_2\delta\phi_4+(\delta\phi_4)^2\right]$$

when use is made of the above value of $\Delta\theta$ and the fact that $\delta\theta=\epsilon_1-\epsilon_2$. Finally

making use of the above value of $\cos\theta$ in terms of the $\delta\phi$'s and also making the above substitutions for the ν_1^α and ν_2^α in the right member of (1), we find, after reduction, that

$$-2\Delta\delta\theta = A(\delta\phi_1)^2 - 2B\delta\phi_1\delta\phi_3 + 2A(\delta\phi_2)^2 + A(\delta\phi_3)^2 + 2A(\delta\phi_4)^2$$

to within quadratic terms in the variations $\delta\phi$. Here $A = \epsilon_1 - \epsilon_2 > 0$ and $B = (\epsilon_1 - \epsilon_3) + (\epsilon_2 - \epsilon_3) > 0$. The matrix of the coefficients of this (symmetric) quadratic form in the $\delta\phi$'s is

$$\left\| \begin{array}{cccc} A & 0 & -B & 0 \\ 0 & 2A & 0 & 0 \\ -B & 0 & A & 0 \\ 0 & 0 & 0 & 2A \end{array} \right\|$$

and the algebraic signs of the successive diagonal determinants are $+, +, -, -$. Hence in accordance with the usual test this form is neither positive nor negative definite, and hence the value $\delta\theta = \epsilon_1 - \epsilon_2$ under consideration is neither a relative maximum nor minimum. It follows that a similar conclusion holds for the negative of this value of $\delta\theta$ namely $\delta\theta = -(\epsilon_1 - \epsilon_2)$.

By rotating the coordinate axes as indicated by $(x^3, x^1, x^2) \rightarrow (x^1, x^2, x^3)$ and making the corresponding substitution on the quantities ϵ_α namely $(\epsilon_3, \epsilon_1, \epsilon_2) \rightarrow (\epsilon_1, \epsilon_2, \epsilon_3)$ we can pass from the case just treated to that for which the vectors ν_1 and ν_2 bisect the quadrants of the x^2, x^3 plane. Since the matrix derivable from the above matrix by this substitution on the ϵ's is neither positive nor negative definite, it follows that the values $\delta\theta = \pm(\epsilon_2 - \epsilon_3)$ corresponding to vectors ν_1 and ν_2 bisecting the quadrants of the x^2, x^3 plane are neither relative maxima nor minima. However, the above matrix passes into one corresponding to a definite quadratic form under the substitution $(\epsilon_3, \epsilon_1, \epsilon_2) \rightarrow (\epsilon_2, \epsilon_3, \epsilon_1)$ which gives an explicit proof of the fact already observed that the values $\delta\theta = \epsilon_1 - \epsilon_3$ and $\delta\theta = -(\epsilon_1 - \epsilon_3)$ are respectively maximum and minimum values of $\delta\theta$.

We can now sum up our results for Case I as follows. *Vectors ν_1 and ν_2 bisecting the first and second or the third and fourth quadrants of the x^1, x^3 plane give a maximum $\delta\theta = (\epsilon_1 - \epsilon_3)$ while those bisecting the first and fourth or the second and third quadrants of this plane give a minimum value $\delta\theta = -(\epsilon_1 - \epsilon_3)$. Vectors ν_1 and ν_2 bisecting adjacent quadrants of the x^1, x^2 or x^2, x^3 planes give stationary values of $\delta\theta$ but these values are neither relative maxima nor minima.* In making this statement it is of course to be understood (as also in the following) that the vectors ν_1 and ν_2 have their initial points at the origin of the rectangular coordinate system for which $e_{\alpha\beta} = \epsilon_\alpha \delta_{\alpha\beta}$ and that moreover the coordinate axes have been enumerated so that the inequalities $\epsilon_1 > \epsilon_2 > \epsilon_3$ hold.

Case II. *Two of the ϵ's are equal.* This case breaks up into the following subcases, namely (α) $\epsilon_1 = \epsilon_2 > \epsilon_3$ and (β) $\epsilon_1 > \epsilon_2 = \epsilon_3$. We shall give the discussion in detail under the assumption (α).

Since two of the bracket expressions in (13) must be equal to zero we must

have $(\epsilon_1+k_3)^2-k^2=0$ and hence (a) $\epsilon_1=k-k_3$ or (b) $\epsilon_1=-k-k_3$. If $k=0$ there is no distinction between (a) and (b). We treat this special case first. Then equations (12) reduce to $(\epsilon_3+k_3)\nu_2^3=0$ and $(\epsilon_3+k_3)\nu_1^3=0$. But $\epsilon_3+k_3=\epsilon_3-\epsilon_1<0$ and hence $\nu_1^3=\nu_2^3=0$. Hence ν_1 and ν_2 can be taken as any pair of perpendicular vectors in the x^1, x^2 plane.

Now assume $k\neq 0$ and suppose (a) to hold. Then equations (12) become

$$(16)\qquad \begin{cases} \nu_1^1+\nu_2^1=0, & \nu_1^2+\nu_2^2=0, \\ k\nu_1^3+(\epsilon_3+k_3)\nu_2^3=0, & (\epsilon_3+k_3)\nu_1^3+k\nu_2^3=0. \end{cases}$$

If we combine the equation expressing the fact that the vectors ν_1 and ν_2 are perpendicular with the equations (16) we find

$$k(\nu_1^1)^2+k(\nu_1^2)^2+(\epsilon_3+k_3)(\nu_1^3)^2=0.$$

But,

$$k(\nu_1^1)^2+k(\nu_1^2)^2+k(\nu_1^3)^2=k,$$

since ν_1 is a unit vector, and hence

$$[k-(\epsilon_3+k_3)](\nu_1^3)^2=k \quad\text{or}\quad (\epsilon_1-\epsilon_3)(\nu_1^3)^2=k.$$

Since $\epsilon_1-\epsilon_3\neq 0$ and $k\neq 0$ it follows that $\nu_1^3\neq 0$. Hence the values of the quantities ν_1^3 and ν_2^3 satisfying the two equations in the second line of (16) cannot both be zero and hence the determinant of the coefficients in these equations must vanish. But this determinant is

$$(\epsilon_3+k_3)^2-k^2=(\epsilon_3+k_3-k)(\epsilon_3+k_3+k)=(\epsilon_3-\epsilon_1)(\epsilon_3+k_3+k)=0.$$

Then since $\epsilon_3-\epsilon_1\neq 0$ we must have $\epsilon_3+k_3+k=0$. Hence the two equations in the second line of (16) reduce to the single equation $\nu_1^3=\nu_2^3$.

We have thus shown under the above condition (a) that equations (12) reduce to $\nu_1^1=-\nu_2^1$, $\nu_1^2=-\nu_2^2$ and $\nu_1^3=\nu_2^3$. Under condition (b) we find in a similar manner that we may also have $\nu_1^1=\nu_2^1$, $\nu_1^2=\nu_2^2$ and $\nu_1^3=-\nu_2^3$. These results can be expressed as follows: The vectors ν_1 and ν_2 can be taken as any pair of perpendicular vectors which issue from the origin of the coordinate system and which lie along generators of the complete circular cone C, the generators of which make an angle of $\pi/4$ with the axis, and whose axis coincides with the x^3 coordinate axis. Under condition (a) both vectors ν_1 and ν_2 lie in the same nappe of the cone, while under condition (b) the vectors ν_1 and ν_2 lie in different nappes.

Calculation of the values of $\delta\theta$ corresponding to the above determinations of the vectors ν_1 and ν_2 is readily carried out and leads to the following results. Corresponding to $k=0$ for which ν_1 and ν_2 are perpendicular vectors in the x^1, x^2 plane we find $\delta\theta=0$. When ν_1 and ν_2 have the determination under condition (a) above, we have $\delta\theta=\epsilon_1-\epsilon_3$ and when the vectors ν_1 and ν_2 are determined according to condition (b) then $\delta\theta=-(\epsilon_1-\epsilon_3)$. The above pairs of perpendicular vectors ν_1 and ν_2 constitute all pairs of directions for which $\delta\theta$ has stationary values. Also we have max. $\delta\theta=\epsilon_1-\epsilon_3$ and min. $\delta\theta=-(\epsilon_1-\epsilon_3)$.

We can readily observe that the value $\delta\theta = 0$ determined by any pair of perpendicular vectors $\nu_1 = \nu_1^0$ and $\nu_2 = \nu_2^0$ is neither a relative maximum nor minimum. To show this let us first determine all pairs of perpendicular vectors ν_1 and ν_2 for which $\delta\theta = 0$. The required conditions on such vectors are obtained by equating the right members of (6) to zero. This gives

$$\sum \epsilon_\alpha \overset{\alpha}{\nu_1}\overset{\alpha}{\nu_2} = \epsilon_1(\overset{1}{\nu_1}\overset{1}{\nu_2} + \overset{2}{\nu_1}\overset{2}{\nu_2}) + \epsilon_3\overset{3}{\nu_1}\overset{3}{\nu_2} = (\epsilon_3 - \epsilon_1)\overset{3}{\nu_1}\overset{3}{\nu_2} = 0.$$

Since $\epsilon_3 - \epsilon_1 \neq 0$ it follows that $\overset{3}{\nu_1} = 0$ or $\overset{3}{\nu_2} = 0$, *i.e.* one of the vectors ν_1 or ν_2 must lie in the x^1, x^2 plane. Now take a pair of perpendicular vectors ν_1 and ν_2 lying along generators in the same nappe of the cone C, for example in the nappe lying in the semispace $x^3 \geqq 0$. For these vectors we then have $\delta\theta = \epsilon_1 - \epsilon_3 > 0$. Displace ν_1 and ν_2 continuously, but without altering the condition of perpendicularity into ν_1^0 and ν_2^0 respectively in such a way that the end points of the vectors always lie in the same semispace with respect to the x^1, x^2 plane. It follows from the result just proved that we shall then have $\delta\theta > 0$ throughout this displacement. Similarly if we start with perpendicular vectors ν_1 and ν_2 lying along generators of C, but in different nappes of C, and displace them continuously under the condition of perpendicularity into ν_1^0 and ν_2^0 in such a way that the end point of each vector always remains in the same semispace with respect to the x^1, x^2 plane, then we shall have $\delta\theta < 0$ throughout the displacement. Hence in any neighborhood of ν_1^0 and ν_2^0 there exist vectors ν_1 and ν_2 for which $\delta\theta > 0$ and vectors ν_1 and ν_2 for which $\delta\theta < 0$. It follows that the value $\delta\theta = 0$ determined by ν_1^0 and ν_2^0 is neither a relative maximum nor minimum.

Corresponding results are obtained under the assumption (β) for which $\epsilon_1 > \epsilon_2 = \epsilon_3$. Now instead of the above cone C we have the analogous cone C' with axis along the x^1 coordinate axis. We give the results in the following statement which contains the complete results for Case II. *If $\epsilon_1 = \epsilon_2 > \epsilon_3$ perpendicular vectors ν_1 and ν_2 lying along generators in the same nappe of the cone C yield max. $\delta\theta = \epsilon_1 - \epsilon_3$ while those perpendicular vectors along generators in different nappes of C give min. $\delta\theta = -(\epsilon_1 - \epsilon_3)$. A stationary value $\delta\theta = 0$ is given by any pair of perpendicular vectors ν_1 and ν_2 in the x^1, x^2 plane, this value being neither a relative maximum nor minimum. If $\epsilon_1 > \epsilon_2 = \epsilon_3$ those perpendicular vectors ν_1 and ν_2 which lie along generators in different nappes of the cone C' give the max. $\delta\theta = \epsilon_1 - \epsilon_2$ and perpendicular vectors ν_1 and ν_2 along generators in the same nappe of C' give the min. $\delta\theta = -(\epsilon_1 - \epsilon_2)$. When the vectors ν_1 and ν_2 are any pair of perpendicular vectors in the x^2, x^3 plane we obtain a stationary value of $\delta\theta = 0$ which is neither a relative maximum nor a minimum.*

Case III. *All ϵ's have the same value.* This case can be handled quickly. Let ν_1 and ν_2 be any two unit vectors not necessarily perpendicular. Then since $\epsilon_1 = \epsilon_2 = \epsilon_3$ it is immediately seen that the right member of (1) is equal to zero. *Hence $\delta\theta = 0$ for arbitrary directions* ν_1 and ν_2.

This completes the determination of all directions giving angles of maximum and minimum variation under small displacements.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A MATTER OF MOTIVATION

H. V. CRAIG, University of Texas

It is well known that vector analysis is admirably suited to the exposition of a miscellany of topics in elementary mathematics and advanced calculus. Usually, the vectorial treatments are short, simple, and *well motivated.* A case in point that so far as the writer knows has escaped attention is the Lagrange multiplier method for determining critical points. Roughly, the problem involved, in its simplest form, is to find a necessary condition that a point P on a surface $S(x, y, z)=0$ give a function $F(x, y, z)$ an extreme value relative to the surface—we assume that all functions are of class c' and that $\nabla S\neq 0$. Evidently, if P is such a point; and C $(x=x(s),\ y=y(s),\ z=z(s))$ is a curve passing through P and lying entirely on $S=0$; then the derivative dF/ds vanishes at P. The traditional development of Lagrange's method of attack is fairly concise but emphatically it is not well motivated. The fundamental expression $F(x, y, z)+\lambda S(x, y, z)$ is essentially extracted from thin air. With regard to its introduction Osgood* says: "The motif lies in the purely algebraic situation which arises when we do this thing."

Let us now examine the problem from the standpoint of vector analysis. If s is arc length on C and $\boldsymbol{\tau}$ is the unit tangent vector to C at P, then since F has an extreme value at P and since $S(x, y, z)=0$ on the surface, we have

$$0=\left.\frac{dF}{ds}\right|\text{ at }P=\nabla F\cdot\boldsymbol{\tau}\Big|\text{ at }P;\qquad 0=\left.\frac{dS}{ds}\right|\text{ at }P=\nabla S\cdot\boldsymbol{\tau}\Big|\text{ at }P.$$

But C is any curve on the surface and therefore $\boldsymbol{\tau}$ is any unit vector tangent to the surface at P. Consequently, either ∇F vanishes at P or it, as well as ∇S, is normal to the surface. In either case we have at P, $\nabla F=\lambda\nabla S$. This last equation together with $S=0$, comprises the multiplier method.

The case of two added conditions $S_1(x, y, z)=S_2(x, y, z)=0$ may be treated similarly. We suppose that these equations determine a curve. The choice of C is now limited to this curve, and we conclude that if ∇F at P is not zero, then ∇F, ∇S_1, and ∇S_2 are each perpendicular to $\boldsymbol{\tau}$. In either case we have at P, $\nabla F=\lambda_1\nabla S_1+\lambda_2\nabla S_2$, assuming ∇S_1 and ∇S_2 are not zero at P.

* See W. F. Osgood, Advanced Calculus, Macmillan, 1925, p. 180.

ON THE CALCULATION OF BOND YIELDS

H. D. LARSEN, University of New Mexico

An important problem in the mathematics of finance is that of determining the yield rate on a bond given its purchase price. Tables of bond values are published for yield rates proceeding by one twentieth of 1%, and simple interpolation in these tables furnishes an approximation to the yield which suffices for most purposes. When extensive tables of bond values are not at hand, recourse is made to annuity tables such as those generally found in textbooks on the mathematics of finance. However, simple interpolation in the latter tables does not always give the accuracy desired. Many methods have been proposed to remedy this situation. In general, these methods depend on formulas of finite differences or on successive approximations, and involve considerable computing. It is the purpose of this note to present a simple recursion formula for calculating successive approximations to the yield of a bond. The accuracy of the formula is limited only by the accuracy of the logarithm tables used in its connection, a five-place table supplemented by a seven-place table of log $(1+i)$ being sufficient to give the yield rate to at least six places of decimals.

Consider a bond redeemable at the end of n periods (not necessarily at par), and carrying a dividend at the rate of g per period per unit of its redemption price. Let the bond be purchased at a premium k per unit on its redemption price, where $|k|<1$. The restriction on k is not serious, as it requires only that the purchase price of the bond be less than twice the redemption price, the usual situation in practice. If i denotes the required yield rate per period, then

$$k = (g - i)a, \tag{1}$$

where

$$a = a_{\overline{n}|} \quad \text{at} \quad i.$$

Solving (1) for i, we obtain

$$i = g - ka^{-1}. \tag{2}$$

Equation (2) suggests the recursion formula,

$$i_{m+1} = g - k\overset{-1}{a_m}, \tag{3}$$

where $a_m = a_{\overline{n}|}$ at i_m, and i_1 is any convenient first approximation to i. In order to prove that the successive approximations given by (3) actually converge to i, it suffices to show that

$$\lim_{m\to\infty} |\epsilon_{m+1}| = \lim_{m\to\infty} |i_{m+1} - i| = 0.$$

From (2) and (3),

$$\epsilon_{m+1} = k(\overset{-1}{a} - \overset{-1}{a_m}). \tag{4}$$

If $i > i_m$, then

$$0 < \overset{-1}{a} - \overset{-1}{a_m} = (i + \overset{-1}{s_{\overline{n}|}} \text{ at } i) - (i_m + \overset{-1}{s_{\overline{n}|}} \text{ at } i_m) < i - i_m,$$

since

$$\overset{-1}{s_{\overline{n}|}} \text{ at } i < \overset{-1}{s_{\overline{n}|}} \text{ at } i_m; \qquad i > i_m.$$

Similarly, if $i < i_m$,

$$0 < \overset{-1}{a_m} - \overset{-1}{a} < i_m - i.$$

In either case,

$$\left| \overset{-1}{a} - \overset{-1}{a_m} \right| < \left| i_m - i \right| = \left| \epsilon_m \right|,$$

whence, by (4),

$$\left| \epsilon_{m+1} \right| < \left| k \right| \cdot \left| \epsilon_m \right|. \tag{5}$$

A successive application of (5) gives

$$\left| \epsilon_{m+1} \right| < \left| k \right|^m \cdot \left| \epsilon_1 \right|,$$

so that, since $|k| < 1$,

$$\lim_{m \to \infty} \left| \epsilon_{m+1} \right| = 0.$$

The above discussion discloses the nature of the errors, ϵ_m. It is clear from (5) that *every* approximation is better than the preceding one, and that the rapidity of the convergence increases as $|k|$ decreases. Further, since $a_m^{-1} - a^{-1}$ has the same sign as $i_m - i = \epsilon_m$, it follows from (4) that ϵ_{m+1} and ϵ_m have like signs or opposite signs according as $k < 0$ or $k > 0$. Thus, if $k < 0$, the successive approximations are all too large or all too small; if $k > 0$, the succcessive approximations are alternately too large and too small. It should be noted also that a mistake in any particular approximation does not vitiate the ultimate convergence of the subsequent approximations. The incorrect value merely becomes a new first approximation to i.

A first approximation to i may be obtained in a variety of ways. In the example below, i_1 is computed from the formula,*

$$i_1 = \frac{g - k/n}{1 + (n + 1)k/2n}, \tag{6}$$

which is derived from (2) by expanding a^{-1} in a power series and neglecting all terms above the first degree in i. It is easy to show that the error ϵ_1 resulting from the use of (6) always has the same sign as k. Consequently, if $k < 0$, then $\epsilon_1 < 0$, $\epsilon_2 < 0$, $\epsilon_3 < 0$, *etc.*; if $k > 0$, then $\epsilon_1 > 0$, $\epsilon_2 < 0$, $\epsilon_3 > 0$, *etc.*

As an example, let it be required to find the yield on a bond bearing interest

* Ralph Todhunter, Institute of Actuaries Text-Book, Part I. The Theory of Compound Interest and Annuities Certain, New Edition, revised. Charles and Edwin Layton, London, 1915, pp. 104–105.

at $4\frac{1}{2}$ per cent, payable half-yearly, redeemable in 25 years at $112\frac{1}{2}$, and bought at a price of 120. (Todhunter, p. 105.) Here,

$$n = 50; \qquad g = 0.0225/112.5 = 0.02; \qquad k = 7.5/112.5 = 1/15.$$

Hence,

$$i_1 = \frac{0.02 - 1/750}{1 + (51/100)(1/15)} = 0.018053,$$

$$a_1^{-1} = \frac{0.018053}{1 - (1.018053)^{-50}} = 0.030535,$$

$$i_2 = 0.02 - (1/15)(0.030535) = 0.017964,$$

$$a_2^{-1} = \frac{0.017964}{1 - (1.017964)^{-50}} = 0.030477,$$

$$i_3 = 0.02 - (1/15)(0.030477) = 0.017968.$$

The value of i_3 agrees with the true yield rate to six places of decimals.

AN IMPROPER INTEGRAL

E. J. Camp, Macalester College

In Osgood's *Advanced Calculus** the integral

$$\int_0^\infty \frac{\cos mx}{1 + x^2}\,dx$$

appears on page 490 along with a suggested transformation for its evaluation. If one performs the transformation and then makes use of two other improper integrals which have been worked out on previous pages, the integral can be evaluated. The transformation is rather artificial and is not one that is suggested by a sequence of logical steps. The method of this paper will be to evaluate the above integral by the use of a second order differential equation.

Let the function $u(m, \beta)$ be defined by the equation

$$u(m, \beta) = \int_0^\infty \frac{e^{-\beta x}\cos mx}{x^2 + 1}\,dx, \tag{1}$$

where β is regarded as constant. The derivatives† of this function are

$$\frac{du}{dm} = \int_0^\infty \frac{-xe^{-\beta x}\sin mx}{x^2 + 1}\,dx, \tag{2}$$

$$\frac{d^2u}{dm^2} = \int_0^\infty \frac{-x^2e^{-\beta x}\cos mx}{1 + x^2}\,dx. \tag{3}$$

If equation (1) is subtracted from equation (3) the result is the equation

* See W. F. Osgood, Advanced Calculus, Macmillan, 1925, p. 490.

† The integrals 1, 2, and 3 are uniformly convergent and so the differentiation under the integral sign is justified.

(4)
$$\frac{d^2u}{dm^2} - u = \frac{-\beta}{\beta^2 + m^2}.$$

Solving equation (4) by the method of variation of parameters, we get

(5)
$$u = c_1e^m + c_2e^{-m} + \tfrac{1}{2}e^m\int_0^m \frac{-\beta e^{-t}}{\beta^2 + t^2}dt + \tfrac{1}{2}e^{-m}\int_0^m \frac{\beta e^t}{\beta^2 + t^2}dt, \quad m > 0.*$$

If m is set equal to 0 in equation (5) the right member has the value c_1+c_2. The same function in the form (1) has the value

$$\int_0^\infty \frac{e^{-\beta x}}{x^2 + 1}dx.$$

The right member of equation (2) has the value 0 when $m=0$. The derivative du/dm obtained from equation (5) is c_1-c_2 for $m=0$. These considerations lead to the equations

(6)
$$c_1 + c_2 = \int_0^\infty \frac{e^{-\beta x}}{x^2 + 1}dx,$$
$$c_1 - c_2 = 0,$$

the solutions of which are

(7)
$$c_1 = c_2 = \frac{1}{2}\int_0^\infty \frac{e^{-\beta x}}{x^2 + 1}dx.$$

The problem is now solved except for evaluating the integrals in (7) and (5) for $\beta=0$. The integrals in (7) have the value $\pi/4$.

The integrals in equation (5) which belong to the particular integral will now be evaluated. Thus if we set $t=\beta\tau$ we have

$$\frac{1}{2}e^m\int_0^m \frac{-\beta e^{-t}}{\beta^2 + t^2}dt = -\frac{1}{2}e^m\int_0^{m/\beta} \frac{e^{-\beta\tau}}{1 + \tau^2}d\tau$$

and the limit of this integral for $\beta=0$ is $-\frac{1}{2}e^m(\pi/2)$. Similarly

$$\lim_{\beta=0}\left[\frac{1}{2}e^{-m}\int_0^m \frac{\beta e^t}{\beta^2 + t^2}dt\right] = \frac{\pi}{4}e^{-m}.$$

On substituting each of these values in equation (5) we have

$$\lim_{\beta=0} u(m,\beta) = \frac{\pi}{4}e^m + \frac{\pi}{4}e^{-m} - \frac{\pi}{4}e^m + \frac{\pi}{4}e^{-m} = \frac{\pi}{2}e^{-m}, \qquad m > 0.$$

It follows from the third footnote that where m is any real number,

$$\int_0^\infty \frac{\cos mx}{1 + x^2}dx = \frac{\pi}{2}e^{-|m|}.$$

* If m is negative then the equation will be of the same form with m replaced by $|m|$.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Kappa Mu Epsilon, Kansas State Teachers College, Pittsburg

The activities for the year consisted of six open meetings held at the homes of faculty members, two picnics, and two closed fraternity meetings. Twelve new members were initiated at the summer fraternity meeting and forty-six new members were initiated at the mid-year meeting. The programs of these meetings were followed by the initiation services. Attendance was greatly increased by the presence of a Navy V-12 unit on the campus.

Included among the topics discussed at meetings were:

The practical application of mathematics to engineering, by E. R. Sutterfield, draftsman at the Pittsburg-McNally Manufacturing Corporation

The rehabilitation program at K. S. T. C., by Professor W. H. Matthews, Director of Adult Education

Types of non-euclidean geometry, by Professor R. G. Smith

Recent discoveries in astronomy, by Professor J. A. G. Shirk

Mathematics in the war, by Marie Cowley, Instructor of Physical Science

Mathematics after the war, by Professor F. C. German.

Student members of the club talked on the following topics:

Mathematical prodigies

Trisection of angles

The history of mathematics from 600 B.C. to 200 B.C.

The history of mathematics to the present day.

Officers were: President Archimedes, Helen Kriegsman; Vice-President Plato, Mrs. Mildred Bradshaw; Secretary Lagrange, Mary Lou Ralston; Treasurer Thales, Virginia Prussing; Corresponding Secretary, Professor W. H. Hill, K. S. T. C., Pittsburg, Kansas; and Sponsor, Professor J. A. G. Shirk.

Mathematics Club, Brown University

Monthly meetings were held at eight o'clock in the evening, and were followed by refreshments. Papers presented during the fall semester were as follows:

Ancient number systems, with lantern slides, by Professor Otto Neugebauer, R. P. Breeding presiding

The isograph—a mechanical root finder, with film and lantern slides, by R. E. Jacobson, Jr., S. L. Ehrlich, presiding

Skeleton division, by Anna Irene La Banca, and

Practical applications of special mathematical curves, by Samuel Millman, Winifred O'Connell presiding

From March to May, 1944, the program included the following talks:

On some properties of a circle, by Professor J. D. Tamarkin, R. P. Breeding, presiding

The cell of the honey bee, by Shirley Gallup, and

The number pi, by M. E. Carlson; Frances Weeden presiding

Arithmetical prodigies, by L. M. Pease, and

The duodecimal system of numeration, by C. V. Harding, Jr., Alice Nancy Noyes presiding

During the summer semester 1944, three papers were presented:

Growth and mathematics, by Professor J. Walter Wilson of the Department of Biology, Shirley Gallup presiding

Mathematical fallacies, by F. W. Suffa, and

Map coloring problem, by A. B. Novikoff, C. V. Harding presiding.

A club picture was taken after the April meeting, and picnics were scheduled for May 27 and September 9. The club was directed by a Committee on Program and Arrangements, consisting of a Faculty Representative: Professor R. C. Archibald; a Student Chairman: R. P. Breeding (until spring '44), Shirley Marily Gallup (summer '44); and the following committee members: Barbara Cotter, Stanley Ehrlich, Shirley Gallup, Frances Weeden (until spring '44); and Leslie Carson, Stanley Ehrlich, Clifford Harding, Jr., Evelyn Lindsay (summer '44).

Kappa Mu Epsilon, Texas Technological College, Lubbock

A membership of fifty-eight active members, including twenty-nine new initiates, held regular meetings on the second Thursday of each month. At the first meeting, the local officers discussed the mathematicians whom they represented:

Lobatchevsky, by Virginia Bowman, president

Agnesi, by Brac Biggers, vice-president

Noether, Cayley, by Betty Grace Pugh, secretary-treasurer

Descartes, by Lida B. May, corresponding secretary.

Initiations, followed by refreshments were held in November. The December meeting at the home of Dr. and Mrs. F. W. Sparks, was a Christmas party, including games, puzzles, card tricks, refreshments, and Christmas carols. A very successful talk was presented at the February meeting, entitled

The sun, our source of light and energy, by Donald Robbins.

The spring initiation banquet was held at Cheri Casa. Projects for the year included the sending of Christmas cards to former members now in the armed service, and to past presidents of the chapter; and the purchase of War Savings Stamps and a twenty-five dollar War Bond. Plans have been initiated for making an annual award to the outstanding freshman in mathematics. Officers elected for 1944–45 are: President, Betty Grace Pugh; Vice-President, Tom Hassell; Secretary, Beverly Price; Treasurer, Betty Jane Morris; Corresponding Secretary, Lida B. May; Sponsor, Dr. D. L. Webb.

Mathematics Club, University of Dayton

Weekly meetings were held during the summer 1943, and semimonthly meetings during the next three terms, at which members presented papers. At the last meeting of each term the *Dean of Science Award* was presented to the student who delivered the most interesting paper before the Club. The awards were copies of the following books: *Men of Mathematics* by E. T. Bell (summer and fall 1943); *What is Mathematics?* by Courant and Robbins (winter 1944); *Isaac Newton—a Biography*, by L. T. More (summer 1944). The *Mathematics Club Alumni Awards for Excellence in Advanced Mathematics* were conferred at the December commencement upon Richard O'Brien (a junior), and at the summer commencement upon Lloyd Weeks (a senior) and Dennis Griffin (a junior).

Student papers during the summer term 1943, were:

The life of Leonard Euler, by William Fitzgibbons

The life of Sir Isaac Newton, by William McHugh

The nine point circle theorem, by Godfrey Kampner

Duodecimal arithmetic, by Allan Braun, prize winner

The history of the Mathematics Club, by Patrick Murphy, given as the vice-president's charge to the ten new members at the formal ceremony.

At the final meeting an alumnus of the club, Mr. Jack Homan, spoke on *The theory of mechanical vibration.*

Student papers during the fall term 1943 were:

Non-euclidean geometries, by Richard Welsh, prize winner

Mathematics and biology, by Edward Buescher

The origin and history of pi, by Robert Reef

Three classical problems of Greek antiquity, by Margaret Magin

The nature and importance of zero, by Dennis Griffin.

Two faculty talks were presented during the fall as follows:

The nature and history of the mathematical tripos, by Dr. K. C. Schraut

The aims of education and research in mathematics, by Professor J. L. Synge of Ohio State University.

Student papers during the winter term 1944 were:

The algebra of sets, by Dennis Griffin

The theory of groups, by Edward Buescher

Curves, by George Igel

The graphical solution of algebraic problems, by Edward Freeh

Mathematical recreations, by David Tom

The Mathematics Club, past, present, and future, by James Schuler, prize winner, given as the vice-president's charge to the eleven new initiates, and subsequently published in the University's literary magazine, the *Exponent.*

Two faculty talks were presented during the spring as follows:

Caustic curves, by Professor K. C. Schraut

Geometrography, by Professor W. J. Bellmer.

Student papers during the summer term 1944 were:

The application of vectors to the proof of plane geometry theorems, by David Timmer

The real number system, by Dennis Griffin, prize winner

The theory of matrices, by Lloyd Weeks

The history of numerals, by Allan Braun

The future of the mathematics club, by George Igel, given as the charge to eleven new members at the formal initiation ceremony.

Two faculty papers were:

Gombert's curve, by Mr. C. G. Peckham

The theory of numbers, by Professor I. A. Barnett of the University of Cincinnati.

Social events of the year were a camp supper at Hills and Dales Park during the fall term, a banquet at Wishing Well Inn during the winter term, and a summer picnic at Madden Park.

Officers of the club during the several terms were the following: Presidents: William Fitzgibbons (S'43), William McHugh (F'43), David Tom (W'44), George Igel (S'44); Vice-Presidents: Patrick Murphy (S, F'43), Richard Welsh (F'43), James Schuler (W'44), Allen Braun (S'44); Secretary-Treasurers: Edward Buescher (S, F'43), Allan Braun (W'44, Sec'y S'44), Kenneth Trimbach (Treas. S'44). Professor K. C. Schraut was Faculty Adviser.

Pi Mu Epsilon, Northwestern University

At the *Mathematics Festival*, a whole day devoted to mathematics, the Illinois Beta Chapter of *Pi Mu Epsilon* was installed at Northwestern University on May 20, 1944. The *Mathematics Club* which had been active at Northwestern University for many years served as the nucleus for the honorary group. Several hundred people helped celebrate this installation at the *Mathematics Festival* and heard distinguished guest speakers. The banquet culminated the day's activities and included the initiation of 70 charter members, 15 of whom are faculty members, as well as the presentation of the charter by Professor Tomlinson Fort of Lehigh University, Director-General of *Pi Mu Epsilon*. Papers presented during the day's program were:

Adventure vs. mathematics, by Professor E. R. Smith, Iowa State College

The concept of area in plane geometry, by Professor C. C. MacDuffee, University of Wisconsin

Alignment charts and projective geometry, by Professor L. R. Ford, Illinois Institute of Technology

Stirling's numbers and Taylor's formula, by Professor Tomlinson Fort, Lehigh University

Demonstration and heuristic reasoning in mathematics, by Professor L. M. Graves, University of Chicago.

Among the papers presented to the Club during 1943 were:

The mathematical theory of utility, by Professor H. T. Davis

The analytic theory of continued fractions, by Professor H. S. Wall

Geographic maps and their scales, by Professor D. R. Curtiss

Systems of linear differential equations, by Dean Walter Bartky

Some interesting facts about integers, by Professor L. W. Griffiths.

Papers presented to the Chapter at meetings in the summer of 1944, scheduled to fit in with the accelerated programs of both the civilian students and Navy V-12 alike, were:

Pi by chance, by Dr. Elliot Buell

The dynamics of the solar system, by Professor Oliver Lee

Alexandria, home of mathematics, by Professor H. T. Davis.

Officers for 1943 were: President, Frank Morehouse; Vice-President, James Murrin; Secretary-Treasurer, Beth Henry; Social Chairman, Alice Christiansen; Assistant Social Chairman, Marilyn Arms; Faculty Adviser, H. A. Simmons. Officers for 1944–45 are: President, Isabel Emma Barrett; Vice-President, John Pederson; Secretary, Elizabeth McDonald; Treasurer, Paul Axt; Social Chairman, Priscilla Mae Williams. Correspondence should be addressed to Professor E. H. C. Hildebrandt.

Mathematics Club, University of Buffalo

Three double-feature meetings were held in each semester, at which the following programs were presented:

The nine point circle, by Marjorie Easterbrook, and

The theory of relativity, by Anatole Shapiro.

The value and uses of the slide rule, by Judy Ullmann, and

Some mathematical puzzles, presented for the students to work, by Lois Obenauer.

Locks, by Jessie Brown and Joanne Yunker, featured by a display of present day locks, and slides of old time locks; and

A Chinese mystery, presented for the audience to solve, by John Euller.

At the February meeting guest speakers were the following former students who described how their background in mathematics had helped them in their work: Ruth Euller Heintz, a Spencer Lens worker; Virginia McCausland, a Curtiss Cadet; and Joan Searles, a teacher.

The cloudchamber, a paper by John Euller, was followed at the March meeting by some tricky mathematics problems, presented by Phyllis Valentine and Jane Noller.

How maps are made, was presented jointly by Audrey Strabel and Richard Salemi at the final April meeting, to which students of various high schools were invited. A discussion followed the lecture.

Social events sponsored by the Club during the year were a bowling party (at which scores ranged from 50 to 150); and a scavenger hunt held at the home of Dr. Gehman, at which the prize-winning team consisted of John Euller, Jane Noller, Marie Alessi, and Mary Jane Gill.

Officers were: President, Bernice Cohen; Vice-President, John Euller; Secretary-Treasurer, Jane Noller.

TWO RECTANGLES IN A QUARTER-CIRCLE

B. M. STEWART, Michigan State College

1. The problem. Consider the rectangle A, bounded by $x=0$, $x=\cos a$, $y=0$, $y=\sin a$, and of area $A=\cos a \sin a$; and the rectangle B, bounded by $x=0$, $x=\cos b$, $y=\sin a$, $y=\sin b$, and of area $B=\cos b\,(\sin b-\sin a)$; the angles, a and b, being subject to the restriction $0<a<b<90°$. (See Figure 1.) The problem is to maximize the function $z=A+B$, subject to the auxiliary condition $A=B$.

2. The interest of the problem. A substitution shows $z=2A=\sin 2a$, so that at first sight the solution seems to be $a=45°$, but the restriction on the companion angle b—which must also be in the first quadrant—makes this solution not acceptable. Here then is a situation of interest to students of the calculus, for the ordinary method of finding a maximum has failed because of a restriction on the range of the variable.

3. The source of the problem. Furthermore, the restriction is not artificial, for this problem arises in electrical engineering in the design of a winding-core with the figure described above representing one-fourth of the total cross section. (See Figure 2.)

4. General methods of solution. The student with broad interests will want to investigate the general methods for solving problems of the type here encountered—namely, to maximize the function $z=f(a, b)$ subject to the auxiliary condition $g(a, b)=0$. These general methods are presented in advanced calculus texts under the topics of partial differentiation, or differential geometry, or Lagrange's method of multipliers. Even with these methods caution must be used if there are restrictions imposed on the range of the variables.

5. A solution by elementary calculus. But this problem is of a special character. It is clear from the figure that if angle a is between 45° and 90°, then area B cannot be as great as area A. Since the function $z=\sin 2a$ is increasing for $0<a<45°$, the problem is to find the greatest a in the range $0<a<45°$ for which the equation $A=B$ has a solution b satisfying $0<a<b<90°$. Conceivably this may be solved by treating the relation $A=B$ as in elementary calculus for a maximum of a—namely, by setting $da/db=0$. The following equations are obtained:

$$A = B; \qquad \cos a \sin a = \cos b \sin b - \cos b \sin a; \tag{1}$$

$$da/db = 0; \qquad \cos^2 b - \sin^2 b + \sin a \sin b = 0. \tag{2}$$

By squaring and by using simple trigonometric identities it is possible to eliminate b from equations (1) and (2) and to obtain the following equation in which $Q=\sin^2 a$:

$$(Q-1)(15Q^3+3Q^2+10Q-1) = 0. \tag{3}$$

From equation (3) one and only one value of a, with an acceptable companion b, can be obtained. The approximate solution of the problem is as follows:

$$a = 18° \ 2' \ 27'', \qquad b = 52° \ 4' \ 10''.$$

The meaning of this solution is clarified by constructing the graph of the relation $A = B$ (see Figure 3), a task which is much simplified by noticing the symmetry in the points (0, 0) and (90°, 90°) and the symmetry in the line $b = a + 90°$.

6. Another solution by elementary calculus. The following method is indirect, yet worthy of a brief description. For each fixed choice of angle a there can be found by solving $dB/db = 0$ just one value of b such that $0 < a < b < 90°$ and such that B has a maximum value, say $M(a)$, so designated because it depends upon a. It is evident from the figure that for small values of a the value of $M(a)$ exceeds A, hence it is possible to solve the equation $A = B$; whereas for large values of a the value of $M(a)$ is less than A, hence it is impossible to solve $A = B$. The equation $M(a) = A$ determines the boundary between these cases and determines the largest value of a for which the equation $A = B$ has a solution with acceptable values of a and b. The equation $M(a) = A$ is equivalent to equation (3). Since the solution $a = 18° \ 2' \ 27''$ is in the range $0 < a < 45°$ where z is increasing, this maximum a serves to maximize z under the auxiliary condition $A = B$.

7. A solution without calculus. The maximum a with a companion b satisfying $A = B$ and $0 < a < b < 90°$ may also be determined by using the theory of equations.

Consider the rectangles in the figure. For small angles a there are two choices of b such that $A = B$. For large angles a there are no choices of b. Between these cases is a certain angle a for which there is one choice of b. This situation may be studied algebraically by replacing the equation (1) after some rearrangement, squaring, and use of identities by the following equation which may be described as a reduced quartic in the variable cos b with coefficients involving cos a:

$$\text{(4)} \quad \cos^4 b - \cos^2 a \cos^2 b + 2(\cos a - \cos^3 a) \cos b + (\cos^2 a - \cos^4 a) = 0.$$

First, Descartes' rule of signs shows there are two or no real positive roots, confirming the geometric intuition used above. Secondly, the inbetween case of two real and positive but identical roots can be determined by the vanishing of the discriminant of the quartic. The computation of the discriminant for equation (4) is not an easy matter, but after considerable simplification the discriminant may be written in the following way with $P = \cos^2 a$:

$$-16P^2(P - 1)(15P^3 - 48P^2 + 61P - 27).$$

Under the substitution $P = \cos^2 a = 1 - \sin^2 a = 1 - Q$ the discriminant takes the form which follows:

$$-16Q(1 - Q)^2(15Q^3 + 3Q^2 + 10Q - 1).$$

Setting the discriminant equal to zero leads to but one acceptable solution, in agreement with that obtained from equation (3).

8. A trigonometric oddity. In the equation $A = B$ one of the two solutions for b, when $a = 18°$, is $b = 54°$, exactly! The proof of this fact is an interesting exercise in trigonometry. The second solution of $A = B$, when $a = 18°$, is given by $b = 50° \ 7'$, approximately.

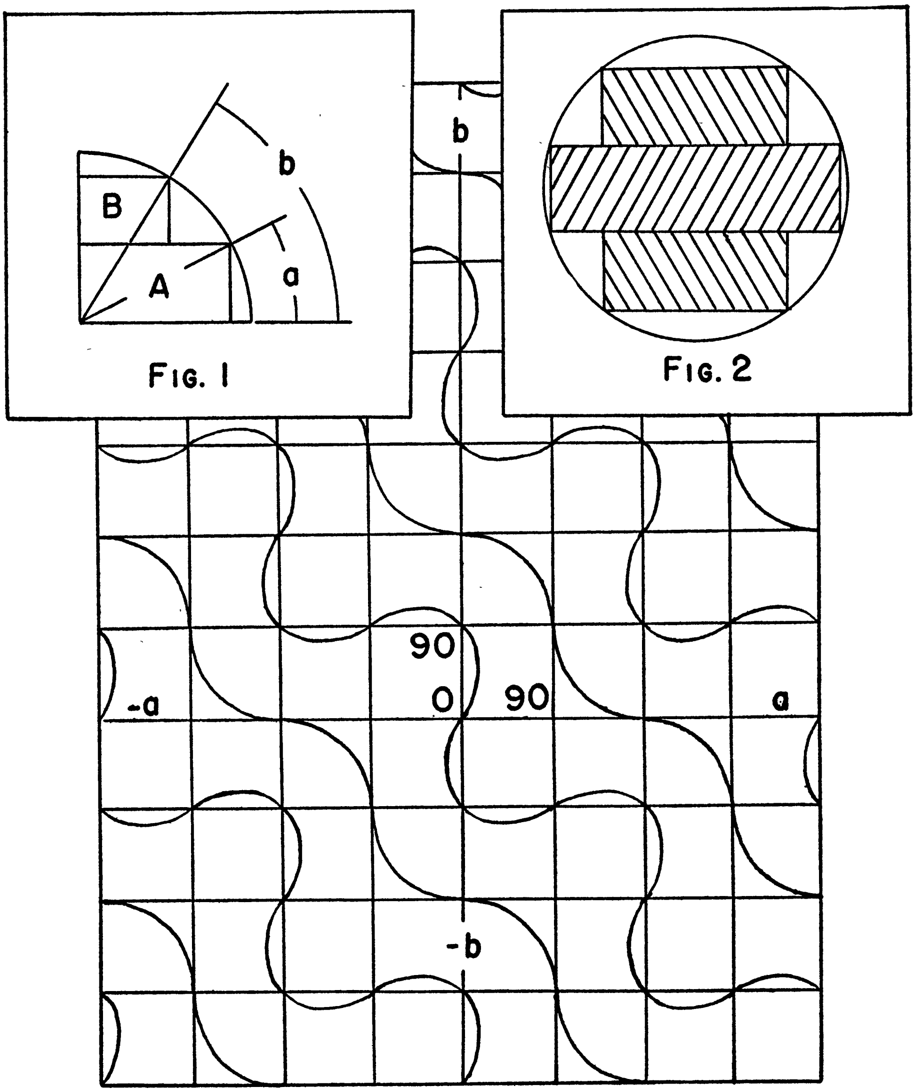


FIG. 1

FIG. 2

FIG. 3

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 656. *Proposed by S. H. Gould, National Research Council, Ottawa*

Each member of a very large office-staff agrees to buy a fifty-dollar war bond. One of the members suggests that each bond, instead of being given to its purchaser, be disposed of by lot. He wishes to wager one dollar that he himself will suffer loss through his own suggestion. How much should be wagered against him?

E 657. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the locus of centers of spheres passing through two given points and touching a given sphere.

E 658. *Proposed by Norman Anning, University of Michigan*

Find three three-digit numbers in geometrical progression which can be derived from one another by cyclic permutation of digits.

E 659. *Proposed by R. A. Staal, University of Toronto*

Show that, if one conic is self-reciprocal with respect to another, then the two conics belong to a symmetrical set of four, each of which is self-reciprocal with respect to any of the other three. (However, not more than three of the four conics can be real.)

E 660. *Proposed by C. D. Olds, Purdue University*

Let $z_0, z_1, \cdots, z_k$ be $k+1$ different complex numbers, all contained in the circle $|z| \leqq r$. Let

$$B_{kp} = \begin{vmatrix} 1 & z_0 & z_0^2 \cdots z_0^{k-1} & z_0^{k+p} \\ 1 & z_1 & z_1^2 \cdots z_1^{k-1} & z_1^{k+p} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ 1 & z_k & z_k^2 \quad z_k^{k-1} & z_k^{k+p} \end{vmatrix}.$$

Prove that

$$\left| \frac{B_{kp}}{B_{k0}} \right| \leqq \binom{k+p}{p} r^p.$$

SOLUTIONS

Sums of Products of Digits

E 622 [1944, 285]. *Proposed by V. Thébault, Tenne, Sarthe, France*

Find the smallest four-digit number such that the sum of products of pairs of digits is equal to the sum of products of sets of three.

Solution by Frank Hawthorne, New Rochelle, N. Y. Trying $a=1$ in the equation $ab+ac+ad+bc+bd+cd=abc+abd+acd+bcd$, we obtain

$$b + c + d = bcd.$$

If we exclude the trivial case $b=c=d=0$, the smallest solution is 1, 2, 3, giving the desired number 1123.

Also solved by Murray Barbour, D. H. Browne, W. E. Buker, H. N. Carleton, M. I. Chernofsky, E. D. Schell, E. P. Starke, W. R. Talbot, Hazel Schoonmaker Wilson, and the proposer. Mrs. Wilson remarks that, if duplicate digits are not allowed, the number is 2036.

Homothetic Tetrahedra and Coaxal Spheres

E 623 [1944, 285]. *Proposed by N. A. Court, University of Oklahoma*

The circumsphere of a tetrahedron $ABCD$ meets four "cevians" LA, LB, LC, LD in the points A', B', C', D'; $A''B''C''D''$ is a tetrahedron homothetic to the tetrahedron $A'B'C'D'$ with respect to the point L. If P and Q are two points in space, show that the four spheres $AA''PQ$, $BB''PQ$, $CC''PQ$, $DD''PQ$ are coaxal.

Solution by the Proposer. We have

$$LA\cdot LA' = LB\cdot LB' = LC\cdot LC' = LD\cdot LD' = p,$$

where p is the power of the point L for the sphere $ABCD$. Again, we have

$$LA'' = kLA', \qquad LB'' = kLB', \qquad LC'' = kLC', \qquad LD'' = kLD',$$

where k is the value of the homothetic ratio. Hence

$$LA\cdot LA'' = LB\cdot LB'' = LC\cdot LC'' = LD\cdot LD'' = pk.$$

Thus the point L has equal powers with respect to the four spheres $AA''PQ$, $BB''PQ$, $CC''PQ$, $DD''PQ$, and LPQ is the radical plane of any two of these four spheres. This proves the proposition.

A Property of Subfactorials

E 624 [1944, 285]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the integer nearest to $n!/e$ is a multiple of $n-1$.

Solution by Irving Kaplansky, Columbia University. The error made in stopping the expansion of e^{-1} with the term $(-1)^n/n!$ is less than $1/(n+1)!$. Hence the integer nearest to $n!/e$ is

$$P_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right\} = \Delta^n 0!,$$

the well known solution of the *Problème des Rencontres*. The divisibility property in question can be verified as follows:

$$\begin{aligned} P_n &= nP_{n-1} + (-1)^n = (n-1)P_{n-1} + P_{n-1} + (-1)^n \\ &= (n-1)P_{n-1} + (n-1)P_{n-2} + (-1)^{n-1} + (-1)^n \\ &= (n-1)(P_{n-1} + P_{n-2}). \end{aligned}$$

Also solved by Murray Barbour, J. B. Kelly, L. M. Kelly, Norman Miller, E. D. Schell, E. P. Starke, and the proposer.

An Extension of E 481

E 625 [1944, 285]. *Proposed by C. A. B. Smith, Trinity College, Cambridge, England*

Let X_i, Y_i, Z_i be the cofactors of x_i, y_i, z_i in the general third-order determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = D.$$

Prove that

$$\begin{vmatrix} X_2X_3 & Y_2Y_3 & Z_2Z_3 \\ X_3X_1 & Y_3Y_1 & Z_3Z_1 \\ X_1X_2 & Y_1Y_2 & Z_1Z_2 \end{vmatrix} = -D^2 \begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}.$$

Solution by C. V. L. Smith, P. G. School, U. S. Naval Academy. Multiplying the respective rows of the determinant

$$\begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}$$

by x_1, x_2, x_3, and then subtracting the first row from the second and third, we find that it is equal to

$$\frac{1}{x_1x_2x_3}\begin{vmatrix} x_1x_2x_3 & x_1y_2y_3 & x_1z_2z_3 \\ 0 & -y_3Z_3 & z_3Y_3 \\ 0 & y_2Z_2 & -z_2Y_2 \end{vmatrix} = y_3z_2Y_2Z_3 - y_2z_3Y_3Z_2.$$

By a well-known result on determinants (see, *e.g.*, Bôcher, *Introduction to Higher Algebra*, p. 33, Cor. 1), the cofactors of the terms of the adjoint of D are equal to the corresponding terms of D multiplied by the value of D. Hence from the result already obtained it follows that

$$\begin{vmatrix} X_2X_3 & Y_2Y_3 & Z_2Z_3 \\ X_3X_1 & Y_3Y_1 & Z_3Z_1 \\ X_1X_2 & Y_1Y_2 & Z_1Z_2 \end{vmatrix} = D^2(Y_3Z_2y_2z_3 - Y_2Z_3y_3z_2)$$

$$= -D^2 \begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}.$$

Also solved by A. R. Stokes.

The Position of a Ship

E 626 [1944, 347]. *Proposed by W. E. Buker, Perry High School, Pittsburgh*

A ship at sea attempts to determine its position by plotting lines of position through observation of three stars. Due to inaccuracy of observation, these lines are not concurrent but form a triangle. What is the most probable position of the ship, assuming (a) that the errors are due to defective instruments and are of the same kind, (b) that the errors are due to inaccurate observation and may not all be of the same kind?

Solution by R. H. Wilson, Jr., U. S. Naval Academy. The three lines of position represent the observed bearings of three objects whose positions, with reference to the coordinates of the chart, are known. (Such definition would then include landmarks as well as stars; but we shall assume that, as in the case of stars, parallactic differences are negligible.) Consider the probable bearing of the ship as viewed from these exterior points. It must lie toward the interior of the triangle as viewed from at least two of the points, if the angular value of the correction is to be arithmetically equal for all three. This requires that at least two errors must be of the "same kind."

(a) When all three errors are of the same kind, the true position must lie within the triangle, at such a point that the angular corrections are equal. Thus the most probable position of the ship is a point equidistant from the three sides of the triangle, namely the *incenter.*

(b) When the error in any one bearing may be of the opposite kind, the most probable position will be a point toward the exterior of the triangle as viewed from the object on which the error of the opposite kind was made. Again assuming equality of absolute values of all errors, the probable point will be an excenter, *viz.*, the center of the escribed circle exterior to the side on which this error of the opposite kind was made. Assuming equal probability for an error of the opposite kind on any one of the three bearings or on none at all, the three *excenters* and the *incenter* are positions of equal weight. The most probable position of the ship is then the mean position of these four points.

The resulting probable position is, in many cases, surprisingly far from the incenter (the point usually assumed by navigators), especially when one angle is obtuse. However, practical navigation will include simultaneously both conditions (a) and (b); so the position of the incenter should be weighted more heavily. This discrepancy partly explains the navigators' traditional prejudice in favor of an equilateral triangle.

Also solved by R. A. Rosenbaum, who remarks that a complete discussion is contained in an article entitled "Accurate Determination of the Position at Sea," in the *Hydrographic Review* for Nov., 1931.

An Enneagon

E 627 [1944, 347]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct an enneagon, given the centers of exterior squares described on the sides.

Solution by Robert Steinberg, University of Toronto. Suppose the nine given centers are $A_{01}, A_{12}, \cdots, A_{80}$. Then we will have solved the problem when we have found one vertex, A_0, of the enneagon $A_0 A_1 \cdots A_8$. Start with an arbitrary point P_0 in the plane; join it to A_{01} and mark off $A_{01}P_1 = A_{01}P_0$ on the perpendicular to $A_{01}P_0$ at A_{01}, so that $\sphericalangle P_0A_{01}P_1$ is one right angle in the positive sense. Next, join P_1 to A_{12} and mark off $A_{12}P_2 = A_{12}P_1$ on the perpendicular to $A_{12}P_1$ at A_{12}, so that $\sphericalangle P_1A_{12}P_2$ is one right angle. Continuing in this way, we get a sequence of points $P_0, P_1, \cdots, P_8, P_9 = P'$ (say). If P' coincides with P_0, we have finished, and $P_0P_1 \cdots P_8$ is the required enneagon. If not, let Q_0 be a point different from P_0. Going through the same construction as before, we get a sequence of points $Q_0, Q_1, \cdots, Q_8, Q'$. If we can determine Q_0 so that Q' coincides with Q_0, then this will be our required vertex.

Now, let us regard $P_0Q_0, P_1Q_1, \cdots, P'\,Q'$ as different positions of a directed segment PQ, and let us see what happens to this segment in moving from P_0Q_0 to P_1Q_1, and so on. Since $A_{01}P_0Q_0$ and $A_{01}P_1Q_1$ are congruent triangles, we have $P_1Q_1 = P_0Q_0$. Also $\sphericalangle(P_0Q_0, P_1Q_1)$ is a positive right angle. Thus, in going from P_0Q_0 to P_1Q_1, PQ has remained the same in length and has turned through one right angle. Hence, in going from P_0Q_0 to P_9Q_9 or $P'Q'$, PQ remains the same in length and is turned through nine right angles, or effectively one right angle. If $Q' = Q_0$, we must have $P_0Q_0 = P'Q_0$, and $\sphericalangle P_0Q_0P'$ equal to one right angle. This determines $Q_0 = A_0$ uniquely as the third vertex of one of the isosceles right-angled triangles standing on P_0P' as hypotenuse, namely the one for which $\sphericalangle P_0A_0P'$ is a positive right angle.

We can extend this solution to the general case of an n-gon; but we have to distinguish four cases, depending on the residue of n modulo 4. The above solution typifies the case when $n \equiv 1 \pmod 4$. If $n \equiv 3 \pmod 4$, PQ will be turned through an angle congruent to three right angles, and the vertex A_0 will be uniquely determined as the third vertex of the *other* isosceles right-angled triangle standing on P_0P' as hypotenuse, namely the one for which $\sphericalangle P_0A_0P'$ is a negative right angle. If $n \equiv 2 \pmod 4$, PQ will be turned through an angle congruent to two right angles, *i.e.*, reversed in direction, and the vertex A_0 will be uniquely determined as the midpoint of P_0P'. Finally, if $n \equiv 0 \pmod 4$, PQ will undergo several complete turns, making $P'Q'$ parallel and equal to P_0Q_0. Then $Q' = Q_0$ if and only if $P' = P_0$; so we have either no solution or else an infinite number of solutions (with A_0 an arbitrary point in the plane). To sum up, the solution is unique or poristic according as $n \not\equiv 0$ or $n \equiv 0 \pmod 4$.

Our construction can readily be extended to solve the more general problem of constructing a polygon $A_0A_1 \cdots A_{n-1}$ such that the triangles $A_0A_{01}A_1$, $A_1A_{12}A_2, \cdots$, shall be similar to n given triangles, where $A_{01}, A_{12}, \cdots$ are n given points. A particular case of this more general problem is mentioned in the *Editorial Note* to 4076 [1944, 411].

Also solved by Howard Eves, J. B. Kelly, Joseph Rosenbaum, and the proposer.

A Highly Composite Number

E 628 [1944, 348]. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

Find the smallest positive integer, one-half of which is a square, one-third of which is a cube, and one-fifth of which is a fifth power.

Solution by C. B. Barker, Jr., University of New Mexico. Clearly, the desired number is of the form $N=2^p3^q5^r$.

Since $N/2$ is a square, $q\equiv r\equiv 0 \pmod 2$ and $p\equiv 1 \pmod 2$.
Since $N/3$ is a cube, $p\equiv r\equiv 0 \pmod 3$ and $q\equiv 1 \pmod 3$.
Since $N/5$ is a fifth power, $p\equiv q\equiv 0 \pmod 5$ and $r\equiv 1 \pmod 5$.
The smallest values of p, q, r satisfying these conditions are

$$p = 15, \qquad q = 10, \qquad r = 6.$$

So the smallest value of N is

$$N = 2^{15}3^{10}5^6 = 30{,}233{,}088{,}000{,}000.$$

Also solved by Murray Barbour, Shepard Bartnoff, H. W. Becker, R. G. Blake, Colin Blyth, D. H. Browne, W. E. Buker, L. H. Bunyan, H. N. Carleton, F. M. Carpenter, Olive E. Cory, J. S. Cromelin, Monte Dernham, C. W. Emmons, Howard Eves, Daniel Finkel, Clifford Gardner, George Grossman, R. W. Hamming, Frank Hawthorne, Irving Kaplansky, Norbert Kaufman, J. B. Kelly, R. T. Luginbuhl, G. W. Petrie, E. D. Schell, E. P. Starke, W. R. Talbot, Hazel Schoonmaker Wilson, R. H. Wilson, Jr., H. J. Zimmerberg, and the proposer. Kaplansky remarks that, if it is further required that one-seventh of N be a seventh power, then the smallest value of N is

$$2^{105}3^{70}5^{126}7^{120},$$

a number with 255 digits.

A Finite Series

E 629 [1944, 348]. *Proposed by H. S. M. Coxeter, University of Toronto*

Sum the series

$$\sum_{r=0}^{[n/2]} (-\tfrac{1}{4})^r \binom{n-r}{r}.$$

Solution by G. W. Petrie, USNR Midshipmen's School, Notre Dame, Ind. Let the sum of the given series be S_n. Since

$$\binom{n-r}{r} = \binom{n-r-1}{r} + \binom{n-r-1}{r-1},$$

we find

$$S_n = S_{n-1} - \tfrac{1}{4}S_{n-2}. \tag{1}$$

This may be simplified by the substitution $S_n = U_n/2^n$, which gives

$$U_n = 2U_{n-1} - U_{n-2},$$

whence

$$U_n - U_{n-1} = U_{n-1} - U_{n-2} = \cdots = U_1 - U_0 = 2S_1 - S_0 = 2 - 1 = 1,$$

$$U_n - U_0 = n,$$

$$U_n = n + 1,$$

$$S_n = (n+1)/2^n.$$

Also solved by H. W. Becker, C. S. Gardner, Irving Kaplansky, J. B. Kelly, Norman Miller, and M. Wyman. D. H. Browne remarks that, for the series with 1 instead of $-\frac{1}{4}$, (1) becomes

$$S_n = S_{n-1} + S_{n-2};$$

hence the expression

$$\sum_{r=0}^{[n/2]} \binom{n-r}{r}$$

for Fibonacci numbers.

Editorial Note. In terms of $U_n(x)$, the Tschebyscheff polynomial of the second kind, we have

$$2^n S_n = U_n(1) = n + 1.$$

This problem occurs as "Ex. I" in Lucas, *Théorie des Nombres*, vol. 1 (Paris, 1891), p. 464. It can be derived from Bernoulli's expansion

$$\frac{\sin (n+1)\theta}{\sin \theta} = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} (2 \cos \theta)^{n-2r}$$

by making θ tend to zero. Bernoulli's expansion itself may be verified by means of the recurrence formula

$$U_n = 2 \cos \theta U_{n-1} - U_{n-2},$$

or obtained directly by equating coefficients of t^n in

$$2 \sum_{n=1}^{\infty} \frac{t^n \cos n\theta}{n} = -\log (1 - 2t \cos \theta + t^2) = \sum_{k=1}^{\infty} \frac{(2t \cos \theta - t^2)^k}{k}$$

and then differentiating with respect to θ.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4148. *Proposed by B. M. Stewart, Michigan State College*

A variation of the game of Solitaire is this game of the Red Cross, which is played with checkers on a field of squares arranged in six rows and five columns. During the play there can be at most one piece on a square, and at the outset one square is free so that the play, which consists of a series of jumps, may begin. A jump may be made if in three consecutive squares A, B, C of any row or column there are pieces on A and B while C is free. The jump is moving the piece on A over B to C and removing the piece on B from the play. Jumps along the diagonals of the field are forbidden. The object of the play is to remove all the pieces from the field except five, which are to be left in the form of the Red Cross, as in figure 1.

By the known theory for Solitaire (for example, see this MONTHLY, April, 1941, p. 228) a necessary condition is that the initially empty square be either as in figure 2 or 3. Show that in each of these cases a solution is possible.

x x x x x	0 0 0 0 0	0 0 0 0 0
x x 0 x x	0 0 0 0 0	0 0 0 0 0
x 0 0 0 x	0 0 x 0 0	0 0 0 0 0
x x 0 x x	0 0 0 0 0	0 0 0 0 0
x x x x x	0 0 0 0 0	0 0 0 0 0
x x x x x	0 0 0 0 0	0 0 x 0 0
Figure 1	Figure 2	Figure 3

4149. *Proposed by N. A. Court, University of Oklahoma*

The powers of the vertices of a tetrahedron (T) with respect to the spheres determined by the centroid of (T) and the circles of intersection of the respectively opposite faces of (T) with a sphere (L), of arbitrary radius, concentric with the circumsphere of (T), are equal.

4150. *Proposed by V. Thébault, Tenne, Sarthe, France*

In a tetrahedron $ABCD$ with the orthocenter H, the perpendiculars at A to the faces ACD, ABD, ABC meet respectively the planes HCD, HBD, HBC in (A_1, A_2, A_3), and similarly for the points (B_1, B_2, B_3) *etc.* Show that the planes $A_1A_2A_3$, $B_1B_2B_3$, *etc.*, are perpendicular to the medians of $ABCD$.

SOLUTIONS

A Special Triangle Transversal

4102 [1943, 638]. *Proposed by Hüseyin Demir, Columbia University*

Let O and I be respectively the circumcenter and incenter of a given triangle ABC. Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l. Prove synthetically that: (1) The points A_0, B_0, C_0 are collinear; (2) The sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l; (3) the line $A_0B_0C_0$ is perpendicular to the line OI.

Solution by the Proposer. (1) The locus of points whose sum of distances to the sides CA, AB is l, is a straight line passing through A_0, and perpendicular to AI. Let B_c, C_b be points where this locus cuts CA, AB. Similarly we consider two other loci corresponding to B_0, C_0. Let $A'B'C'$ be the triangle formed by these three loci. We shall prove that the last triangle is in perspective with ABC, I being the center of perspective. This is obvious, because since A' is the intersection of two loci, its distances to CA, AB are equal, that is, A' belongs to AI. Similarly B', C' belong respectively to BI, CI. Thus applying Desargue's theorem we have collinearity of A_0, B_0, C_0.

(2) Let M be a point of $A_0B_0C_0$ with x, y, z its distances to BC, CA, AB. We shall prove that $x+y+z=l$. Consider the locus of points with $y+z=Cst.$ This locus MQ (see figure) is parallel to B_cC_b, and $QQ_1=y+z$. Now, C_b, A_b having equal distances l to CA (see (1)) what we have to prove is that $QQ_2=x$. Draw MP parallel to BC, then $x=MX=PP_1$. Since $A'A_b$ is the bisector of $A_0A_bC_b$, we have $x=PP_1=PP_2$. It remains to prove that $QP \| C_bA_b$. This is true because the two triangles QMP, $C_bA_0A_b$ have two sides parallel, namely QM, C_bA_0 and MP, A_0A_b and they are in perspective, with C_0 as center of perspective. Therefore their third sides QP, C_bA_b must be parallel, that is $x=PP_2=QQ_2$.

(3) We shall prove two things: (a)—$A_0B_0C_0$ is the radical axis of circles (ABC) and $(A'B'C')$. (b)—The center O' of $(A'B'C')$ lies on OI, thus property (2) will be proved.

(a)—For, observe the relation $\overline{C_0A}\cdot\overline{C_0B}=\overline{C_0A'}\cdot\overline{C_0B'}$. This is true because the quadrilateral $ABB'A'$ is cyclic. (Note the equality of angles $A_bB'B=A'AB=\frac{1}{2}A$). Thus C_0 has equal powers with respect to the two circles. A similar property holds for A_0, B_0.

(b)—To prove that O' belongs to OI we shall remark that the locus of O' is a straight line when $A'B'C'$, whose sides are perpendicular to AI, BI, CI, varies, and since $A'B'C'$ is always in perspective with ABC, with I the center of perspective, O' will describe a straight line passing through I. It also passes through O. For, let A' be taken at the point where AI meets the circle (ABC). It is easy to see that B', C' will be similar points on the same circle. Thus O', the center of $(A'B'C')$, coincides with O, the center of (ABC). Therefore the radical axis $A_0B_0C_0$ of (ABC) and $(A'B'C')$ is perpendicular to the line OI passing through the centers O and O'.

Editorial Note. The first two theorems follow from similar triangles. The

case of an isosceles ABC may be discarded. For, if say the sides AB, AC have equal lengths, then in consequence of symmetry about AI the same is true for AB_0, AC_0, B_0 and C_0 being respectively on AC and AB. It then follows that B_0C_0 is perpendicular to OI; the converse is true as well as parts (1) and (2), but A_0 has an exceptional position. The points A_0, B_0, C_0 are uniquely determined by the given constant l. The distances x_b, y_b, z_b for B_0 are such that $y_b=0$, $x_b+z_b=l$, *etc.* Let P be a point on the straight line of C_0B_0 and let it divide this segment in the ratio $\lambda:1$. Then we have

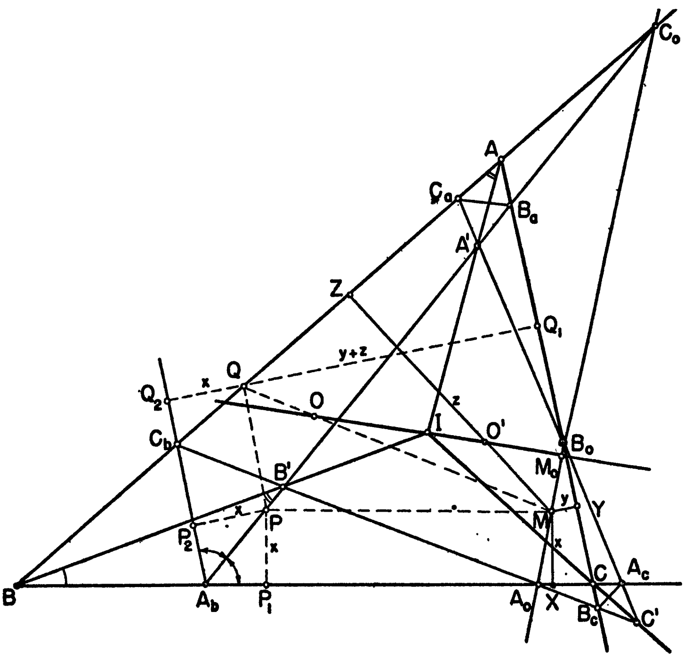
$$(\lambda+1)x = x_c + \lambda x_b, \qquad (\lambda+1)y = y_c, \qquad (\lambda+1)z = \lambda z_b,$$

where x, y, z are the distances for P. By addition we have

$$(\lambda+1)(x+y+z) = (x_c+y_c) + \lambda(x_b+z_b) = (\lambda+1)l;$$

and, if P is a finite point $x+y+z=l$. The straight line C_0B_0 meets BC in a finite point for which $x_a=0$ and $y_a+z_a=l$; hence this point is A_0. The two straight lines $A_0B_0C_0$ for different values of l are parallel; for, if they meet in a finite point, this point would have the sum of its distances equal to two different values. If P is a point not on B_0C_0 the line through it parallel to the latter meets the two sides in points different from B_0 and C_0. Hence the sum of its distances must be different from l; and this proves that the locus of points for a given l is the straight line $A_0B_0C_0$ for that value of l.

In the special case where $A'B'C'$ is inscribed in (O) the polar of C_0 passes through $(AA', BB')=I$, similarly, the polar of B_0 passes through I. Hence the polar B_0C_0 of I is perpendicular to OI.



NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. E. G. Begle of Yale University has been promoted to an assistant professorship.

Assistant Professor H. R. Branson of Howard University has been promoted to an associate professorship.

Associate Professor W. H. Brothers of Talladega College, Talladega, Alabama, has been promoted to a professorship.

J. O. Brown of Hampton Institute, Hampton, Virginia, has been promoted to an assistant professorship of physics.

Assistant Professor R. E. Byrne and Associate Professor A. E. Taylor of the University of California, Los Angeles, have been granted leaves of absence, the former for war research work and the latter to serve as operational research consultant with the Army Air Forces in England.

Dr. E. W. Cannon of the University of Delaware has been granted leave of absence for military service with the rank of Lieutenant Commander at Philadelphia Navy Yard.

Sister M. Loyola Conlan of the College of Mt. St. Vincent, New York City, has been promoted to a professorship and the acting chairmanship of the department.

Associate Professor C. M. Cramlet of the University of Washington has been granted leave of absence to serve with the Twenty-first Bomber Command operating in the Pacific.

Professor A. R. Crathorne of the University of Illinois has retired with the title professor emeritus.

R. E. Fullerton of Yale University has been appointed to an assistant professorship at the University of Oklahoma.

Dr. B. E. Gillam of the University of Missouri has been appointed to an assistant professorship at Drake University, Des Moines, Iowa.

Associate Professor Mary Goins of Western College has been appointed to an associate professorship at DePauw University.

F. F. Helton of the University of Illinois has been appointed to an associate professorship at Central College, Fayette, Missouri.

Dr. A. W. Jones of Yale University has been appointed to an assistant professorship at Michigan State College.

Assistant Professor J. F. Kubis of Fordham University has been promoted to an associate professorship.

Assistant Professor J. N. McClelland of Drake University is now engaged in war work at California Institute of Technology.

Professor W. H. McCrea of Queen's University, Belfast, Northern Ireland, has been appointed to a professorship at the University of London.

Associate Professor Florence M. Mears of George Washington University has been promoted to a professorship.

Associate Professor Cronan Mullen of Siena College, Loudonville, New York, has been promoted to a professorship in physics and the acting chairmanship of the Science Department.

Dr. G. M. Robison of Susquehanna University has been promoted to an assistant professorship.

Assistant Professor A. E. Ross of St. Louis University has been promoted to an associate professorship.

Dr. T. H. Southard of Wayne University has been promoted to an assistant professorship.

Assistant Professor W. G. Warnock of the University of Alabama has been promoted to an associate professorship.

Associate Professor S. S. Wilks of Princeton University has been promoted to a professorship.

J. F. Wyckoff of Trinity College has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
Oregon State College: Dr. S. P. Avann
Queens College: Dr. Ruth O. Goodman, Dr. J. C. R. Li
University of Michigan: Dr. O. G. Owens
University of Saskatchewan: Dr. W. J. R. Crosby

Professor Paul Capron, formerly of the United States Naval Academy, died September 29, 1944. He was a charter member of the Mathematical Association.

Dr. R. R. Fleet, formerly head of the mathematics department of William Jewell College, Liberty, Missouri, died December 1, 1944. He was a charter member of the Mathematical Association.

Lieutenant F. E. Riley, Jr., formerly a teacher in North Phoenix High School, Arizona, was killed in action in the China Burma India theater October 22, 1944.

Professor Emeritus C. L. Thornburg of Lehigh University died October 14, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE ESMWT PROGRAM OF PENNSYLVANIA STATE COLLEGE

The ESMWT Program supervised by the Pennsylvania State College has been of considerable interest to educators because of its extent and several novel experiments in organization. The following brief report treats those features of the Program of especial interest to mathematicians.

The Pennsylvania State College has offered work under ESMWT in all departments of engineering, in a number of branches of the mineral industries, and in chemistry, physics, mathematics, and economics. Some of the work has been done in residence, but most of it has been carried on by extension in more than a hundred communities throughout Pennsylvania. All extension classes are supervised by members of the regular residence departments of the College. A separate operating division has handled the business details of ESMWT, together with a number of regular extension activities. This central plan of organization has been quite effective, and has been particularly helpful in the case of courses, such as the one on quality control, which required the coordination of facilities to meet the needs of several departments.

Many courses are offered in communities where the registration is drawn chiefly from several local industries. In communities where large industries are located, registration is frequently limited to the employees of a single company. This is especially true of certain of the more specialized and advanced courses.

The standard courses offered in mathematics start with elementary algebra and trigonometry, and continue through calculus and differential equations. A course in higher mathematics upon the level of advanced calculus and another in the theory of functions of a complex variable have been given at the request of particular engineering groups. In general, however, most of the demand has been for classes on an elementary level.

Several courses have been given which draw their subject matter from two fields. For instance, an introductory course in physics and mathematics has been offered under the title "Foundations of Engineering, I." This course has been requested most frequently of the varous combinations, and interest in it has been at a high level ever since its introduction in 1941; at times thirty or forty sections have been running simultaneously. Laboratory equipment has always been provided for the physics portion of the work, and this has added considerable interest to the project. Mathematics has also been combined with engineering mechanics, with electrical engineering, with radio, and with drafting. Each of these combinations serves a specific type of industrial group; con-

sequently demand for these courses has been less than for the general mathematics and physics course. A combination course in quality control was set up cooperatively by the departments of mathematics and industrial engineering. The course is generally regarded as satisfactory, but requests for it have been limited to those companies which have already made some application of quality control in their inspection procedures. The cooperation of departments in the writing of an outline for a single course has appreciably increased the mutual understanding of the problems of the several departments.

There has been little attempt as yet to evaluate the results of combination courses from such points of view as efficiency of teaching, concentration of emphasis upon essentials of course content, and student interest. However, there appears to be some basis for the belief that combined courses can be used to advantage in the usual college curriculum as well as in the college extension program.

THE NAVY CAMPAIGN FOR RADAR TECHNICIANS

In recent months the Navy has been conducting an intensive campaign to obtain recruits who will study to become radio technicians, with the emphasis upon radar. Voluntary enlistments are open to men of 17 and to those who are 38 and over. Each inductee who passes his pre-induction physical examination and who meets Navy standards may request to take the Eddy Aptitude Test. Those who pass are assured of assignment to the Navy for radio technician training.

No extensive technical background is necessary to pass the Eddy Test. It is essential, however, to have a knowledge of elementary mathematics and physics. Some acquaintance with the principles of radio is also helpful.

Men who pass the Eddy Test become Seamen First Class, and are sent to a Naval Training Center for indoctrination. After that, each man receives nine months of specialized training and a Petty Officer rating. The program may be of interest to young men who have been able to study a term or two of college mathematics and physics before induction.

THE EDUCATIONAL PROGRAM FOR CANADIAN VETERANS

The leaders of Canada in the future must come, in large measure, from those young Canadians who have volunteered for active service in this war. This is a principle which the Dominion has recognized in setting up its plans for post-war rehabilitation of the armed services, with the result that full opportunity to resume education is given to the young ex-serviceman, or woman, whose ambition to go to university was interrupted by the war. Under the Post-Discharge Reëstablishment Order of October 1, 1941, a single man can obtain a maintenance grant of $60 per month. Married men receive $80 per month, and, in addition, dependents' allowances may be granted. Moreover, university fees are paid.

With 47 per cent of service personnel in this war having high or technical school training as compared with 13 per cent in the last war, the number of those

who might qualify for university training is large. All who were qualified for university admission at the time of enlistment, or who can qualify within fifteen months after discharge, may receive this opportunity.

In addition to the fact that initial educational standards are higher, the avenue of educational opportunity within the services has been opened more widely through the establishment of Directorates of Education in each of the services and through the opportunities provided by the Canadian Legion Educational Services.

Length of service, in the first instance, is the governing factor in the period of time for which an exserviceman or woman applying for university training may receive Government assistance. Grants are awarded on a month-for-month basis, for the time veterans were in the service, providing progress is satisfactory. For instance, an ex-serviceman with two full years in the armed services could receive twenty-four months of assistance. This would mean that he could complete three academic years (each year figured on eight school months) with Government assistance. However, the Government is of the opinion that the opportunity to complete the requirements for a degree should be available to the outstanding student. If, then, at the conclusion of the period of training to which he is entitled by his war service, a student has demonstrated the necessary ability, the assistance may be extended until he has completed his university course. Moreover, if his scholarship is sufficiently outstanding, he may be assisted in post-graduate studies. Post-graduate opportunities also are available to those who had entered upon, or were about to enter upon post-graduate studies at the time of their enlistment.

The regulations provide that no one may repeat a year's work for which benefits have already been paid, nor may he exhaust benefits in university study and then expect to receive vocational training.

This post-discharge training program is now in effect, and a number of exservicemen and women discharged from the armed forces in this war are attending Canadian universities and normal schools. Under a recent amendment, university education was authorized in exceptional cases at any university or college "of educational standards approved by the Minister" when suitable facilities are not available in Canada.

Provision has also been made for a wide program of vocational training for ex-service personnel. Under the Order, maintenance funds are provided, provisions are made for the selection of trainees, and courses of training are authorized. If ex-service men or women believe that a course of vocational training will assist in their rehabilitation, they first have a preliminary interview with the Veterans' Welfare Officer of the Department of Pensions and National Health stationed in the Employment and Selective Service Office in most of the larger centers of the Dominion. Where there is no Welfare Officer immediately available, local Canadian Legion Branches have taken on the responsibility for these preliminary interviews. If it is felt, as a result of these first conferences, that applicants will benefit by training, they are referred to the training officer or

counsellor for further interview or recommendation. As a result of this interview, the case comes before the District Rehabilitation Board which has the power to approve grants and training, and the ex-serviceman or woman is then granted a suitable course of vocational training if it is found that such training will assist in rehabilitation.

Maintenance grants for those taking vocational training are the same as for those studying in universities. These grants, however, will not be paid in most cases for any period in excess of 52 weeks. In an exceptional case, if the period of service is longer, the grant may be extended sufficiently to enable a veteran to complete a course of training, provided the full training period does not exceed the length of service. Veterans taking treatment in hospitals may be permitted to take approved correspondence courses with fees paid. This privilege may also be extended to those employed, in cases where part time or evening courses are not available.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-eighth annual meeting of the Mathematical Association of America was held at Chicago, Illinois, on Saturday and Sunday, November 25 and 26, 1944, in conjunction with meetings of the American Mathematical Society. About two hundred and thirty persons attended the meetings, including the following one hundred and sixty members of the Association:

L. K. ADKINS, State Teachers College, La Crosse, Wis.
V. W. ADKISSON, University of Arkansas
A. A. ALBERT, University of Chicago
E. F. ALLEN, Oklahoma A. and M. College
BEULAH M. ARMSTRONG, University of Illinois
MAX ASTRACHAN, Antioch College
H. G. AYRE, Western Illinois State Teachers College

RUTH M. BALLARD, Wright Junior College
R. H. BARDELL, University of Wisconsin Extension Division
C. F. BARR, University of Wyoming
WALTER BARTKY, University of Chicago
GRACE E. BATES, University of Illinois
FELIX BERNSTEIN, New York University
H. R. BEVERIDGE, Monmouth College
S. F. BIBB, Illinois Institute of Technology
H. L. BLACK, Westminster College
G. A. BLISS, University of Chicago
HENRY BLUMBERG, Ohio State University
M. G. BOYCE, Western Reserve University
R. W. BRINK, University of Minnesota
E. L. BUELL, Northwestern University
R. S. BURINGTON, Case School of Applied Science
HERBERT BUSEMANN, Illinois Institute of Technology
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
C. S. CARLSON, St. Olaf College
W. B. CARVER, Cornell University
W. B. CATON, Illinois Institute of Technology
E. W. CHITTENDEN, University of Iowa
R. V. CHURCHILL, University of Michigan
MARY D. CLEMENT, Wells College
R. H. COLE, University of Western Ontario
J. J. CORLISS, De Paul University
H. M. COX, University of Nebraska
D. R. CURTISS, Northwestern University

PAUL D'ARCO, De Paul University
JAMES EDGAR DAVIS, University of Illinois
JOHN DECICCO, Illinois Institute of Technology
J. E. DOTTERER, Indiana Central College

PAUL EBERHART, Washburn University
J. P. ESPOSITO, Crane Technical High School, Chicago
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

WILL FELLER, Brown University
EDNA M. FELTGES, Woodrow Wilson Junior College
W. L. FIELDS, Louisville Municipal College
J. V. FINCH, Capt., U. S. Army
N. J. FINE, Lukas-Harold Corporation, Indianapolis
L. R. FORD, Illinois Institute of Technology
J. S. FRAME, Michigan State College

H. M. GEHMAN, University of Buffalo
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G. D. GORE, Central Y. M. C. A. College, Chicago
L. M. GRAVES, University of Chicago
LAURA Z. GREENE, Washburn University
V. G. GROVE, Michigan State College

D. W. HALL, University of Maryland
P. R. HALMOS, Syracuse University
R. W. HAMMING, University of Illinois
E. G. HARRELL, State Teachers College, Platteville, Wisconsin
W. L. HART, University of Minnesota
J. O. HASSLER, University of Oklahoma
E. D. HELLINGER, Northwestern University
FRITZ HERZOG, Michigan State College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
J. D. HILL, Michigan State College
T. R. HOLLCROFT, Wells College
R. C. HUFFER, Beloit College
H. K. HUGHES, Purdue University
C. C. HURD, U. S. Coast Guard Academy

M. H. INGRAHAM, University of Wisconsin

R. D. JAMES, University of British Columbia
L. S. JOHNSTON, University of Detroit
B. W. JONES, Cornell University

SAMUEL KARLIN, Illinois Institute of Technology
DORA E. KEARNEY, Iowa State Teachers College
KATHARINE B. KEPPLER, Foxcroft School, Middleburg, Virginia
E. C. KIEFER, James Millikin University
H. R. KINGSTON, University of Western Ontario
J. R. KLINE, University of Pennsylvania
L. A. KNOWLER, University of Iowa
J. C. KOKEN, Parks Air College
W. C. KRATHWOHL, Illinois Institute of Technology
W. H. KURZIN, Herzl Junior College

A. E. LAMPEN, Hope College
JOSEPH LANDIN, University of Notre Dame
E. P. LANE, University of Chicago
R. E. LANGER, University of Wisconsin
D. H. LEAVENS, University of Chicago
A. T. LONSETH, Northwestern University

C. C. MACDUFFEE, University of Wisconsin
H. F. MAC NEISH, Brooklyn College
MORRIS MARDEN, University of Wisconsin
W. T. MARTIN, Syracuse University
MARGARET E. MAUCH, Michigan State College
J. R. MAYOR, Southern Illinois Normal University
KARL MENGER, University of Notre Dame
K. W. MILLER, Commonwealth Edison Company, Chicago
H. J. MISER, Lawrence College
W. L. MISER, Vanderbilt University
A. C. MOELLER, Marquette University
G. E. MOORE, University of Illinois
C. W. MORAN, Lane Technical School, Chicago
EUGENIE M. MORENUS, Sweet Briar College
E. J. MOULTON, Northwestern University
J. R. MUSSELMAN, Western Reserve University

A. L. NELSON, Wayne University
C. V. NEWSOM, Oberlin College
IVAN NIVEN, Purdue University
E. P. NORTHROP, University of Chicago
F. S. NOWLAN, University of British Columbia

E. B. OGDEN, Union College
RUFUS OLDENBURGER, Illinois Institute of Technology
J. M. H. OLMSTED, University of Minnesota
F. W. OWENS, Pennsylvania State College

GORDON PALL, McGill University
P. M. PEPPER, University of Notre Dame

George Piranian, Northwestern University
J. C. Polley, Wabash College
Adrien Pouliot, Laval University

J. F. Randolph, Oberlin College
W. R. Ransom, Tufts College
Ruth B. Rasmusen, Wilson Junior College
C. B. Read, University of Wichita
F. A. Reiber, Chicago
W. T. Reid, Northwestern University
Haim Reingold, Illinois Institute of Technology
P. R. Rider, Washington University
W. H. Roever, Washington University
E. H. Rothe, University of Michigan

Hans Samelson, Syracuse University
R. G. Sanger, University of Chicago
K. C. Schraut, University of Dayton
Tryphena H. Scibiorski, Detroit
M. E. Shanks, University of Missouri
H. A. Simmons, Northwestern University
R. C. Simpson, Jr., University of Wisconsin
E. R. Smith, Iowa State College
G. W. Smith, University of Kansas
Z. L. Smith, University of Chicago
L. W. Stark, William Jewell College
R. C. Stephens, Knox College
B. M. Stewart, Michigan State College
E. B. Stouffer, University of Kansas
E. G. Swafford, Park College

H. P. Thielman, Iowa State College
T. Y. Thomas, Indiana University
C. J. Thorne, Louisiana State University

Henry Van Engen, Iowa State Teachers College
J. I. Vass, University of Wisconsin Extension Division
John von Neumann, Institute for Advanced Study

H. S. Wall, Northwestern University
K. W. Wegner, Carleton College
Marie J. Weiss, Sophie Newcomb College
W. M. Whyburn, Texas Technological College
L. R. Wilcox, Illinois Institute of Technology
J. W. Wiley, Anderson College and Theological Seminary
K. P. Williams, Indiana University
R. S. Wolfe, Northwestern University
Alice K. Wright, Southern Illinois Normal University

J. H. Zant, Oklahoma A. and M. College
Oscar Zariski, Johns Hopkins University
Sister M. Claudia Zeller, College of St. Francis

The meetings were held in the auditorium of the Museum of Science and Industry in Jackson Park at 57th Street and South Shore Drive, and the hotel headquarters were at the Hotels Windermere. Lunches were available at a convenient cafeteria in the Museum.

A joint dinner for the two organizations was held at 6:00 p.m. on Saturday at the Hotel Windermere West. President W. M. Whyburn of Texas Technological College acted as toastmaster. He introduced Professor R. E. Langer of the University of Wisconsin who gave a short talk on post-war problems. At the close of his talk he called attention to the loss to American mathematics in the recent death of Professor G. D. Birkhoff of Harvard University, and the guests at the dinner stood for a moment of silent tribute to the memory of this outstanding member of the two mathematical organizations. Professor W. T. Martin of Syracuse University offered resolutions expressing thanks to Major L. R. Lohr, Director of the Museum of Science and Industry, to Mrs. Florence Carroll of the staff of the Museum, to the management of the Hotels Windermere, and to the Committee on Arrangements, for their cooperation and help in making the meetings pleasant and successful; and these resolutions were adopted by a rising vote.

The sessions of the American Mathematical Society were held on Friday and on Saturday morning and afternoon. On Friday at 7:45 p.m. the eighteenth

Josiah Willard Gibbs Lecture was given by Professor John von Neumann, the title being "The ergodic theorem and statistical mechanics." On Saturday at 2:00 p.m., by invitation of the Program Committee, Professor Will Feller gave an address on "Limit theorems in the theory of probability."

The Mathematical Association held sessions Saturday evening and Sunday morning, the program having been prepared by a committee consisting of L. M. Graves, *Chairman*; L. M. Blumenthal, E. R. Smith. The program follows.

First Session of the Association

Symposium on undergraduate mathematical curricula.

"The changing curriculum in secondary schools and the preparation of teachers," by Professor Raleigh Schorling, University of Michigan.

"The integration of undergraduate mathematics and physics," by Professor Michael Ference, University of Chicago.

"Applied courses for students majoring in mathematics," by Professor R. V. Churchill, University of Michigan.

Second Session of the Association

Symposium on undergraduate mathematical curricula (continued).

"Mathematics in a liberal education," by Professor E. P. Northrup, University of Chicago.

"The needs of the mathematics major and the graduate student," by Professor E. W. Chittenden, University of Iowa.

"The problem of the returning veteran," by Professor C. V. Newsom, Oberlin College.

Meeting of the Board of Governors

The Board met Saturday afternoon at 3:30 at Hotel Windermere West. Sixteen members of the Board were present, including eight Regional Governors.

The following thirty-five persons were elected to membership on applications duly certified:

Sister Winifred Ahl, A.B. (St. Joseph's Coll., Md.) St. Joseph's Coll., Emmitsburg, Md.

G. B. Banks, Ph.D.(Niagara Univ.) Prof., Physical Sci., Niagara Univ., Niagara University, N. Y.

R. A. Beaumont, Ph.D.(Illinois) Asst. Prof., Univ. of Washington, Seattle, Wash.

Grace L. Bolton, B.S.(N. J. Coll. for Women) Instr., New Jersey Coll. for Women, New Brunswick, N. J.

W. C. Bornmann, A.M.(Columbia) Instr., Georgia School of Tech., Atlanta, Ga.

P. F. Byrd, M.S.(Chicago) 1st Lt., U.S. Army Air Corps.

J. D. Daugherty, A.M.(Pennsylvania) Head of Math. Dept., Eastside High School, Paterson, N. J.

H. P. Fawcett, Ph.D.(Columbia) Prof., Educ., Ohio State Univ., Columbus, Ohio.

Harry Ferguson, B.S.(Boston Univ.) Instr., Tufts Coll., Medford, Mass.

Abe Gelbart, Ph.D.(Mass. Inst. of Tech.) Asst. Prof., Syracuse Univ., Syracuse, N. Y.

T. D. Howe, Jr., B.S. in C.E.(Harvard) T. D. Howe Construction Co., Houston, Tex.

J. B. Jeffries, M.S.(Chicago) Jr. Chemist,

Metallurgical Lab., Univ. of Chicago, Chicago, Ill.

S. A. JENNINGS, Ph.D.(Toronto) Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Can.

R. E. JOHNSON, Ph.D.(Wisconsin) Asst. Prof., Mount Holyoke Coll., South Hadley, Mass.

ROBERTA F. JOHNSON, Ph.D.(Cornell) Asso. Prof., Wilson Coll., Chambersburg, Pa.

H. L. KRALL, Ph.D.(Brown) Asso. Prof., Pennsylvania State Coll., State College, Pa.

SAUL KRAVETZ. Student, Brooklyn Coll., Brooklyn, N. Y.

REV. B. J. KUHN, A.M.(St. Bonaventure) Prof., Siena Coll., Loudonville, N. Y.

W. C. McDANIEL, Ph.D.(Wisconsin) Asso. Prof., Southern Illinois Normal Univ., Carbondale, Ill.

ROBERT McLARREN. Managing Editor, Air Age, Inc., 551 Fifth Ave., New York 17, N. Y.

REV. W. J. MILLER, Ph.D.(Harvard) Prof., Woodstock Coll., Woodstock, Md.

E. E. MOISE, A.B.(Tulane) Lt. (j.g.), U.S.N.R.

D. C. MURDOCH, Ph.D.(Toronto) Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Can.

G. M. NAUL, B.Ch.E.(Pratt Inst.) Shift Supervisor, Celanese Corp. of America, Cumberland, Md.

ALBERT NEWHOUSE, Ph.D.(Chicago) Instr., Rice Inst., Houston, Tex.

W. H. NORRIS, JR., A.B. (George Washington) Teacher, Maury High School, Norfolk, Va.

D. O. PATTERSON, Ph.D.(Minnesota) Head of Dept., State Teachers Coll., Valley City, N. D.

C. R. PERISHO, A.M. (Haverford Coll.) Instr., Junior Coll., McCook, Nebr.

BERNARD RASOF, M.S.(Calif. Inst. of Tech.) Research Engr., California Inst. of Tech., Pasadena, Calif.

ERIC REISSNER, Ph.D.(Mass. Inst. of Tech.) Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.

E. H. ROTHE, Ph.D.(Berlin) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

E. J. STULKEN, A.M.(Texas) Statistician, Geophysical Service, Inc., 1311 Republic Bank Bldg., Dallas 1, Tex.

E. G. SWAFFORD, A.M.(Syracuse) Asst. Prof., Park Coll., Parkville, Mo.

L. I. WADE, JR., Ph.D.(Duke) Instr., Duke Univ., Durham, N. C.

D. W. WESTERN, A.M.(Michigan State) Instr., Brown Univ., Providence, R. I.

The Secretary reported the deaths of the following members of the Association:

G. D. Birkhoff, Professor of mathematics, Harvard University. (November 12, 1944)

T. S. Fiske, Professor emeritus of mathematics, Columbia University. (January 10, 1944)

Harris Hancock, Professor emeritus of mathematics, University of Cincinnati. (March 19, 1944)

Edward Helly, Instructor in signal corps training program, Illinois Institute of Technology. (November 28, 1943)

Sister Thomas Marie Maloney, Instructor in mathematics, Trinity College, Washington, D. C. (1943)

J. S. Miller, Professor of mathematics, Emory and Henry College. (March 16, 1944)

Ruth Newlin, Teacher, Junior College and High School, Creston, Iowa. October 20, 1944.

H. L. Rietz, Professor of mathematics, University of Iowa. (December 7, 1943)

T. R. Rosebrugh, Professor of electrical engineering, University of Toronto. (January 24, 1943)

J. A. Shohat, Professor of mathematics, University of Pennsylvania. (October 8, 1944)

D. E. Smith, Professor emeritus of mathematics, Columbia University. (July 29, 1944)

J. R. Wilton, Professor of mathematics, University of Adelaide, Australia. (April 1944)

Clyde Wolfe, Mathematician, Radiation Laboratory, University of California. (March 25, 1944)

The Board elected W. F. Cheney, Jr., as Second Vice-President for a term of two years beginning January 1, 1945.

The Board received a report from the Committee on the Chauvenet Prize consisting of Philip Franklin, Chairman; Saunders Mac Lane, and G. T. Whyburn. The committee recommended that the prize for the years 1941–43 be awarded to R. H. Cameron for his paper, "Some introductory exercises in the manipulation of Fourier transforms," *National Mathematics Magazine*, vol. 15 (1941), pp. 331–356. Acting on this recommendation, the Board voted to award the prize to Professor Cameron.

The Board voted to accept the invitation from McGill University to hold the summer meeting of the Association at Macdonald College of McGill University on Saturday, June 23, 1945. The American Mathematical Society will meet there on Sunday and Monday, June 24–25. These meetings will follow the Canadian Mathematical Congress which will meet at McGill University during the week of June 18–23.

Certain suggestions from the Cooperative Committee on the Teaching of Science were presented to the Board. After some general discussion the suggestions were referred to the Association's Conference Committee on Education for study and subsequent report to the Board.

The resignation of Professor W. D. Cairns from the War Policy Committee was presented, and the Board voted to accept the resignation to be effective January 1, 1945. The Board then elected Professor C. C. MacDuffee to succeed Professor Cairns on this joint committee of the Association and the Society.

On nomination of the Editor-in-Chief the Board elected the following associate editors of the MONTHLY for the year 1945:

E. F. BECKENBACH	OTTO DUNKEL	MARJORIE GROVES
L. M. BLUMENTHAL	H. P. EVANS	B. W. JONES
N. B. CONKWRIGHT	B. F. FINKEL	C. V. NEWSOM
H. S. M. COXETER	J. S. FRAME	P. R. RIDER
W. M. DAVIS	ORRIN FRINK, JR.	MARIE J. WEISS

After President Cairns had retired from the meeting, the Secretary presented a recommendation from the Executive Committee. In view of the long and unique service of Professors Slaught and Cairns to the Association, the committee recommended that the same recognition that had been given to Professor Slaught in 1933 be accorded at this time to Professor Cairns. The Board voted unanimously to recommend to the Association at its annual business meeting that the By-Laws of the Association be suspended and that Professor W. D. Cairns be elected Honorary President for life with the privileges of honorary life membership.

ANNUAL BUSINESS MEETING

The annual business meeting of the Association was held on Sunday morning at 9:30, President Cairns presiding.

On recommendation from the Board of Governors, the Association voted the following amendments to the By-Laws, the effect of these amendments being to abolish institutional memberships in the Association:

ARTICLE II

Section 2 to be eliminated.

In Section 3, after the word "individual" delete the phrase "or institution" and after the word "endorsed" delete the phrase "in the case of individuals."

Sections 3 and 4 to be renumbered as 2 and 3.

ARTICLE VII

In Section 1, delete the word "individual."

In Section 2, delete the word "individual."

Section 3 to be eliminated.

Sections 4, 5 and 6 to be renumbered as 3, 4 and 5.

The results of the election of officers were announced as follows:

President for the term 1945–46: C. C. MacDuffee, University of Wisconsin.

Governors at Large for the term 1945–47: Walter Bartky, University of Chicago; H. S. M. Coxeter, University of Toronto.

An announcement was made of the award by the Board of Governors of the Chauvenet Prize for 1941–43 to R. H. Cameron, Massachusetts Institute of Technology.

The Secretary presented the recommendation from the Board of Governors that the By-Laws be suspended and that W. D. Cairns be made Honorary President for life with the privileges of life membership. The recommendation was adopted by a unanimous rising vote.

W. B. CARVER, Secretary-Treasurer

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held on Saturday, April 15, 1944, at the George Washington University in Washington, D. C. Professor J. H. Taylor, Chairman of the Section, presided at the morning and afternoon sessions.

There were fifty-six persons in attendance, including the following thirty-three members of the Association: O. S. Adams, Harriet W. Allen, M. W. Aylor, T. E. Berry, Archie Blake, Lillian O. Brown, G. R. Clements, Abraham Cohen, J. A. Duerksen, E. J. Finan, B. H. Gere, Michael Goldberg, D. W. Hall, E. H. Hanson, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, J. A. Larrivee, Carol V. McCamman, E. J. McShane, Florence M. Mears, J. F. Milos, T. W. Moore, W. K. Morrill, C. R. Phelps, Grace Shover Quinn, O. J. Ramler, R. E. Root, A. D. Sollins, J. H. Taylor, C. H. Wheeler, III, E. W. Woolard.

At the business meeting the following officers were elected for the coming year: Chairman, C. H. Wheeler, III, University of Richmond; Secretary, W. K. Morrill, Johns Hopkins University; Executive Council, J. H. Taylor, George Washington University, and Michael Goldberg, Bureau of Ordnance; Regional Member of the Board of Governors, D. W. Hall, University of Maryland.

The following program was presented:

1. *Pythagorean representations*, by Lt. (j.g.) C. R. Phelps, U. S. Navy.

2. *A three-space linkage network*, by Michael Goldberg, Bureau of Ordnance, U. S. Navy Department.

It has been proved by A. B. Kempe (1878) that each of the four links of a plane movable quadrilateral linkage can be linked to a common pivot without restricting the movability of the linkage. No three pivots of the resulting configuration need be collinear. Mr. Goldberg demonstrated, with the aid of mechanical models, his three-space analogue of Kempe's theorem. Specifically, he showed that each of the four links of a hinged movable skew quadrilateral linkage can be linked by hinges to a common hinge without restricting the movability of the linkage. No two hinges of the resulting configuration need be parallel or concurrent. No three hinges need have a common perpendicular.

3. *Vertices of plane curves*, by Dr. S. B. Jackson, introduced by the Secretary.

A closed curve of class C'' which is not a circle has two vertices, by virtue of the continuity of the curvature. It was the purpose of this paper to characterize geometrically those curves with exactly two vertices. Let a curve be called normalized if it contains no complete circles, and let a simple closed arc of the curve which is never met again by the curve be called a simple loop. The following facts were established for any normalized curve C having two vertices: (a) the curve C may be divided into two simple arcs; (b) all double points are simple; (c) the curve contains exactly two simple loops, one containing each vertex; (d) none of the regions of the plane bounded by C are bounded always in the same sense except those regions bounded by the loops; (e) at any point of tangency the directed tangents coincide. Slightly different results were obtained for curves which are not normalized. The methods employed were elementary, extensive use being made of the invariance of vertices under direct circular transformations.

4. *Green's functions associated with problems in beam deflection*, by Dr. H. T. Muhly, U. S. Naval Academy, introduced by the Secretary.

The differential equation of the elastic curve of a transversally loaded beam is $y'' = -(1/s)M(x)$, where $s = EI$ is the stiffness coefficient, and $M(x)$ is the bending moment at the point x of the beam. The boundary conditions most frequently associated with this equation are $y(0) = y(L) = 0$, where L is the length of the beam. A widely used formula (a corollary to Castigliano's theorem) for the solution of this problem is $y(x) = (1/s)\int_0^L M(t)m(x, t)dt$, where $m(x, t)$ denotes the bending moment set up by a unit load at x. It was pointed out that this formula has the inherent disadvantage that the computation of $m(x, t)$ involves the solution of an indeterminate problem whenever the beam is statically indeterminate. As an alternative it was suggested that Green's function be used to exhibit the solution. One would then obtain

$$y(x) = \frac{1}{sL}\int_0^x M(t)t(L - x)dt + \frac{1}{sL}\int_x^L M(t)x(L - t)dt.$$

It was shown that this representation enabled one to derive in one step the familiar "three moment theorem" of Clapeyron. The generality of the method was brought out by applying it to the cases in which the beam carried an axial compression or tension in addition to the transverse load. The Green's functions for these cases were given, and the solutions obtained thereby were used to derive generalizations of Clapeyron's theorem to these more difficult cases.

5. *Mathematics as she are taught*, by Dr. L. B. Tuckerman, introduced by the Secretary.

The speaker pointed out many erroneous and ambiguous details in mathematics texts and other publications. He urged that all teachers make an effort to be more precise in their teaching.

W. K. MORRILL, *Secretary*

The Editor-in-Chief wishes to express his indebtedness to the following persons who have served as referees of papers during 1944.

E. F. Beckenbach, L. M. Blumenthal, W. C. Brenke, R. W. Brink, B. H. Brown, H. E. Buchanan, W. E. Buker, R. S. Burington, J. W. Campbell, W. B. Carver, W. B. Caton, N. B. Conkwright, A. H. Copeland, H. S. M. Coxeter, C. C. Craig, John DeCicco, H. J. Ettlinger, J. S. Frame, M. G. Gaba, M. R. Hestenes, T. H. Hildebrandt, Dunham Jackson, W. C. Krathwohl, Mayme I. Logsdon, C. C. MacDuffee, G. M. Merriman, E. J. Moulton, J. R. Musselman, Rufus Oldenburger, Isaac Opatowski, Haim Reingold, P. R. Rider, M. A. Sadowsky, I. M. Sheffer, E. B. Stouffer, T. Y. Thomas, T. L. Wade, Marie J. Weiss, and F. E. Wood.

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA, Cedar Rapids, April 14, 1945
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, Brooklyn, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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COLLEGIATE MATHEMATICS

VOLUME 52 NUMBER 3

CONTENTS

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NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the SECRETARY-TREASURER, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 10th of the month in which the change becomes effective.

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A FOUR-DIMENSIONAL ANALOGUE OF PASCAL'S THEOREM FOR CONICS

B. SEGRE, University of Manchester

1. Introduction. As an extension of Pascal's theorem for conics, the following general problem may be investigated.

Let us consider in [r]* *all sets S of linear spaces given in number and dimensions, subject possibly to given incidence conditions; and a continuous system of algebraic varieties V. Supposing that no V contains the general S, one has to decide whether and how it is possible to express the condition for a set S to lie on a variety V, in the form of linear conditions for certain spaces, linearly deduced from the spaces of S taken in an arbitrary order.*

A less circumstantial problem for the particular case of ten points of an ordinary quadric was proposed in the year 1825 by the Academy of Brussels, and studied since then by several authors.† But the results obtained are intricate, and far from possessing the elegance of Pascal's theorem.

The case of seven points of a twisted cubic curve has been considered by Chasles, whose result is simple but unsymmetrical.‡ In addition one knows a few cases of a different, rather trivial kind. For instance, in ordinary space six skew lines lie on a cubic surface, if and only if they have a complementary sextuplet, forming a double-six with them.§

In the present paper I deal with the above problem in another particular case, giving for this a non-trivial extension of Pascal's theorem, in which all the aesthetic qualities are preserved. The result concerns five lines lying on a three-dimensional quadric of [4] (Part II). It suggests some interesting questions (§11), and easily yields applications of various types, both to plane and three-dimensional geometry (Part III). Some preliminary questions on double-fours of lines, having some interest also in themselves and leading to another extension of Pascal's theorem, are studied in Part I.

PART I. ON DOUBLE-FOURS OF LINES

2. The three types of double-fours of lines. We say that two quadruplets of lines a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 constitute a *double-four*, if both the lines a and the lines b are two by two skew, while a line a and a line b are skew or incident according as they have or have not the same index. Then *e.g.* the lines a_1, a_2, a_3 are incident with b_4, so that the linear space joining these four lines can only have dimension 3 or 4. From the incidences among the lines a and b, we see that this linear space contains also b_1, b_2, b_3 and a_4. Hence:

* The symbol [r] designates a projective space of r dimensions. An [$r-1$] in an r-dimensional space is hereinafter called a "prime" of this space.

† *Cf.* the interesting paper by H. W. Turnbull and A. Young in Trans. Cambridge Phil. Soc. 23, 1923–28, pp. 265–301, where extensive historical references are also given.

‡ M. Chasles, Aperçu historique, Paris 1865 (reprinted from the Mém. cour. par l'Ac. de Bruxelles, 1837), p. 404.

§ *Cf. e.g.* B. Segre, The non-singular cubic surface, Oxford, 1942, pp. 7, 35.

A double-four of lines lies either in a four-dimensional space or in an ordinary space.

In the first case the double-four will be called of *the first kind.* In the second case the double-four will be called of *the second* or *third kind,* according as it lies, or does not lie on a cubic surface.

3. The double-fours of the first kind. We obtain the more general double-four of the first kind, by taking four lines a_1, a_2, a_3, a_4 in a general position in [4], and considering the lines b_1, b_2, b_3, b_4 uniquely defined by the conditions of meeting $a_2a_3a_4$, $a_3a_4a_1$, $a_4a_1a_2$, $a_1a_2a_3$ respectively. For *e.g.* the lines a_2, a_3, a_4 are two by two skew and do not lie in a prime, so that they have a single common transversal b_1, the intersection of the primes joining them in pairs. Moreover, it is easily seen that the lines b_1, b_2, b_3, b_4 are two by two skew, and are also skew to a_1, a_2, a_3, a_4 respectively. The two quadruplets $a_1a_2a_3a_4$, $b_1b_2b_3b_4$ clearly form a double-four of the first kind; their relation is a reciprocal one, and they will be said to be mutually *complementary.*

Since the lines in [4] constitute an irreducible ∞^6 system, and a double-four is uniquely defined by either of its two complementary quadruplets, it follows that:

The double-fours of the first kind, of a given four-dimensional space, form an irreducible system of dimension 24.

This agrees with the remark that *any two such double-fours are transformed one into the other by* $2\cdot 4! = 48$ *collineations.* The double-four $(a_1a_2a_3a_4, b_1b_2b_3b_4)$ is transformed into another given one $(a_1'a_2'a_3'a_4', b_1'b_2'b_3'b_4')$, *e.g.* by the collineation defined by the conditions of transforming the six points a_2b_3, a_3b_2, a_3b_1, a_1b_3, a_1b_2, a_2b_1, into the six points $a_2'b_3'$, $a_3'b_2'$, $a_3'b_1'$, $a_1'b_3'$, $a_1'b_2'$, $a_2'b_1'$ respectively.

We give now a new and simple proof of the following well known results, due to C. Segre.*

If $d = (a_1a_2a_3a_4, b_1b_2b_3b_4)$ *is a double-four of the first kind, the four primes* a_1b_1, a_2b_2, a_3b_3, a_4b_4 *have a line in common. This line is therefore uniquely defined by either of the complementary quadruplets of* d, *and is said to be* associated *to it. The planes meeting four general lines in* [4] *constitute an irreducible* ∞^2 *system, and each of them also meets the associated line.*

It is easily seen that, since d is of the first kind, the primes a_1b_1, a_2b_2, a_3b_3 cannot have a plane in common, so that they intersect in a line r. The results stated will follow, if we prove that the general point O of r lies in the prime a_4b_4; and that there are ∞^1 planes through O meeting $a_1a_2a_3a_4$, and ∞^1 planes through O meeting $b_1b_2b_3b_4$.

Now all this is immediately seen, on projecting the lines a_i, b_i from O upon a prime, into a_i', b_i' say $(i = 1, 2, 3, 4)$. For the lines a_1', a_2', a_3' are two by two skew, and each of them meets b_1', b_2', b_3'. Therefore these six lines lie on an ordi-

* *Cf.* C. Segre, Alcune considerazioni elementari sull'incidenza di rette e piani nello spazio a quattro dimensioni, Rend. Circ. Mat. Palermo 2, 1888, p. 45, or H. F. Baker, Principles of geometry, IV, Cambridge, 1925, pp. 113–123.

nary quadric Q. This contains a_4', b_4' as generators of opposite systems, since a_4' meets b_1', b_2', b_3', and b_4' meets a_1', a_2', a_3'. Hence a_4' and b_4' are incident, and so the prime a_4b_4 contains in fact the point O. The planes projecting from O the two systems of generators of Q, give the two ∞^1 systems of planes having the required properties.

4. The general projection of a double-four of the first kind upon a prime. With the eight lines of a double-four of the first kind, $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$, one can form two mutually residual skew quadrilaterals, in the following three different ways: $a_1b_3a_2b_4$ and $b_1a_3b_2a_4$, $a_1b_4a_3b_2$ and $b_1a_4b_3a_2$, $a_1b_2a_4b_3$ and $b_1a_2b_4a_3$. Since each quadrilateral lies in a prime, we see that d lies in three quadrics, each consisting of two such primes. The eight lines of d obviously constitute the complete intersection of these three quadrics. Hence the latter determine a *net*, N say, formed by all the quadrics of [4] containing d.*

The general quadric of N is irreducible, and so contains ∞^3 lines. We thus obtain in [4] an ∞^5 *algebraic complex of lines*, K say, formed by the lines lying on some quadric of N. We shall prove that:

The complex K is of the third order, and contains all the lines meeting one or other of the lines of the double-four d.

The second part of this theorem is obvious, since a quadric of N containing two points generally chosen upon a line meeting the base curve d of N, clearly contains this line. We prove the first part, by noticing that the lines of K lying on a general plane π constitute an envelope of the third class, which is in fact the Cayleyan envelope of the net of conics cut out by N on π.

Moreover, from a well known property of the Cayleyan envelope, it follows that the lines of K are those lines of [4] on which N cuts out pairs of an ordinary involution. Hence:

The lines of K can be characterized by the property that the pairs of primes a_1a_2 and a_3a_4, a_1a_3 and a_4a_2, a_1a_4 and a_2a_3 cut out on them three pairs of points in involution.

From a previous result we see that the lines of K passing through a given general point O constitute a *cubic three-dimensional cone, containing the planes projecting the eight lines of d from O.* Hence, on account of §2, we can say that:

The general projection of a double-four of the first kind upon a prime is a double-four of the second kind.

We shall reciprocate this result in §7.

5. The prime sections of a double-four of the first kind. Let us consider in [4] a general prime, ω say, and denote by A_i, B_i its intersections with a_i, b_i respectively ($i=1, 2, 3, 4$). The lines $a_ia_jb_hb_k$, where i, j, h, k is any arrangement of the numbers 1, 2, 3, 4, are in a prime (§4); this intersects ω in a plane, on which the points $A_iA_jB_hB_k$ must consequently lie. Hence the lines A_iA_j and B_hB_k are incident.

* Such a net N can clearly be obtained, in two different ways, as the totality of the quadrics of [4] containing four general lines.

On calling *incident* two tetrahedra $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ when each edge A_iA_j of the former is incident with the complementary edge B_hB_k of the latter, we now prove the following theorem:

Two incident tetrahedra $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ lie in the same ordinary space. Moreover, their eight vertices are associated with respect to the quadrics of this space (i.e., are the base points of a net of quadrics).

The ordinary spaces α, β of the two tetrahedra have in common the points of incidence of the six pairs of complementary edges A_iA_j, B_hB_k. Hence α and β coincide, since these six points are neither collinear nor coplanar. For, on the one hand, there is no line meeting all the edges of a tetrahedron. On the other hand, if the six points A_iA_j, B_hB_k lie on a plane π, which is therefore contained both in α and in β, then π cannot be a common face of the two tetrahedra; for, otherwise, the edges of these tetrahedra lying on π would give two triangles, each inscribed in the other, and two such triangles do not exist. Hence π is distinct from the faces of at least one tetrahedron, say of $A_1A_2A_3A_4$. It follows that π intersects the plane $A_2A_3A_4$ along a line, which contains the points where A_3A_4, A_4A_2, A_2A_3 meet B_1B_2, B_1B_3, B_1B_4 respectively. Since the last three edges are non-coplanar, we see that B_1 must lie on π. The same conclusion can likewise be reached with regard to B_2, B_3, B_4, so that the four vertices of the tetrahedron $B_1B_2B_3B_4$ lie on π. This contradiction proves the first part of the the theorem.

The second part follows at once, on noticing that the eight vertices of the two tetrahedra can be distributed in three different ways in two quadruplets of coplanar points:

$A_1A_2B_3B_4$ and $B_1B_2A_3A_4$, $A_1A_3B_4B_2$ and $B_1B_3A_4A_2$, $A_1A_4B_2B_3$ and $B_1B_4A_2A_3$.

Hence the eight points lie in three reducible quadrics, not belonging to a pencil, and are the base points of the net determined by them.

From the previous results it follows that:

The two complementary quadruplets of a double-four of the first kind cut out the vertices of two incident tetrahedra on the general prime of their space. This is the only means of obtaining two incident tetrahedra, by taking as vertices eight points lying on the eight lines of the double-four.

We shall say that such a pair of incident tetrahedra is *inscribed in the double-our.*

6. The double-fours of the second and third kinds. We determine the more general double-four of an ordinary space ω, by taking four generic lines $a_1a_2a_3a_4$ in ω, and further choosing the lines b_1, b_2, b_3, b_4 as generators of the quadrics defined by the directrices $a_2a_3a_4$, $a_3a_4a_1$, $a_4a_1a_2$, $a_1a_2a_3$ respectively. The number of parameters on which the double-four depends is therefore

$$4 + 4 + 4 + 4 + 1 + 1 + 1 + 1 = 20.$$

The cubic surfaces in ω constitute an ∞^{19} linear system. Moreover, the general cubic surface contains 27 lines, and a finite number ($=540$) of double-fours,

which are all equivalent with respect to the group of the 27 lines.† Hence:

The double-fours of the second and third kinds of an ordinary space constitute two irreducible systems, of dimensions 19 *and* 20 *respectively; the latter system contains the former one.*

There is consequently a *single condition*, expressing that a double-four $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ in ordinary space is of the second kind, *i.e.* lies on a cubic surface. The condition can be stated as an *incidence condition* as follows. There are two lines, say b', b'', meeting $a_1a_2a_3a_4$, and two lines, say a', a'', meeting $b_1b_2b_3b_4$. A cubic surface F containing d must obviously also contain the lines a', a'', b', b'', and the two sextuplets of lines a and b constitute a double-six on it.† Hence *the line a' meets either b' or b'', and the line a'' then meets b'' or b' respectively.* Conversely, it is easily seen that if one of these two incidence conditions holds, then d lies on a cubic surface, and the other condition also holds.

It is worth noticing that the lines a', a'', b', b'' are not linearily deducible from the double-four; hence the above criterion for d to lie on a cubic surface does not give a Pascal's theorem, in the sense of the Introduction. We shall obtain a theorem having the requisite property, at the end of next paragraph. Now we remark that:

A double-four of lines in ordinary space cannot lie on two different cubic surfaces.

For, otherwise, the two cubic surfaces would have 12 lines in common, and so they would have a common component. Now this is impossible, since it is easily seen that no double-four of lines lies on any reducible cubic surface.

7. A Pascal's theorem for double-fours in ordinary space. Let us consider in [4] a double-four of the first kind, $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$, a general point O and a general prime ω. We denote by α_i, β_i the planes projecting a_i, b_i from O, and by A_i, B_i, a_i', b_i' the intersections of ω with a_i, b_i, α_i, β_i respectively ($i=1, 2, 3, 4$). Then $d'=(a_1'a_2'a_3'a_4',\ b_1'b_2'b_3'b_4')$ is a double-four of the second kind lying in ω (§4), and $T=(A_1A_2A_3A_4,\ B_1B_2B_3B_4)$ is a pair of incident tetrahedra inscribed both in d and in d' (§5). We now prove the following theorem:

The double-four d' is the projection from O upon ω of an infinity of double-fours of the first kind lying in [4]; *these double-fours constitute an ∞^5 irreducible system, and can all be obtained by transforming d with the homologies of* [4] *of center O. There is an infinity of pairs of incident tetrahedra inscribed in d'; they constitute an ∞^4 irreducible system, and can all be obtained as projections from O upon ω of the prime sections of d.*

It is clear from §3 that the transforms of d by means of the ∞^5 homologies of [4] of center O constitute an ∞^5 irreducible system of double-fours of the first kind, each of which has d' as projection from O upon ω. Conversely, in order to determine in [4] a double-four $d^*=(a_1^*a_2^*a_3^*a_4^*,\ b_1^*b_2^*b_3^*b_4^*)$ of the first kind, having d' as projection from O upon ω, we can first of all choose b_4^* in a general posi-

† *Cf. e.g.* B. Segre, *op. cit.*, p. 7 and Ch. II.

tion on β_4. Then the lines a_1^*, a_2^*, a_3^* must lie in the planes α_1, α_2, α_3, and belong to the pencils having the points where b_4^* meets α_1, α_2, α_3 respectively as centers. We can choose a_1^*, a_2^*, a_3^* generally in such pencils, and then we have at most a single possibility for the remaining lines b_1^*, b_2^*, b_3^*, a_4^* of d^*, since these must contain the sets of points

$$(a_2^*\beta_1,\ a_3^*\beta_1),\qquad (a_1^*\beta_2,\ a_3^*\beta_2),\qquad (a_1^*\beta_3,\ a_2^*\beta_3),\qquad (b_1^*\alpha_4,\ b_2^*\alpha_4,\ b_3^*\alpha_4)$$

respectively. It follows that d^* can be uniquely defined as transform of d by means of the homology of center O, having b_4 and b_4^* as corresponding lines, and whose fundamental space is the join of the points $a_1a_1^*$, $a_2a_2^*$, $a_3a_3^*$, $b_4b_4^*$.

From §5, a general prime section of d gives a pair of incident tetrahedra inscribed in d; hence the projection of such a pair from O upon ω is a pair of incident tetrahedra inscribed in d'. Conversely, let $T^*=(A_1^*A_2^*A_3^*A_4^*,\ B_1^*B_2^*B_3^*B_4^*)$ be a pair of incident tetrahedra inscribed in d'. We shall shortly see that T^* is the section of ω with a double-four of the first kind, say d^*, having d' as projection from O upon ω. Hence d^* is transformed into d by a homology of center O. This homology transforms ω into a prime ω^*, and it is clear that T^* is the projection from O upon ω of the section of d with ω^*.

The result employed during the previous deduction is included in the following:

Let $d'=(a_1'a_2'a_3'a_4',\ b_1'b_2'b_3'b_4')$ be any double-four lying in an ordinary space ω, and $T=(A_1A_2A_3A_4,\ B_1B_2B_3B_4)$ be a pair of incident tetrahedra inscribed in d'. Then, on considering a space [4] *containing ω, and a point O lying in* [4] *but not in ω, it is possible to construct in* [4] *a double-four of the first kind having d' as projection from O upon ω, and intersecting ω in T.*

We denote by α_i, β_i the planes projecting a_i', b_i' from O $(i=1, 2, 3, 4)$, and by $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ the required double-four. We can choose b_4 generally in β_4 through the point B_4. Then we define the lines a_1, a_2, a_3 as joins of the points $(A_1,\ \alpha_1b_4)$, $(A_2,\ \alpha_2b_4)$, $(A_3,\ \alpha_3b_4)$ respectively, the lines b_1, b_2, b_3 as joins of the points $(a_2\beta_1,\ a_3\beta_1)$, $(a_1\beta_2,\ a_3\beta_2)$, $(a_1\beta_3,\ a_2\beta_3)$ respectively, and the line a_4 as the line of [4] incident to b_1, b_2, b_3. Hence $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ is a double-four of the first kind, having $d^*=(a_1'a_2'a_3'a_4^*,\ b_1'b_2'b_3'b_4')$ as projection from O upon ω, and $T^*=(A_1A_2A_3A_4^*,\ B_1^*B_2^*B_3^*B_4)$ as section with ω, where a_4^* is a line of ω, and B_1^*, B_2^*, B_3^*, A_4^* are points of ω, the three first of which lie on b_1', b_2', b_3' respectively. We need only to prove that B_1^*, B_2^*, B_3^*, A_4^*, a_4^* coincide with B_1, B_2, B_3, A_4, a_4' respectively.

From the incidence conditions for T and for T^*, we see that both B_1B_4 and $B_1^*B_4$ are lines through B_4 meeting A_2A_3; moreover, both these lines meet b_1', at B_1, B_1^* respectively. Since A_2A_3, b_1' are two skew lines, and B_4 lies on neither of them, it follows that $B_1=B_1^*$. Likewise we see that $B_2=B_2^*$, $B_3=B_3^*$. Hence seven of the eight vertices of the pair of incident tetrahedra T are also vertices of T^*, so that (from §5) also the remaining vertices of T, T^* coincide, *i.e.* $A_4=A_4^*$. Finally, from the incidence conditions for d', d^* it follows that a_4' and a_4^* coincide, since both these lines contain the point $A_4=A_4^*$ and are incident with b_1', b_2', b_3'.

Let us consider again in [4] a prime ω and a point O not lying on ω. We know that the double-fours of the first kind of [4] form an ∞^{24} irreducible system (§3). In general, the projection of one of these double-fours from O upon ω is a double-four of the second kind (§4), and can be obtained by such a projection from exactly ∞^5 double-fours of the first kind lying in [4]. Hence the system formed by these projections has the dimension $24-5=19$, and therefore coincides with the ∞^{19} irreducible system formed by *all* the double-fours of the second kind lying in ω (§6). This proves the *converse* of the final result of §4. We see, moreover, that:

A double-four of lines lying in ordinary space is of the second kind (i.e. it lies on a cubic surface) if, and only if, there is a pair of incident tetrahedra inscribed in it. There are then ∞^4 such pairs of tetrahedra, and we determine uniquely one of them, by choosing four of their eight vertices in general positions on any four lines of the double-four.

This result gives a *linear test* for ascertaining whether a double-four of lines in ordinary space lies on a cubic surface. Hence, according to the Introduction, it can be considered as a *generalized Pascal's theorem.*

PART II. QUINTUPLETS OF RELATED LINES IN [4]

8. Pascal's theorem for five related lines in [4]. There are ∞^{14} three-dimensional quadrics in [4], and three conditions are required for a quadric to contain a given line. Hence five lines in [4] do not generally lie on a quadric, and *a single condition* is required for the existence of a quadric containing them. We shall say that *five lines lying on a quadric in [4] are related*, and prove the following *extension of Pascal's theorem.*

*Five lines in [4] are related if, and only if, on considering them in an arbitrary cyclic arrangement, $a_1a_2a_3a_4a_5$ say, and intersecting each line a_i with the prime $a_{i-1}\,a_{i+1}(i=1, 2, 3, 4, 5, a_0=a_5, a_6=a_1)$, we obtain five points of a prime.**

We shall prove this theorem in §10, by means of algebraic-geometric considerations, after having investigated (in §9) its analytical substratum.† A simple, purely geometric proof of the *necessity* for the stated condition will be indicated in §14.

From §4 we see immediately that:

Five lines $a_1a_2a_3a_4a_5$ in [4] are related if, and only if, the pairs of primes a_1a_2 and a_3a_4, a_1a_3 and a_4a_2, a_1a_4 and a_2a_3 cut out on a_5 three pairs of points in involution. This property also holds if we interchange the line a_5 with any of the remaining lines.

9. An identity between two determinants. Let us introduce in [4] homogeneous projective coordinates, and define the line a_i as the join of two distinct points

* We could substitute the prime $a_{i-1}\,a_{i+1}$ in this statement with $a_{i-2}\,a_{i+2}$ (assuming $a_{-1}=a_4$, $a_7=a_2$), since this would simply be tantamount to applying the stated result on considering the five lines in the order $a_1a_3a_5a_2a_4$.

† A proof on the same line can also be given for Pascal's theorem for conics.

$$(1)\qquad P_i(x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}), \qquad Q_i(y_{i1}, y_{i2}, y_{i3}, y_{i4}, y_{i5}) \qquad (i = 1, 2, 3, 4, 5).$$

The general quadric in [4] has the equation

$$(2)\qquad \begin{aligned} &k_{11}x_1^2 + k_{22}x_2^2 + k_{33}x_3^2 + k_{44}x_4^2 + k_{55}x_5^2 \\ &\quad + 2k_{52}x_5x_2 + 2k_{13}x_1x_3 + 2k_{24}x_2x_4 + 2k_{35}x_3x_5 + 2k_{41}x_4x_1 \\ &\quad + 2k_{12}x_1x_2 + 2k_{23}x_2x_3 + 2k_{34}x_3x_4 + 2k_{45}x_4x_5 + 2k_{51}x_5x_1 = 0, \end{aligned}$$

where the k's are 15 parameters not all zero; and we express that it contains the line a_i, by writing the conditions that each of the points (1) lies on (2), and that P_i and Q_i are conjugate with respect to this quadric ($i=1, 2, 3, 4, 5$). This gives altogether 15 linear equations for the k's, whose compatibility condition is

$$(3)\qquad D = \begin{vmatrix} x_{11}^2 & \cdots & 2x_{15}x_{12} & \cdots & 2x_{11}x_{12} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_{11}^2 & \cdots & 2y_{15}y_{12} & \cdots & 2y_{11}y_{12} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{11}y_{11} & \cdots & x_{15}y_{12} + x_{12}y_{15} & \cdots & x_{11}y_{12} + x_{12}y_{11} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

In the determinant D we have written explicitly only three rows, arising as indicated from the conditions for the points (1) for $i=1$; the other rows arise likewise in succession for $i=2, 3, 4, 5$. Moreover, the arrangement of the columns in D can be taken to be the one corresponding to that of the terms in (2). The condition (3) is of the third degree in each of the ten sets of variables (1), and gives *the necessary and sufficient condition for the lines $a_1a_2a_3a_4a_5$ to be related.* It is geometrically obvious that the determinant D must be expressible as a cubic form in the Plücker coordinates of these five lines.

We consider next the point

$$(4)\qquad R_i(z_{i1}, z_{i2}, z_{i3}, z_{i4}, z_{i5})$$

intersected on a_i by the prime $a_{i-1}\, a_{i+1}$ ($i=1, 2, 3, 4, 5$). It is easily seen that R_i has the coordinates

$$(5)'\quad z_{ih} = \begin{vmatrix} x_{i1}y_{ih} - x_{ih}y_{i1} & x_{i2}y_{ih} - x_{ih}y_{i2} & x_{i3}y_{ih} - x_{ih}y_{i3} & x_{i4}y_{ih} - x_{ih}y_{i4} & x_{i5}y_{ih} - x_{ih}y_{i5} \\ x_{i-1,1} & x_{i-1,2} & x_{i-1,3} & x_{i-1,4} & x_{i-1,5} \\ x_{i+1,1} & x_{i+1,2} & x_{i+1,3} & x_{i+1,4} & x_{i+1,5} \\ y_{i-1,1} & y_{i-1,2} & y_{i-1,3} & y_{i-1,4} & y_{i-1,5} \\ y_{i+1,1} & y_{i+1,2} & y_{i+1,3} & y_{i+1,4} & y_{i+1,5} \end{vmatrix},$$

which could also be expressed as trilinear functions of the Plücker coordinates of a_{i-1}, a_i, a_{i+1}. The condition for the five points (4) to be in a prime is

$$\Delta = \begin{vmatrix} z_{11} & z_{12} \cdots z_{15} \\ z_{21} & z_{22} \cdots z_{25} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ z_{51} & z_{52} \cdots z_{55} \end{vmatrix} = 0, \tag{6}$$

and is clearly of the third degree in each of the ten sets of variables (1).

In §10 we shall establish geometrically the *remarkable identity*

$$D = 32\,\Delta. \tag{7}$$

This implies that the conditions (3), (6) are equivalent, whence the first theorem of §8 follows immediately.

A purely algebraic proof of (7) seems not to be easy. Hence, in the present paragraph, we confine ourselves to verifying (7) directly for the following numerical values of the variables (1). We assume all these variables to be zero, excepting

$$x_{11} = x_{22} = x_{33} = x_{44} = x_{55} = 1,$$

and

$$y_{15} = y_{12} = y_{21} = y_{23} = y_{32} = y_{34} = y_{43} = y_{45} = y_{54} = y_{51} = 1.$$

Then in the determinant D all the elements of the first five rows are zero, excepting those of the principal diagonal, which are 1; moreover, all the elements of the successive five rows which are not in the first five columns are zero, excepting those of the principal diagonal, which are 2. Hence

$$D = 2^5 \begin{vmatrix} x_{11}y_{12}+x_{12}y_{11} & x_{12}y_{13}+x_{13}y_{12} & x_{13}y_{14}+x_{14}y_{13} & x_{14}y_{15}+x_{15}y_{14} & x_{15}y_{11}+x_{11}y_{15} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{51}y_{52}+x_{52}y_{51} & x_{52}y_{53}+x_{53}y_{52} & x_{53}y_{54}+x_{54}y_{53} & x_{54}y_{55}+x_{55}y_{54} & x_{55}y_{51}+x_{51}y_{55} \end{vmatrix}$$

$$= 2^5 \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = 2^5 \cdot 2.$$

From (5), (6) we deduce at present without difficulty

$$\Delta = \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{vmatrix} = 2.$$

The last two equations prove (7) for the indicated numerical values of the variables (1), and show incidentally that *neither* (3) *nor* (6) *holds identically*.

10. Proof of the previously stated results. From §9 we see that, in order to establish (7), we need only to prove that the ratio D/Δ is a constant, *i.e.* is independent of each of the ten sets of variables (1). It is obviously sufficient to demonstrate the independence of D/Δ from one of these sets, *e.g.* from

$$x_{51},\ x_{52},\ x_{53},\ x_{54},\ x_{55}, \tag{8}$$

since the same argument can be applied for the nine remaining sets. This will clearly follow if we prove that the equations (3), (6) represent *the same algebraic variety*, when we interpret the variables (8) as coordinates of a point P_5 variable in [4] and the remaining variables as parameters.

First of all, on fixing generically the sets (1) for $i=1, 2, 3, 4$ we fix the quadruplet of lines $a_1a_2a_3a_4$ in a general position, and so we can consider the complementary quadruplet, $b_1b_2b_3b_4$ say, defined as in §3. Then (3) is a cubic equation for the Plücker coordinates of the line $a_5=O_5P_5$, and represents the cubic complex of lines, K say, defined by the double-four $d=(a_1a_2a_3a_4,\ b_1b_2b_3b_4)$ as in §4. On fixing in (3) also the coordinates $(y_{51}, y_{52}, y_{53}, y_{54}, y_{55})$ of the point O_5, and considering (8) there as coordinates of the variable point P_5, we clearly obtain the equation of the cone of the complex K having O_5 as vertex. As already noticed in §4, this is a cubic cone containing the eight planes projecting d from O_5.

Secondly we remark that, from §9, the equation (6) represents likewise a cubic cone of vertex O_5; and we have to prove that this cone coincides with the previous one. This follows by projection from the final result of §6, if we show that also the cone represented by (6) contains the eight planes projecting d from O_5. We therefore need only to prove that (6) holds, *i.e.* that the five points (4) are in a prime, if a_5 meets one or other of the eight lines of d. We do so now, on distinguishing three cases.

(i) If a_5 meets a_1, then both R_5 and R_1 coincide with the point a_1a_5, so that (6) obviously holds. A similar argument can be applied if a_5 meets a_4.

(ii) If a_5 meets a_2, the prime a_5a_2, and therefore also R_1, is indeterminate. Then (5) shows in fact that $z_{11}=z_{12}=z_{13}=z_{14}=z_{15}=0$, and so (6) holds. A similar argument can be applied if a_5 meets a_3.

(iii) If a_5 meets b_1, the five points (4) lie on the prime a_5a_2. This is obvious for R_5, R_2 which lie on a_5, a_2 respectively, as well as for R_1 which is the intersection of a_1 with this prime. As for R_3, R_4, it is easily seen that these points coincide now with the intersections of b_1 with a_3, a_4 respectively; hence they lie on the prime a_5a_2, since this prime contains the line b_1, which meets both a_5 and a_2. A similar argument shows that, if a_5 meets either b_2, or b_3, or b_4, then the five points (4) lie on the prime a_1a_3, or a_2a_4, or a_3a_5 respectively.

11. On some questions suggested by §3 and §8. The final result of §8 gives a *linear test* for five related lines in [4]. This can be taken as the *definition* of related lines, without the need of mentioning the existence of a quadric contain-

ing the five lines. Another *linear test* is given by the first theorem of §8, which we can restate in a slightly different form as follows:

Let us consider in [4] *a set of five lines arranged cyclically, and suppose that the five points of intersection of each line with the join of its two adjacent lines are in a prime. Then, and only then, the five lines are related, and the same property holds for every cyclical arrangement of the five lines.*

Five related lines thus define an interesting configuration,* which, however, we do not intend to study in detail in this paper. We only remark that the 12 possible cyclical arrangements of the five lines give 12 primes in [4]. The points of intersection of these primes with the five lines are only 30 in number, since each such point lies in two distinct primes. In fact each of the five lines contains 6 of the 30 points, given by its intersections with the primes joining in pairs the remaining four lines, and consisting of three pairs of points in involution (§8). Hence *the configuration defined in* [4] *by five related lines has the characters* (30_2, 12_5).

By duality in [4] we deduce immediately properties of *five related planes*, which for the sake of brevity will not be stated explicitly. Each quintuplet of related lines is transformed into a quintuplet of related planes, by the polarity with respect to the quadric containing its five lines. The relation between two such quintuplets is a reciprocal one, and deserves further investigation.

We now point out a few questions of a different type, also suggested by the results of §3 and §8. We may define the *associated quintuplet* of an arbitrary quintuplet of lines in [4], as the one formed by the lines associated to the five quadruplets of lines contained in the given quintuplet (§2). The new quintuplet of lines has in its turn an associated one, and so on. Thus *sequences of quintuplets* are obtained; and one could investigate those which contain *one or more quintuplets of related lines*, or those (if any) which are *periodic.*

Another interesting problem is that of studying the *sets of* $n>5$ *lines in* [4] *which are five by five related.* A trivial example is offered by n lines in [4] lying on the same quadric.

PART III. SOME APPLICATIONS OF PLANE AND THREE-DIMENSIONAL GEOMETRY

12. On quintuplets of lines in space. We first prove the following theorem:

Let $a_1a_2a_3a_4a_5$ *be five general lines in ordinary space, meeting a given plane at* $A_1A_2A_3'A_4A_5$ *respectively, and denote by* Γ *the conic containing these five points. We construct the point* B_i *in which* a_i *intersects the quadric containing* Γ, a_{i-1}, a_{i+1}, *residually to* $A_i(i=1, 2, 3, 4, 5, a_0=a_5, a_6=a_1)$. *Then the five points* $B_1B_2B_3B_4B_5$ *lie on a quadric passing through* Γ.

* The analogue of the well known configuration of the Hexagrammum Mysticum, defined on a plane by six points of a conic.

The ∞^4 linear system formed by the ordinary quadrics through Γ can be considered, in the well known way, as representative of the prime sections of a quadric Q in [4]. Then $a_1a_2a_3a_4a_5$ represent five lines of [4] lying on Q, and the result stated above is an immediate consequence of the first theorem of §8. Other properties of five lines and a plane in ordinary space, can likewise be deduced from §11.

13. On quintuplets of related circles in space. The ∞^4 linear system formed by the spheres of an ordinary space (*i.e.*, by the ordinary quadrics containing the absolute conic), can be taken as representative of the prime sections of a quadric Q in [4]. It is well known that any two orthogonal spheres thus represent the sections of Q with two primes which are conjugate with respect to Q, and conversely. In this representation, the circles of the ordinary space correspond to the plane sections of Q, since each of the former is the base curve of a pencil of spheres, representing the sections of Q with the primes of a pencil. The sections of Q with a prime and a conjugate plane, are represented in ordinary space by a sphere and a circle orthogonal to it, and conversely; *etc., etc.*

We shall say that *five circles in ordinary space are related*, if the corresponding plane sections of Q lie in five related planes of [4]. This gives a single condition for the five circles. The same condition can obviously be stated otherwise, by saying that the polar lines with respect to Q of the five planes in [4] are related.

From §11 we easily deduce that:

Five circles in ordinary space are related if, and only if, either of the following two properties holds for the five circles taken in a given order, $C_1C_2C_3C_4C_5$ say; then both properties hold for every arrangement of the five circles. (i) *On constructing the sphere through C_i orthogonal to the one orthogonal to C_{i-1} and C_{i+1} ($i=1, 2, 3, 4, 5$, $C_0=C_5$, $C_6=C_1$) we obtain five spheres, and these are orthogonal to another sphere.* (ii) *The centers of the spheres through C_5 orthogonal to the spheres orthogonal to the pairs of circles C_1C_2 and C_3C_4, C_1C_3 and C_4C_2, C_1C_4 and C_2C_3, are three collinear pairs of points in involution.*

14. On quintuplets of points and their polar planes with respect to a null-system. It is well known that the lines of [3] can be represented by the points of a four-dimensional quadric Q^* in [5], in such a way that the lines of a linear complex L in [3] are represented in [5] by the points of the quadric Q section of Q^* with a prime.† Five arbitrary generators of Q are represented in [3] by five arbitrary pencils of lines of L: the centers $A_1A_2A_3A_4A_5$ of these pencils are five arbitrary points of [3], and their respective planes $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ are the polar planes of $A_1A_2A_3A_4A_5$ with respect to L. Finally a prime section of Q is represented in [3] by a congruence of lines of L, *i.e.*, by the totality of the lines of [3] incident with two lines, which are mutually polar with respect to L, and may possibly be infinitely near.

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† This prime does not touch or touches Q^*, according as L is a general or special linear complex.

From the results just recalled and §8, we easily deduce the following theorem:

Let $A_1A_2A_3A_4A_5$ be five general points in ordinary space, and $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ their polar planes with respect to a given (general or special) linear complex of lines. Then the five pairs of planes α_i and $A_{i-1}A_iA_{i+1}$ ($i=1, 2, 3, 4, 5, A_0=A_5, A_6=A_1$) intersect in five lines having two common transversals.

We obtain an alternative proof of this theorem, by noticing that the five lines indicated are both in the given linear complex, and in the linear complex defined by the conditions of having $A_{i-1}A_iA_{i+1}$ as polar plane of A_i ($i=1, 2, 3, 4, 5$). Hence they are in the linear congruence formed by the common lines of the two complexes.

Conversely, from this proof and the previous hyperspatial interpretation, it is easy to deduce a new proof of the necessity for the condition stated in the first theorem of §8.

15. On quintuplets of oriented elements on a plane. It is well known that S. Lie gave a useful representation between the elements of a plane π and the points of an ordinary space.* This representation was then slightly modified by E. Kasner,† into a birational transformation between the oriented elements of π and the points in space. In this transformation, the oriented circles of π correspond to the lines of a general linear complex L in space; hence two oriented elements of π are co-circular if, and only if, their representative points are conjugate with respect to L. A general line in space corresponds to a *turbine* of π, *i.e.*, to ∞^1 oriented elements whose points form a circle and whose directions are equally inclined to that circle; and conversely. Two lines in space mutually polar with respect to L, lead to two turbines having the same circle as point locus, and whose elements are symmetrically related to the elements of the circle.

Let us now apply the theorem of §14 to the linear complex L. We may consider the line of intersection of α_i and $A_{i-1}A_iA_{i+1}$ as the join of A_i with the point of intersection of α_i and $A_{i-1}A_{i+1}$. From the results recalled above, we then deduce the following theorem.

If five oriented elements $e_1e_2e_3e_4e_5$ of a plane are generally given, we can consider the oriented circle containing e_i and having an element in common with the turbine defined by e_{i-1} and e_{i+1}($i=1, 2, 3, 4, 5, e_0=e_5, e_6=e_1$). The five circles thus obtained are isogonal to another circle.

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* Geometrie der Berüngstransformationen, I, p. 238.

† The group of turns and slides and the geometry of turbines, Amer. Journ. of Math., 33, 1911, pp. 193–202.

MATHEMATICS IN A LIBERAL EDUCATION*

E. P. NORTHROP, University of Chicago

1. Introduction. Considerations of the place of mathematics in liberal education have been somewhat neglected during the past three years. The neglect was brought about, of course, by the demand that many men acquire certain mathematical skills in the shortest possible time. Now although the war has brought mathematics before the public eye, it has focussed that eye only on the more immediately useful mathematical skills—skills almost exclusively manipulative in nature. The public may well tend to lose sight of an entirely different kind of skill in mathematics. This skill, which has values in liberal education, is one that the public had already begun to challenge two decades before the war.

2. Some shortcomings and misconceptions. As I reflect on what I have read in the past few years about the place of mathematics in liberal education, and about mathematics curricula designed for such education, I am impressed by several things. One of these is that it is difficult to discover a definition of "liberal education" in many of the papers which use the term. Possibly the writers regard the concept as too simple, or too general, to admit of definition. A number of them imply that for them a liberal education is one which enables the student to solve certain specific problems he will presumably meet later in his vocational or social life. This sort of thing is not education, but a kind of training which may have a place in a school designed to train, but which has little place in a school devoted to the liberal arts.

Then there are writers who discuss mathematics as a discipline, and so come somewhat closer to the point. Many of them are concerned with the problem of "transfer." Here I am impressed by the number of writers who, unable to find any significant transfer of mathematical methods to non-mathematical situations, tend to overlook, among other things, the kind of mathematical instruction received by the students they are trying to test. In this connection they disregard not only the content of such instruction, but its method of presentation as well. I am also impressed by the fact that only a very small number of writers are bold enough to question whether or not the objectives of mathematics in liberal education have yet been clearly stated; or, if stated, whether or not they have yet been measured by means of tests and examinations. Of this small group of critics, some believe that significant tests will eventually be found, but have not been found to date. A very few suspect that significant tests may never be found.

Leaving the question of tests to the examiners, I should like to discuss briefly the general objectives of liberal education, to suggest an appropriate place for mathematics in such an education, and to describe a course designed to fill that place.

* Address presented at the annual meeting of the Mathematical Association of America, Chicago, Illinois, November 26, 1944.

3. Liberal education and mathematics. A liberal education, properly speaking, is one which liberates the student's mind. It does so by providing him with intellectual disciplines of various kinds. He must be taught not only to read and write, but to analyze and interpret what he has read, to look for premises and conclusions of arguments, to recognize them when he has found them, and to discover the presuppositions which lead to the particular choice of the premises used. He must become acquainted not only with a part of humanity's store of knowledge, but with the various methods by which knowledge of different kinds is gained, with methods by which premises are formulated, with methods by which premises lead to conclusions, and with methods by which conclusions are validated. No single discipline can do all of these things. Nor is it correct to assume that there exists a one-to-one correspondence between fields of knowledge and disciplines appropriate to them. There is a variety of disciplines, some of which are appropriate to one field and some to another field, with much overlapping among them.

Now some of these disciplines involve the use of ordinary numerical reckoning. The value of mathematics in this very narrow sense is almost too obvious to mention. But consider mathematics as a discipline in itself—that is to say, as a body of concepts and methods which constitute a way of thinking. Surely mathematics is such a discipline. It deals almost exclusively with premises and conclusions, and with deductive reasoning, which is one of the more important methods of drawing conclusions from premises. Moreover, clarity and precision of definitions and assumptions, and rigor in reasoning, can be more nearly attained and more simply studied in mathematics than in the other disciplines. Is not this the real place of mathematics in a liberal education—not simply as a subject matter, or as a discipline applicable only to its own subject matter, but as a discipline which is applicable to almost every intellectual activity of man?

Mathematics was given without question the place it deserves in liberal education until the last two or three decades. In 1928 Florian Cajori published a compilation of the opinions of all writers known to him, an historian of mathematics, who had expressed themselves on the relation of mathematics to liberal education.* The list includes the names of over seven hundred men, ranging in time from the Greek period to the twentieth century, and ranging in professions from philosophers, mathematicians, and scientists to literary men, statesmen, business men, and lawyers. Over four-fifths of these men placed a high value on mathematics as a part of liberal education. Although this accumulation of opinion may not constitute public opinion, it is certainly representative of it to some degree. At any rate, it is doubtful whether we shall have public opinion so strongly on our side in the years which lie immediately ahead. Returned veterans, for example, having been subjected to accelerated courses in mathematics, often at the hands of inexperienced teachers, will know little or nothing of the value of mathematics as an appropriate study for all students, but will regard

* Mathematics in Liberal Education, Boston, Christopher, 1928.

mathematics as a study relevant only to vocations in which they are no longer engaged. Perhaps our best argument will lie in mathematics courses truly—and thoughtfully—designed for all students, regardless of the occupation or profession they may expect to enter.

4. **A course designed for liberal education.** Since the autumn of 1943 such a course has been offered in the College of the University of Chicago. It should perhaps be pointed out that Chicago differs from most colleges in that students are accepted after they have completed two years of high school. Many of you, however, are also faced with entering students who have had no more than two years of high school mathematics, and what I have to say here applies equally well to them. Obvious modifications of the program for students who enter college with more than two years of mathematics will no doubt occur to you.

Our general course, then, is a one-year course (two semesters, or three quarters), meeting five times per week, and presupposing a knowledge of elementary algebra and plane geometry such as is ordinarily acquired in one-year high school courses in these subjects. Note the expression, "such as is ordinarily acquired . . . ." Rather than bring up the time-worn question of pre-college training in mathematics, we might as well resign ourselves to accepting students who have been exposed to a year of algebra and a year of geometry. Actually, much can be done with such students even though they may have failed to catch that to which they were exposed.

The work of the course falls into four parts: (1) logical structure, (2) geometry, (3) algebra, and (4) coördinate geometry. Of these four parts, the first two together, the third, and the fourth, each occupy about one third of the time devoted to the course.*

Part 1 (logical structure) consists of a general consideration of the role played by definitions, assumptions, and methods of reasoning in fields of thought in general, and in mathematical and scientific fields of thought in particular. Here the student is made aware of the necessity for undefined terms and assumptions. He begins to learn something about the importance of clear and precise formulations of definitions, assumptions, and other propositions. He acquires an elementary understanding of the notions of consistency and independence of postulates. He suddenly realizes that in order to be able to demonstrate anything, he must accept some fundamental logical structure by which to reason. He begins to see what a proof *is*, and how the postulates of contradiction and excluded middle enter into indirect proof. And in his study of relations between propositions, he is led to understand the significance of converses, inverses, contrapositives, contradictories, and necessary and sufficient conditions. Needless to say, the ideas developed here constitute the fundamental framework of the entire course.

Part 2 (geometry) consists of a brief critical examination of Euclid's definitions and postulates, a brief review of plane geometry, and an extension of plane

* The text for the course, prepared by the author of the present article, is Fundamental Mathematics, Chicago, University of Chicago Bookstore, 1944–45.

geometry to certain portions of euclidean space geometry. The postulates used in the review of plane geometry are postulates formulated by the late G. D. Birkhoff.* Based on scale and protractor, they are different from, but equivalent to, the ordinary euclidean postulates. They serve to give the student practice in reasoning from a set of postulates with which most students are unfamiliar. They make it possible to develop, in some ten or twelve theorems, nearly everything of importance to the course concerning rectilinear figures in the plane (Pythagorean theorem, theorems on similarity, angle sums, parallelism and perpendicularity, *etc.*). And resting ultimately upon the properties of real numbers, they lead naturally to the work in algebra and lay the foundation for the work in coördinate geometry. The discussion of euclidean space geometry is designed to show the student how an additional assumption or two is sufficient for the purposes of extending geometry from two dimensions to three, and to give him some understanding of the relations of lines and planes in space.

Part 3 (algebra) consists of a postulational development of certain portions of algebra, including elementary algebra and parts of "intermediate" and "college" algebra. This approach not only serves to bring order into what the student looks upon as a vast collection of dissociated facts and techniques, but offers an opportunity for badly needed remedial work in connection with those techniques. The work here centers around a relatively rigorous development of number systems, which in turn leads to a relatively thorough investigation of variables and functions.

Starting with the natural numbers, these numbers and addition are accepted as undefined concepts. Multiplication is defined in terms of addition, and the ordinary closure, commutative, associative, and distributive postulates are formulated. Subtraction is introduced as the inverse of addition, and the natural numbers extended to the integers through postulation of identity and inverse elements for addition. Division is introduced as the inverse of multiplication, and the integers extended to the rational numbers through postulation of an inverse element for multiplication. Evolution, or root extraction, is introduced as the inverse of involution, or raising to a power, and the necessity for further generalizations of the number system is made clear. Here rigor tends to break down, for the obvious reason that this elementary course is not a course in real variables. Nevertheless, real numbers are introduced by means of the Dedekind cut, a concept which is by no means as difficult for the student to grasp as it is generally thought to be. This contention is particularly true in the case of the student whose geometry is founded on the notion of one-to-one correspondences between points of lines and members of classes of numbers. No attempt is made, however, to develop the real numbers in any complete sense. A return to rigor is accomplished in the extension of real numbers to complex numbers through

* "A set of postulates for plane geometry, based on scale and protractor," Annals of Mathematics, series 2, vol. 33, 1932, pp. 329–345.

postulation of an imaginary unit. Theorems on special products and factoring are introduced in connection with integers, theorems on order in connection with rational numbers, and theorems on exponents in connection with real numbers. The study of number systems closes with an investigation of the Peano postulates, by means of which the natural numbers and addition may be defined, and the original postulates for natural numbers proved. Particular emphasis is laid on Peano's fifth postulate—that of mathematical induction—as a new method of proof.

The second portion of part 3 opens with a discussion of the role played by functional relations in the sciences. Variables and functions are introduced with care, and a wide variety of functions is examined. The material also includes the study of functions of several variables, the construction of functional relations, the introduction of functional notation, and various classifications of functions. Investigation of zeros of functions leads to the general question of the solution of equations, which is treated in detail through polynomials of the second degree. A short time is also spent in the study of functions of positive integral variables: arithmetic and geometric series, and permutations and combinations, together with their applications to probability and the binomial theorem.

Part 4 (coördinate geometry) is the logical outgrowth of parts 2 and 3, the student being well prepared for this union of geometry and algebra. For in geometry he was introduced to the assumption of a one-to-one correspondence between the points of a line and the real numbers, and in the discussion of Dedekind cuts he was able to gain a clearer notion of the implications of such an assumption. Linear coördinates are thoroughly investigated before rectangular coördinates are introduced. Plane rectangular coördinate geometry is then developed in detail through the straight line and the circle, and the graphs of algebraic functions are studied in a general way. More than usual attention is paid to the complete correspondence between geometric concepts, relations, and operations on the one hand, and algebraic concepts, relations, and operations on the other. Considerations of the circle lead naturally to a discussion of the circular functions, their properties, and their applications. Thus the fundamentals of trigonometry are treated, not as a separate subject, but as an appropriate part of the general study of functions and functional behavior.

5. Emphasis on method of presentation. My point in describing this course designed for liberal education is not only to argue for the inclusion in such a course of the particular material cited, or at least for the inclusion of the four parts named, but to argue for the particular method of presentation used. Indeed, the pedagogy is to be stressed far more than the content.

For example, the work of the first part of the course—logical structure—does not end when the second part is begun, but continues on, transformed from something *thought about* to something *used in thinking.*

Again, throughout the course emphasis is laid on the cultivation of an acute critical attitude on the part of the student. He is held to a level of rigor higher

than that which most students of his age are credited with the ability to appreciate. He is not often faced with the lame phrase, "it can be shown that," and when he is faced with it, an attempt is made to show him why.

Emphasis is also laid on the cultivation of skill in fundamental manipulative techniques. This emphasis is not only consistent with, but actually necessary to, the objectives of the course. Full appreciation of mathematics is impossible without such skill, and it is evidently of help to the student in his general courses in the sciences. On the other hand, this emphasis is confined to the more fundamental techniques, and is not allowed to run away with the course.

The students' attention is directed to relevant applications of his work in the other sciences, but no attempt is made to force such references. So-called practical applications of mathematics are generally included in courses for one of two purposes: to train the student for particular vocational work, or to motivate his interest. The first of these purposes is not one of the purposes of a liberal education, and the second is unnecessary in the course described above if it is properly presented.

It should be added that this is actually the first of two general courses in mathematics offered by the College of the University of Chicago, and that the College faculty has voted to make it a required part of their program of liberal education. A second course—an optional one, designed for students who wish to continue their work in mathematics—is in the process of construction. This second course is of necessity more technical than the first, in that it aims to prepare the student for more specialized work in mathematics and science. On the other hand, the method of presentation used in the first course is carried over into the second.

6. Conclusion. The first course, which has been described in detail, is a proposed course in mathematics for purposes of liberal education. I believe that such a course, properly taught, is more appropriate than most conventional courses for the student who plans to study mathematics only one year beyond algebra and geometry. In this I think many of you may agree with me. I wonder how many of you would also agree with me if I were to make the far more radical suggestion that such a course, properly taught, is also more appropriate than most conventional courses at the same level for the student who plans to continue his work in mathematics or science. My belief that this is the case is based on the following grounds: that an order of learning in which first emphasis is placed on the mastery of a conception of mathematical systems and of skill in following their development makes mastery of subsequently studied systems more rapid and more complete. Whereas to place first emphasis on the mastery of a large body of mathematical formulae and of skill in their manipulation makes mastery of subsequent formulae and manipulation almost as difficult as mastery of the first, and makes more and more difficult any later attempts to master and appreciate mathematical systems as such.

THE VALUATION OF RISKS

W. J. MAYS, The Volunteer State Life Insurance Company

1. Introduction. The purpose of this paper is to develop some of the ideas set forth by Laplace in the chapters of his "Theorie Analytique des Probabilités" which treat upon *l'esperance moral*, or the moral expectation that may be assigned to a gain or loss contingent upon the outcome of a doubtful event. These ideas are summarized in the following quotation:*

"The probability of events serves to determine the hopes and fears of persons interested in the outcomes of the events. The word 'expectation' has various acceptations; it expresses generally the advantage to one who expects some fortune merely on basis of a likely supposition. In the theory of risks, that advantage is the product of the sum expected by the probability of its being obtained; it is the partial sum which ought to be exchanged when one cannot run the risk of the outcome, on the supposition that the distribution of the whole sum is made in proportion to the probabilities. This is the only equitable means of distribution when every foreign circumstance is removed, for those who have equal degrees of probability have equal rights to the sum expected. We call this advantage mathematical expectation to distinguish it from moral expectation, which depends, as the former, upon the value expected and upon the probability of its realization, but which is governed by a thousand circumstances that are nearly always impossible to define, and even more so to subject to calculation. Although these circumstances increase or diminish the value expected, the moral expectation itself may be considered as the product of the value by the probability of obtaining it; but then a distinction must be made for the value expected, between its relative value and its absolute value; the latter is independent of the motives which make it desirable, whereas the former increases with such motives.

"No general rule can be given for appraising the relative value; however, it is natural to suppose the relative value of an infinitely small sum to be in direct proportion to its absolute value and in inverse proportion to the fortune of the interested person. Indeed, it is clear that a franc has very little value for one who has a large number of francs and that the most natural method of estimating its relative value is to suppose such value to be in inverse proportion to the number (of francs in the fortune)."

2. Bernoulli's Hypothesis. The last paragraph in the preceding quotation contains a statement of Bernoulli's Hypothesis. Symbolically, if dL represents the infinitesimal relative moral value of an infinitesimal gain of absolute value dA to a person whose fortune† is of absolute value A, Bernoulli's Hypothesis may be stated in the form

* Théorie Analytique des Probabilités, Book II, Chapter 2.

† The term "fortune" as used in this paper is intended to mean that portion of a person's means which is available for speculation, or otherwise properly subject to risk.

$$dL = k\frac{dA}{A} \tag{1}$$

where k is a constant of proportionality.

By integration of (1) we find the relative moral value L of a finite gain to be

$$L = k \log A + c \tag{2}$$

where c is a constant of integration and A is taken to include the gain. To find c, notice that if L is zero, $c = -k \log A_0$, where A_0 is the initial amount of the fortune, that is, the amount before inclusion of any gain. Equation (2) may now be written

$$L = k \log\left(1 + \frac{X}{A_0}\right) \tag{3}$$

where X is the absolute value of the gain.

Suppose that one possessed initially of a fortune A has a probability p_1, of realizing a sum P_1, a probability p_2 of realizing a sum P_2, $\cdots$, a probability p_n of realizing a sum P_n, the probabilities being mutually exclusive. Then a value X of his expectation is determined, on basis of Bernoulli's Hypothesis, from the equation

$$\log\left(1 + \frac{X}{A}\right) = p_1 \cdot \log\left(1 + \frac{P_1}{A}\right) + p_2 \cdot \log\left(1 + \frac{P_2}{A}\right) + \cdots + p_n \cdot \log\left(1 + \frac{P_n}{A}\right), \tag{4}$$

$$\text{or } \frac{X}{A} = \left(1 + \frac{P_1}{A}\right)^{p_1} \cdot \left(1 + \frac{P_2}{A}\right)^{p_2} \cdots \left(1 + \frac{P_n}{A}\right)^{p_n} - 1. \tag{5}$$

The value, X, of the expectation as determined from equation (4) or (5) has been called the physical value of the expectation to distinguish it from the relative moral value and from the mathematical value. The true significance of L, the relative moral value, is that it represents not particularly a value but rather a transformation function which, when assigned, serves to determine the physical value, X, of the expectation. Although the physical value is dependent upon the particular transformation function employed (the logarithmic function, for Bernoulli's Hypothesis), it is measured in the same scale as the absolute values of the fortune and the gains. All that has been said about gains applies similarly to losses provided appropriate changes in signs are made (that is, if P is the amount of a loss, represent the loss by $-P$). It is, of course, assumed that no loss is to be considered which would be in excess of the fortune.

Suppose now that one possessed of a fortune A engages in a play for which there is a probability p of winning a sum P and a probability q of losing a sum Q

such that $pP=qQ$ and $p+q=1$, or in other language a play that is fair in the mathematical sense. Then, on basis of Bernoulli's Hypothesis

$$1+\frac{X}{A}=\left(1+\frac{P}{A}\right)^p\cdot\left(1-\frac{Q}{A}\right)^q. \tag{6}$$

Now, it is easy* to show that the right-hand member of (6) is less than unity; hence, on basis of Bernoulli's Hypothesis, it is disadvantageous for one of limited means to play a "fair" game of chance.

There is an interesting interpretation of equation (6).

If one repeats the play a large number of times, say n times, he will win, on the average, pn times and will lose qn times. Now, if at each play the amounts at risk are so changed that they are always in proportion to his changing fortune, then each time he wins he will multiply his fortune by $(1+P/A)$ and each time he loses he will multiply his fortune by $(1-Q/A)$. The geometric mean factor for the change in fortune at each play is $1+X/A$ (X being negative), which is the constant factor by which he may certainly multiply his fortune n times to arrive at the same end result as if he played n times. The role of the logarithmic transformation is to transform constant values of the amounts at risk into values that are always proportional to the player's changing fortune.

3. A general hypothesis. The results obtained thus far have depended upon the choice of the logarithmic function for the moral value. Evidently, a considerably more general treatment of the subject of moral values is possible, if we use the following hypothesis:

The physical value, X, of gaining (or losing) P_1, with probability p_1, P_2, with probability $p_2, \cdots, P_n$ with probability p_n (the probabilities being mutually exclusive) for one whose fortune is initially A, is determined from the relation

$$f(A+X)=p_1f(A+P_1)+p_2f(A+P_2)+\cdots+p_nf(A+P_n), \tag{7}$$

where f is an arbitrary function except for the following restrictions which are imposed from practical considerations

(i) f is a continuous function of all the variables over their ranges.

(ii) For each pair of values of A, $A_1>A_2$, the corresponding physical values X_1 and X_2, determined from (7) for a fixed mathematical expectation of gain, must satisfy the inequality $X_1>X_2$, and for a fixed mathematical expectation of loss, must satisfy the inequality $X_1<X_2$.

(iii) As A is indefinitely increased, the function f must be such that equation (7) approaches a limiting form and the value of X determined therefrom must approach the mathematical expectation, that is,

$$\lim_{A\to\infty} X = p_1P_1+p_2P_2+\cdots+p_nP_n. \tag{8}$$

A general function that satisfies these restrictions is $(A+X)^m$, where m is

* The geometric mean of the quantities $(1+P/A)$ and $(1-Q/A)$ weighted by p and q, respectively, is less than the correspondingly weighted arithmetic mean, which is unity.

less than unity. If m is zero, the result is indeterminate; in consequence, to obtain continuity, let $\log (A+X)$ be used for the function corresponding to zero value of m.

For a concrete illustration of the hypothesis, consider a person whose fortune is \$10,000, of which \$5,000 is invested in a home. Suppose that the probability of the destruction of his home by fire in a given period of time is 1/100. The physical values of his expectation of loss are shown for several transformation functions in the following table:

Formula: $$f(10{,}000 - X) = \frac{1}{100}\cdot f(5{,}000) + \frac{99}{100}\cdot f(10{,}000)$$

$f(x)$	X (physical value of loss)
x	\$50·00 (mathematical expectation)
$x^{.8}$	53·23
$x^{.6}$	56·64
$x^{.4}$	60·45
$x^{.2}$	64·60
$\log x$	69·07
x^{-1}	99·01

It appears that, if moral valuation is admissible in accordance with the hypothesis, there is a margin between the mathematical expectation and the physical value in which an insuring company (whose resources are large in comparison with the fortune of the individual) may assign a premium that will be profitable to itself and at the same time advantageous to the insured. The hypothesis is thus in agreement with universal experience on this point.

4. The St. Petersburg problem. This classic problem might well be used to introduce the whole subject of moral values. It may be stated as follows:

A person is offered an expectation of winning \$1 if a coin turns up heads on the first throw, \$2 if it turns up heads on the second throw but not before, \$4 if it turns up heads on the third throw but not before, and so on, *ad infinitum*, the prize doubling for each throw. What price may he judiciously pay for this?

The following table shows the physical values of the expectation to one whose fortune is \$100 on the basis of several transformation functions.

Formula: $$f(100) = 1/2f(101 - X) + 1/4f(102 - X) + 1/8f(104 - X) + 1/16f(108 - X) + \cdots$$

Transformation Function $f(x)$	Physical Value of Expectation X
x	∞ (same as mathematical expectation)
$x^{1/2}$	\$5·32
$\log x$	4·35
x^{-1}	3·67
x^{-2}	3·35

5. Subdivision of risks. A question of some interest arises in connection with the subdivision of risks. As before let p be the probability of gaining (or losing) a sum P, with respect to a person whose fortune is initially A. We wish to inquire about the moral advantage of subdividing the single risk into many risks, such that the total mathematical expectation of the subdivided risks is the same as the mathematical expectation of the single risk. The subdivided risks are assumed not to be mutually exclusive.

We shall investigate two particular cases of this general problem.

Case I. Probabilities of subdivided risks:

$$\pi_1 = \pi_2 = \pi_3 = \cdots = \pi_n = p.$$

Amounts of subdivided risks:

$$\Pi_1 = \Pi_2 = \Pi_3 = \cdots = \Pi_n = \frac{P}{n}\cdot$$

If X represents the physical value of all expectations from the subdivided risks, we have

$$
\begin{aligned}
f(A + X) = {} & p^n\cdot f\left(A + \frac{nP}{n}\right) + np^{(n-1)}\cdot(1 - p)\cdot f\left(A + \frac{n-1}{n}P\right) \\
& + \frac{n(n-1)}{2}\cdot p^{(n-2)}\,(1-p)^2\cdot f\left(A + \frac{n-2}{n}P\right) + \cdots \\
& + \frac{n(n-1)}{2}\cdot p^2\cdot(1-p)^{(n-2)}\cdot f\left(A + \frac{2}{n}\cdot P\right) \\
& + np\cdot(1-p)^{(n-1)}\cdot f\left(A + \frac{P}{n}\right) + (1-p)^n\cdot f(A).
\end{aligned}
\tag{9}
$$

By suitably increasing n, we can concentrate the important terms of the series on the right into as narrow range (with reference to the order of magnitude of P) as we please about the term

$$\frac{n!}{(pn)!(n-pn)!}\cdot p^{pn}\cdot(1-p)^{n-pn}\cdot f\left(A + \frac{pn}{n}\cdot P\right).$$

Whence as n is indefinitely increased, the limit of the sum of the series becomes $f(A+pP)$ and the physical value, X, of the expectation from infinite subdivision is pP, which is the same as the mathematical expectation. This result calls to mind the adage about not putting all one's eggs in the same basket.

Case II. Probabilities of subdivided risks:

$$\pi_1 = \pi_2 = \pi_3 = \cdots = \pi_n = \frac{p}{n}\cdot$$

Amounts of subdivided risks:

$$\Pi_1 = \Pi_2 = \Pi_3 = \cdots = \Pi_n = P.$$

The equation from which the physical value of all expectations is determined is:

$$
\begin{aligned}
f(A+X) = {} & \left(1-\frac{p}{n}\right)^{n}\cdot f(A) + \frac{p}{n}\left(1-\frac{p}{n}\right)^{n-1}\cdot n\cdot f(A+P) \\
& + \left(\frac{p}{n}\right)^{2}\cdot\left(1-\frac{p}{n}\right)^{n-2}\cdot\frac{n(n-1)}{2}\cdot f(A+2P) + \cdots \\
& + \left(\frac{p}{n}\right)^{n-2}\cdot\left(1-\frac{p}{n}\right)^{2}\cdot\frac{n(n-1)}{2}\cdot f(A+\overline{n-2}\,P) \\
& + \left(\frac{p}{n}\right)^{n-1}\cdot\left(1-\frac{p}{n}\right)\cdot n\cdot f(A+\overline{n-1}\,P) + \left(\frac{p}{n}\right)^{n}\cdot f(A+nP).
\end{aligned}
\tag{10}
$$

As n is indefinitely increased, the equation becomes in the limit

$$
\begin{aligned}
f(A+X) = e^{-p}\Big\{ & f(A) + p\cdot f(A+P) + \frac{p^2}{2!}\cdot f(A+2P) + \cdots \\
& + \frac{p^k}{k!}\cdot f(A+kP) + \cdots \Big\}.
\end{aligned}
\tag{11}
$$

If $P=\frac{1}{2}A$, $p=1$, and f is the cube root of the argument, then X, as obtained from equation (11) is approximately $.449A$. In general, the moral value of the expectation of gain from this mode of subdivision is less than the corresponding mathematical expectation.

6. Moral disadvantage from possibility of exhaustion of funds in repeated plays. In the quotation at the beginning of this paper Laplace says that moral values are dependent upon a thousand foreign circumstances that are very difficult, if not impossible, to subject to calculation. At least one of these circumstances is amenable to fairly rigorous treatment, provided that a reasonable assumption is made. We refer to the disadvantage to one possessed of limited means that he may exhaust his entire fortune at some stage of a game in which a play is repeated many times. The assumption that will be made is that this disadvantage may be reasonably compensated by adjustment of the probabilities for a play so that the person of limited means may, likely as not, continue the game indefinitely with an opponent of infinite means.

Suppose that two players A and B, possessed initially of fortunes M and N respectively, engage in game consisting of repetitions of the same play. For each play A has a probability p of winning a sum P and B has a probability q of winning a sum Q such that $p+q=1$. The game is to continue until one or the other of the players loses his entire fortune. We require the probability that A will win.

Let U_x represent A's probability of wining the game at a stage when A's fortune is x. Set up the difference equation

$$U_x = pU_{x+P} + qU_{x-Q}. \tag{12}$$

If the indicial equation

$$r^{P+Q} - \frac{1}{p}\cdot r^Q + \frac{1-p}{p} = 0 \tag{13}$$

has no double roots, the solution of the difference equation (12) is

$$U_x = C_1\cdot r_1^x + C_2\cdot r_2^x + C_3\cdot r_3^x + \cdots + C_{P+Q}\cdot r_{P+Q}^x, \tag{14}$$

where $r_1, r_2, r_3, \cdots, r_{P+Q}$ are the roots of the indicial equation and $C_1, C_2, C_3, \cdots, C_{P+Q}$ are constants determined from the $(P+Q)$ conditions:

$$\left\{\begin{array}{l}
C_1 + C_2 + C_3 + \cdots + C_{P+Q} = 0 \\
C_1\cdot r_1 + C_2\cdot r_2 + C_3\cdot r_3 + \cdots + C_{P+Q}\cdot r_{P+Q} = 0 \\
C_1\cdot r_1^2 + C_2\cdot r_2^2 + C_3\cdot r_3^2 + \cdots + C_{P+Q}\cdot r_{P+Q}^2 = 0 \\
\cdot \\
\cdot \\
\cdot \\
C_1\cdot r_1^{Q-1} + C_2 r_2^{Q-1} + C_3\cdot r_3^{Q-1} + C_{P+Q}\cdot r_{P+Q}^{Q-1} = 0 \\
C_1\cdot r_1^{(M+N-P+1)} + C_2\cdot r_2^{(M+N-P+1)} + \cdots + C_{P+Q}\cdot r_{P+Q}^{(M+N-P+1)} = 1 \\
C_1\cdot r_1^{(M+N-P+2)} + C_2\cdot r_2^{(M+N-P+2)} + \cdots + C_{P+Q}\cdot r_{P+Q}^{(M+N-P+2)} = 1 \\
\cdot \\
\cdot \\
\cdot \\
C_1\cdot r_1^{(M+N)} + C_2\cdot r_2^{(M+N)} + \cdots + C_{P+Q}\cdot r_{P+Q}^{(M+N)} = 1
\end{array}\right. \tag{15}$$

If each play is fair in the mathematical sense, that is, if $pP=qQ$, the indicial equation becomes

$$r^{(P+Q)} - \frac{P+Q}{Q}\cdot r^Q + \frac{P}{Q} = 0, \tag{16}$$

which has a double root for $r=1$. The solution of the difference equation for this case is

$$U_x = C_1 + C_2\cdot x + C_3\cdot r_3^x + \cdots + C_{P+Q}\cdot r_{P+Q}^x, \tag{17}$$

where $r_3, \cdots, r_{P+Q}$ are the simple roots of the indicial equation other than unity, and the constants are to be determined from the $P+Q$ conditions

$$\begin{array}{c}
C_1 + C_2\cdot x + C_3\cdot r_3^x + \cdots + C_{P+Q}\cdot r_{P+Q}^x = 0 \\
\text{for } x = 0, 1, 2, \cdots, Q-1 \\
C_1 + C_2\cdot x + C_3\cdot r_3^x + \cdots + C_{P+Q}\cdot r_{P+Q}^x = 1 \\
\text{for } x = (M+N-P+1), (M+N-P+2), \cdots, (M+N).
\end{array} \tag{18}$$

By Pellet's Theorem,* $Q-1$ of the simple roots are of absolute value less

* See Lewis Weisner's paper entitled "Moduli of the roots of polynomials and power series" and reference, page 33, January 1941 issue of this MONTHLY.

than unity. Suppose now that the sum of the fortunes of A and B $(M+N)$ is relatively large in comparison with the amounts at risk for each play, P and Q. Then it can be shown that for large values of x

$$U_x \cong C_1 + C_2 x. \tag{19}$$

For, as x increases, all terms involving roots of absolute value less than unity must approach zero. If any term involving a root of absolute value greater than unity were to contribute finite value to the probability for some value of x, then a larger value of x could be found for which the value of the term would be so large that it would produce an impossible value for the probability U_x. We conclude that when $(M+N)$ becomes very large, all terms of the form $C_i r_i^x$ in which r_i is not unity tend to vanish.

A satisfactory approximation to the values of the coefficients C_1 and C_2 when $(M+N)$ is large may be obtained by solving the $Q+1$ equations.

$$\begin{cases} C_1 + C_{i_1} + C_{i_2} + \cdots + C_{i_{Q-1}} = 0 \\ C_1 + C_2 + C_{i_1} \cdot r_{i_1} + C_{i_2} \cdot r_{i_2} + \cdots + C_{i_{Q-1}} \cdot r_{i_{Q-1}} = 0 \\ C_1 + 2C_2 + C_{i_1} \cdot r_{i_1}^2 + C_{i_2} \cdot r_{i_2}^2 + \cdots + C_{i_{Q-1}} \cdot r_{i_{Q-1}}^2 = 0 \\ \cdot \\ \cdot \\ \cdot \\ C_1 + (Q-1) \cdot C_2 + C_{i_1} \cdot r_{i_1}^{Q-1} + C_{i_2} \cdot r_{i_2}^{Q-1} + \cdots + C_{i_{Q-1}} \cdot r_{i_{Q-1}}^{Q-1} = 0 \\ C_1 + (M+N) \cdot C_2 = 1 \end{cases} \tag{20}$$

where $r_{i_1}, r_{i_2}, r_{i_3}, \cdots, r_{i_{Q-1}}$ are the $Q-1$ roots whose absolute values are less than unity.

The values of C_1 and C_2 thus found yield

$$U_x \cong \frac{-\lambda}{M+N-\lambda} + \frac{x}{M+N-\lambda} \quad \text{where} \tag{21}$$
$$\lambda = \frac{1}{1-r_{i_1}} + \frac{1}{1-r_{i_2}} + \cdots + \frac{1}{1-r_{i_{Q-1}}} \cdot$$

If both x and $M+N$ are very large in comparison to λ, we have $U_x = x/\overline{M+N}$; that is, either player's chance of winning at any stage of the game is the ratio of his fortune at that stage to the sum of his fortune and his opponent's fortune. In particular, A's chance of winning the game in the beginning is approximately $M/\overline{M+N}$.

Suppose now that the probabilities for a play are varied slightly in A's favor, but that the amounts at risk are the same as before. Suppose further that A's fortune is large in comparison to the amounts at risk for each play but small in comparison to his opponent's fortune. Let us find A's chance of winning at a stage of the game when his fortune is x.

The roots of the indicial equation (13) are now all distinct. One root is unity; another root is less than unity by as small amount as we please, depending upon

how little we vary the probabilities for a play in A's favor; $(Q-1)$ of the roots have absolute values less than unity by finite amounts; and the remaining roots have absolute values greater than unity by finite amounts. Denote by $(1-\Delta)$ the root that approaches unity as the probabilities for a play approach mathematically "fair" values.

Then, the solution of the difference equation is approximately for large values of x

$$U_x \cong C_1 + C_2(1-\Delta)^x \tag{22}$$

where C_1 and C_2 are constants to be determined from the $Q+1$ conditions

$$\begin{aligned} &C_1 + C_2\cdot(1-\Delta)^x + C_{i_1}\cdot r_{i_1}^x + C_{i_2}\cdot r_{i_2}^x + \cdots + C_{i_{Q-1}}\cdot r_{i_{Q-1}}^x = 0; \\ &\qquad\qquad\qquad\qquad\qquad\qquad x = 0, 1, 2, 3, \cdots, Q-1, \\ &C_1 + C_2\cdot(1-\Delta)^{M+N} = 1; \end{aligned} \tag{23}$$

$r_{i_1}, r_{i_2}, \cdots, r_{i_{Q-1}}$ denoting as before the roots of absolute value less than unity. The values of C_1 and C_2 thus obtained are

$$C_1 = \frac{1}{1-(1-\Delta)^{M+N}\cdot\dfrac{(1-r_{i_1})(1-r_{i_2})\cdots(1-r_{i_{Q-1}})}{(1-\Delta-r_{i_1})(1-\Delta-r_{i_2})\cdots(1-\Delta-r_{i_{Q-1}})}}$$

and

$$C_2 = -C_1\cdot\frac{(1-r_{i_1})(1-r_{i_2})\cdots(1-r_{i_{Q-1}})}{(1-\Delta-r_{i_1})(1-\Delta-r_{i_2})\cdots(1-\Delta-r_{i_{Q-1}})}.$$

If B's fortune is infinite and if Δ is very small but not vanishingly small, we have approximately for A's probability U_x of winning the game at a stage when his fortune is x

$$U_x \cong 1-(1-\Delta)^x. \tag{24}$$

From equation (24) let us determine how much the probabilities for each play must be varied in A's favor, when A is possessed initially of a fortune of large but finite amount M, in order that A may, likely as not, continue the game indefinitely with an opponent whose fortune is infinite. We have

$$(1-\Delta)^M = \tfrac{1}{2}, \quad \text{and} \quad \Delta \cong \frac{\log 2}{M}.$$

From equation (13), if we assume that Δ is very small, we obtain for the corresponding increase δp in the probability of A's winning a play

$$\delta p \cong \frac{PQ}{2(P+Q)}\cdot\Delta. \tag{25}$$

Upon substituting the value previously found for Δ, we have

$$(26) \qquad \delta p \cong \frac{\log 2}{2} \cdot \frac{PQ}{(P+Q)M}$$

for the increase in A's probability of winning a play over the mathematically "fair" probability, other quantities remaining the same, to place A in position, likely as not, to continue the game indefinitely. We shall refer to the increase, δp, as A's margin of probability. Note that if the amounts at risk are proportionately increased, then A's margin must be increased in direct proportion, in order that he may maintain his advantage.

Let us now find the particular transformation function that produces the moral advantage corresponding to A's margin of probability. Using the same symbols as in the paragraph immediately preceding, we have

$$(27) \qquad f(M) = (p + \delta p) \cdot f(M + P) + (q - \delta p) \cdot f(M - Q).$$

Assume (1) that P and Q are small in comparison to M, (2) that M is so large that the second and higher powers of its reciprocal may be neglected,and (3) that $f(M)$ and its successive derivatives are analytic over the range under consideration. We obtain from the Taylor Series expansion of the members of equation (27) the following differential equation, provided we neglect terms involving third and higher derivatives

$$(28) \qquad f''(M) + \frac{\log 2}{M} f'(M) = 0.$$

Equation (28) has the solution

$$(29) \qquad f(M) = K_1 \cdot M^{(1-\log 2)} + K_2$$

where K_1 and K_2 are constants of integration. It is easy to verify from this solution that neglect of the third and higher order derivatives in the expansion of the members of equation (27) is justifiable. Since values of the constants of integration are not necessary for the determination of the physical value of the expectation, we, therefore, may write

$$(30) \qquad f(M) = M^{.307}$$

for the transformation function which represents the moral disadvantage from the possibility that one of limited means will exhaust his fortune by repetition of a fair venture many times. This function would seem to represent the reasonable minimum of moral advantage (or disadvantage) that would accrue to a person of limited means when he assumes a risk that would augment (or diminish) his fortune.

7. Concluding remarks. In contrast to the transformation function just determined, which seems to represent a reasonable minimum of moral advantage (or disadvantage), there also seems to be a limit in the other direction. Consider

the physical value of probability of entire loss of a fortune. The equation for determining the physical value, X, of such expectation is

$$f(M - X) = p \cdot f(0) + (1 - p) \cdot f(M), \tag{31}$$

where M is the amount of the fortune and p is the probability of loss. If $f(M)$ is a positive power (not exceeding unity) of M, equation (31) yields a value for X less than M. If $f(M)$ is $\log M$, as in Bernoulli's Hypothesis, or a negative power of M, equation (31) yields X equal to M, that is, no matter how small the probability of loss may be, the physical value of the expectation of loss is equal to the fortune itself. Clearly, this last result is untenable, for it is generally agreed that there is some probability, however small, that one may lose his entire fortune, and no one would value the expectation of such loss as highly as the fortune itself, when the probability of the loss is not in the neighborhood of certainty.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Dicussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

AZIMUTHAL EQUIDISTANT PROJECTION OF THE SPHERE*

A. D. BRADLEY, U. S. Naval Air Station, Norman, Oklahoma

The azimuthal equidistant map projection is a graphic means of showing the great circle distances and great circle bearings of points on the earth from any given point. The given point is the center of the map, and any point is mapped by its great circle distance from the center and the initial great circle course from the center to the point.

In Figure 1, O is the point which is to be the center of the map; the coordinates of any point A are the great circle arc OA, and the angle P_NOA of the spherical triangle P_NOA. Angle P_NOA is the initial great circle course from O to A; it is measured clockwise from 000°, north, through 360°. Angle P_NOA is also called the true bearing of A from O.

The azimuthal equidistant projection has rather obvious uses in showing great circle routes from any point. "The projection has been employed in celestial maps, in problems of crystalloptics by the Carnegie Geophysical Laboratory, by the National Geographic Society for maps of the Arctic Regions, North America,

* In accordance with the regulations of the United States Navy, this article is to be understood as reflecting the writer's personal opinions, and is not to be construed as official or as reflecting the views of the Navy Department or the naval service at large.

Asia, and Africa, by the General Electric Company for a map of the world."*

The method for constructing this projection, given by Deetz and Adams, consists in calculating the coordinates of the intersections of the meridians and parallels. It is the purpose of this note to present an alternate method which it is believed will shorten the computations. The notation used follows that of Dutton's *Navigation and Nautical Astronomy*.

To construct a meridian: Let O, latitude L_0, longitude λ_0, be the center of the projection and P_NAP_SB the great circle formed by two meridians differing 180° in longitude (Fig. 1). Let the difference of longitude from O to A, DLO_A, be less than the difference of longitude from O to B, DLO_B.

Construct the great circle AOB perpendicular to the meridians PA and PB. Denote the great circle distance of A from O by D_A and the initial great circle course from O to A by C_A, *i.e.*, the map coordinates of A are C_A and D_A. In right triangle P_NOA (Fig. 2):

$$\sin D_A = \cos L_0 \sin DLO_A, \tag{1}$$

$$\cot C_A = \sin L_0 \tan DLO_A. \tag{2}$$

The coordinates of B are C_B and D_B:

$$C_B = 180° + C_A,$$
$$D_B = 180° - D_A.$$

For other points on the meridian it is easy to determine the value of the initial great circle course which corresponds to a great circle distance D_X, where $D_A < D_X < D_B$. D_X determines two points X_1 and X_2 such that,

$$C_{X_1} = C_A - \alpha_X,$$
$$C_{X_2} = C_A + \alpha_X.$$

From right triangle X_1OA (see Fig. 2),

$$\cos \alpha_X = \tan D_A \cot D_X. \tag{3}$$

Formulas (1), (2), and (3) afford a means of determining the coordinates for a given meridian by assuming values of D_X between D_A and D_B.

To construct a parallel: It is convenient to assume appropriate values of D and calculate the corresponding values of C. For the parallel of L_1,

$$L_0 < L_1 < 90°, \qquad L_1 - L_0 \leqq D \leqq 180° - (L_0 + L_1),$$

$C = 000°$ for both the maximum and minimum values of D. At two points E and F of L_1, the great circles OE and OF are tangent to L_1 (Fig. 3). In right triangle OP_NE:

* C. H. Deetz, and Adams, O. S. Elements of Map Projections, Special Publication No. 68, U. S. Coast and Geodetic Survey, page 167.

(4) $$\cos D_E = \sin L_0 \csc L_1,$$

(5) $$\sin C_E = \sec L_0 \cos L_1,$$

$$D_F = D_E, \qquad C_F = 360° - C_E.$$

Hence for L_1 between L_0 and $+90°$, C varies from 000° to C_E and from 360° to C_F.

For parallels of latitude between L_0 and $-L_0$, C varies from 000° to 360°. The point diametrically opposite to L_0, λ_0, *i.e.*, $-L_0$, $\lambda_0 \pm 180°$, is mapped by the periphery of the projection. For parallels of latitude between $-L_1$ and $-90°$, $C=180°$ for both the maximum and minimum values of D, and C is restricted to a certain range on either side of 180°.

If values of D are assumed for a given parallel, then it is sufficient to solve a spherical triangle of sides $90°-L_0$, D, and $90°-L_1$, for angle C opposite $90°-L_1$. In a spherical triangle of sides a, b, and c,

$$\sin^2 C/2 = \sin (s-a) \sin (s-b) \csc a \csc b.$$

Let $2S=L_1+L_0+D$, $c=90°-L_1$, $b=90°-L_0$, and $a=D$, then $s-b=S-L_1$ and $s-a=90°-S$. It is convenient to replace $\sin C/2$ by the haversine function, $\text{hav } C=(1-\cos C)/2=\sin^2 C/2$. The above formula may be written,

(6) $$\text{hav } C = \sec L_0 \sin (S - L_1) \cos S \csc D.$$

Values of C corresponding to appropriate values of D may be readily computed by formula (6).

Example 1. Center 40° 46′ N., 73° 52′ W.

(a) Meridian of 30° W. 150° E.:

$$D_A = 31°39', \quad D_B = 148°21', \quad C_A = 057°53', \quad C_B = 237°53'.$$

(b) Parallel of 60° N.:

$$D_E = D_F = 41°04', \quad C_E = 041°19', \quad C_F = 318°41'.$$

(c) Parallel of 60° S.:

$$D_E = D_F = 138°56', \quad C_E = 138°41', \quad C_F = 221°19'.$$

Example 2. Center 33° 52′ S., 151° 13′ E.

(a) Meridian of 0°–180°:

$$D_A = 23°35', \quad D_B = 156°25', \quad C_A = 107°02', \quad C_B = 287°02'.$$

(b) Parallel of 60° N.:

$$D_E = D_F = 130°03', \quad C_E = 037°02', \quad C_F = 322°58'.$$

(c) Parallel of 60° S.:

$$D_E = D_F = 49°57', \quad C_E = 142°58', \quad C_F = 217°02'.$$

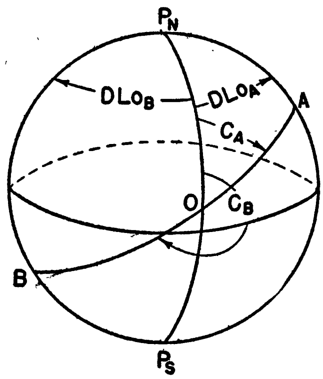


FIG. 1

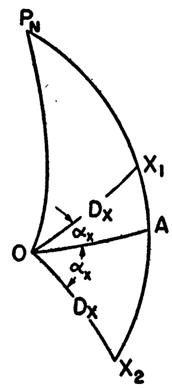


FIG. 2

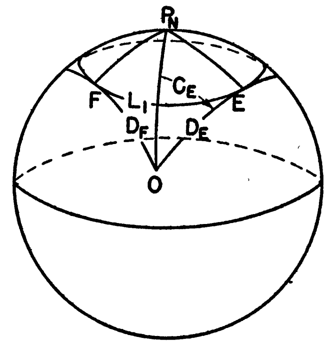
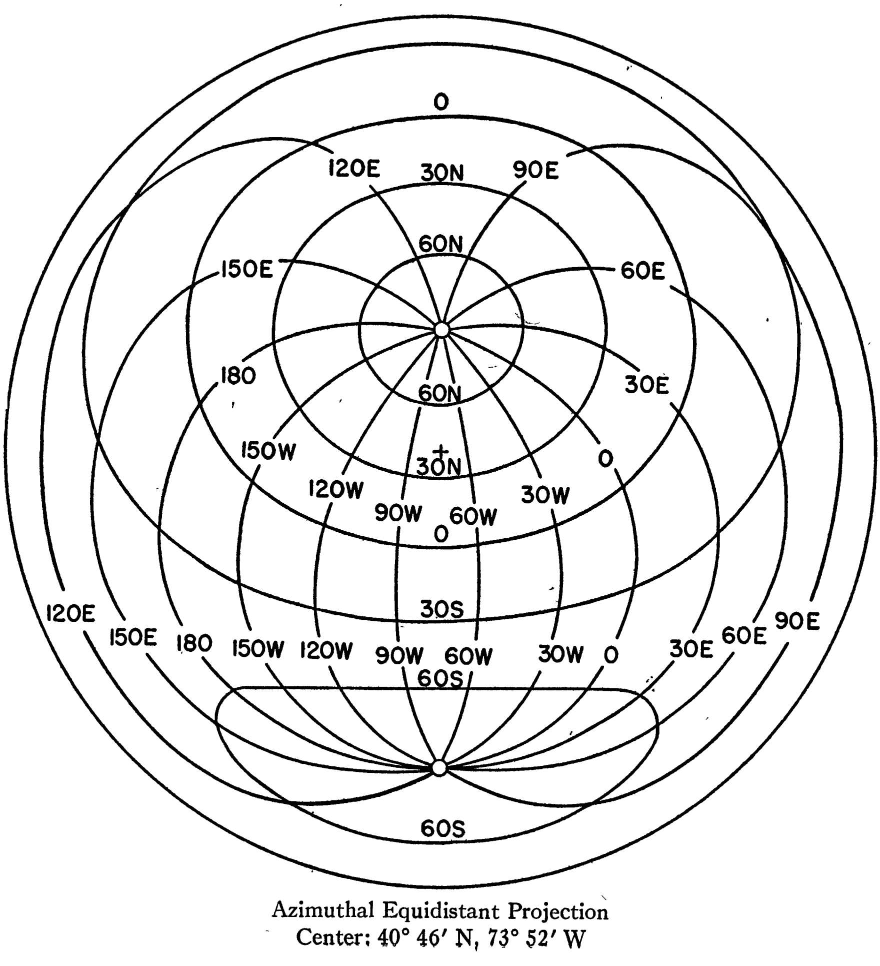


FIG. 3



Azimuthal Equidistant Projection
Center: 40° 46′ N, 73° 52′ W

VARIATIONS OF VECTOR INEQUALITIES

HERBERT RYSER, University of Wisconsin

The vectors $\alpha=(a_1, a_2, \cdots, a_n)$ and $\beta=(b_1, b_2, \cdots, b_n)$ are in an n-dimensional space over the real field. By (α, β), the inner product of α and β, is meant $\sum a_i b_i$, and $\|\alpha\|$, the norm of α, is defined as $(\alpha, \alpha)^{1/2}$. Two inequalities of vector algebra are $|(\alpha, \beta)| \leqq \|\alpha\| \|\beta\|$ (the Schwarz inequality) and $\|\alpha+\beta\| \leqq \|\alpha\| + \|\beta\|$ (the triangle inequality).

From α and β, $\lambda=(M_1, M_2, \cdots, M_n)$ and $\mu=(m_1, m_2, \cdots, m_n)$ are formed by choosing $M_i = \max(|a_i|, |b_i|)$ and $m_i = \min(|a_i|, |b_i|)$, for $i=1, 2, \cdots, n$. These λ, μ vectors offer variations of the above inequalities.

$$|(\alpha, \beta)| \leqq \|\lambda\| \|\mu\| \leqq \|\alpha\| \|\beta\| \tag{1}$$

$$\|\alpha+\beta\| \leqq \|\lambda\| + \|\mu\| \leqq \|\alpha\| + \|\beta\|. \tag{2}$$

To prove (1), apply Schwarz's inequality to λ and μ, giving $|(\lambda, \mu)| \leqq \|\lambda\| \|\mu\|$. Since $|(\alpha, \beta)| \leqq (\lambda, \mu)$, it remains only to establish that $\|\lambda\| \|\mu\| \leqq \|\alpha\| \|\beta\|$. From the choice of λ and μ, $\|\alpha\| \leqq \|\lambda\|$, $\|\mu\| \leqq \|\beta\|$, and $\|\beta\| \leqq \|\lambda\|$, $\|\mu\| \leqq \|\alpha\|$. Hence $|\|\alpha\| - \|\beta\|| \leqq \|\lambda\| - \|\mu\|$ and $\|\alpha\|^2 - 2\|\alpha\| \|\beta\| + \|\beta\|^2 \leqq \|\lambda\|^2 - 2\|\lambda\| \|\mu\| + \|\mu\|^2$. But $\|\alpha\|^2 + \|\beta\|^2 = \|\lambda\|^2 + \|\mu\|^2$. Thus $\|\lambda\| \|\mu\| \leqq \|\alpha\| \|\beta\|$.

For the derivation of (2) note that

$$\begin{aligned} \|\alpha+\beta\|^2 &= (\alpha, \alpha) + 2(\alpha, \beta) + (\beta, \beta) = \|\alpha\|^2 + 2(\alpha, \beta) + \|\beta\|^2 \\ &\leqq \|\lambda\|^2 + 2(\lambda, \mu) + \|\mu\|^2 = \|\lambda+\mu\|^2, \end{aligned}$$

and $\|\lambda+\mu\|^2 \leqq \|\lambda\|^2 + 2\|\lambda\| \|\mu\| + \|\mu\|^2 \leqq \|\alpha\|^2 + 2\|\alpha\| \|\beta\| + \|\beta\|^2$.

The angle θ between α and β is defined as the arc cos $[(\alpha, \beta)/\|\alpha\| \|\beta\|]$, $0 \leqq \theta \leqq \pi$. If ϕ denotes the angle between λ and μ, inequality (1) implies

$$0 \leqq \phi \leqq \theta \leqq \pi. \tag{3}$$

Inequalities which are analogues of (1) and (2) hold for infinite series and functions.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

Pi Mu Epsilon, Marquette University

The activities of *Pi Mu Epsilon* during the past year have been affected by the war and the entrance of many active members into the service. Nevertheless, the fraternity continued its monthly meetings and attempted to further

mathematical interest. Five new members were initiated in November: Donald Bennett, Beverly Ullmer, Gary Himler, Eugene Philipp, and Charles Polzin.

The writing of the history of the local chapter and of the national organization was one of the projects undertaken by the fraternity. Papers presented during the year included:

The significance of exponents, by E. A. Halbach

Celestial navigation, by Dr. Ross Bardell, of the University of Wisconsin in Milwaukee. This was the after dinner speech at the annual banquet in May. At this time Donald La Budde, as winner of the Frumveller contest for high school seniors, was presented with a $100 scholarship to Marquette University.

Officers for the coming year are: Director, A. J. Gillan; Vice-Director, Gary Himler; Corresponding Secretary, Lillian Schnell; Recording Secretary, Lois Ebert; Treasurer, Mrs. Madeline Kennedy; Librarian, Beverly Ullmer.

Mathematics Club, Boston University

The Club held monthly meetings at which the following talks were presented:

Plotting of graphs by mechanical means, by Francis Scheid

Statistics, by Robert Haskins of Harvard University

Picturing fractions, by Peter Franck of Harvard University

Analytical triangles, by Emma De le Vin

As an extra activity, some of the members visited the meetings of the Harvard Mathematics Clubs during the winter. The members also visited the Treasure Room of the Boston Public Library and examined rare mathematics texts preserved there.

Officers for the year 1943–44 were: President, Mary Siteman; Vice-President, Emma De le Vin; Secretary, Marion King; Treasurer, Doris Stovald; Faculty Adviser, Professor Lewis Brigham. Officers elect for 1944–45 are: President Harry Kouyoumjian; Vice-President, Marion King; Secretary, June Westgate; Treasurer, Phyllis Paulsen; Faculty Adviser, Professor Ralph Johanson.

Mathematics Club, Immaculate Heart College

The Mathematics Club of Immaculate Heart College, Hollywood, California, organized in January 1944, held monthly meetings from January to June, at which papers were presented by students. Topics discussed were:

Descartes—Analytic geometry, by Zilda Cross

Desargues—Projective geometry, by Sister Margaret Ann

De Moivre's theorem, by Kitty Campion and Jean Gormaly

Mathematics in science and aircraft, by Peggy Haws.

The Club is directed by Dr. Myrtle Collier. The officers were: President, Zilda Cross; Secretary-Treasurer, Sister Margaret Ann.

RECENT PUBLICATIONS

Edited by H. P. Evans, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Marine and Air Navigation. J. Q. Stewart and N. L. Pierce. Boston, Ginn and Co., 1944. 12+472 pages. $4.50.

At the outbreak of the present war, a large number of short texts on navigation began to come from the publishers. The object was to shorten and clarify the standard texts, Bowditch's *American Practical Navigator* and Dutton's *Navigation and Nautical Astronomy*. The former, written in 1802, was completely revised in 1938. Benjamin Dutton, Commander U. S. Navy, wrote his text in 1926. The last chapter in the fifth edition, 1934, contained forty pages entitled "Aerial Navigation."

Some of the recent texts deal only with air navigation; some pay most attention to the recently developed short methods of navigation. Actually the fundamental principles of marine and air navigation are the same.

The authors of this text—*Marine and Air Navigation*—state: "Our experience in the teaching of marine and air navigation for a number of years has convinced us of the need for a comprehensive and understandable textbook—one which clearly explains present-day methods and practice." Their book is certainly comprehensive. It contains 400 pages of text—two columns to the page of $8\frac{1}{2}\times 11$ paper—18 pages of drill problems with answers, and eight star charts. There are 367 photographs and diagrams which are numbered by chapters and 47 tables including excerpts from Bowditch's tables and various Hydrographic Office publications.

The photographs are numerous and include remarkable shots of naval vessels from official sources, the interiors of army training planes showing navigation instruments, and astronomical photographs to illustrate a chapter on Astronomy for Navigators. One has a feeling that the photographs, copies of charts, and drawings are the best part of the book.

The text covers approximately the same subjects as the standard texts by Bowditch and Dutton, with more emphasis on air navigation. The surface navigator would want to omit much on air navigation and the aviator would not need much of the material on marine navigation. It is easy to see why the Air Corps prefers its own manuals.

Many students object to the difficulty of reading Dutton's book because it includes too much material. This text has the same fault, although the authors try to make it more readable by numerous incidents or words of advice to the reader. We wonder if the extra words make the study of navigation more understandable. The many excellent exercises completely worked out should serve the

purpose sufficiently. It is enjoyable to read the history of navigation which is included.

The star charts are excellent—a big improvement over those furnished in the Nautical Almanac. The necessary astronomical terms for celestial navigation are explained with diagrams. Since the seven pages on Astronomy of General Interest are too brief to be of much interest, the student would find a short text on Astronomy more satisfactory.

Two unusual tables are worth mentioning. One is a double page of Tolerances, giving the degree of accuracy which is to be expected in various phases of navigation. The other is a long-term almanac for the sun and stars. This is a table of corrections to the 1940 almanac and is good until 1955.

The publishers have done an unusual job by producing a text on excellent paper, with large, readable type and free from errors—a notable accomplishment in war time. The photographs make it especially attractive. Since each school has its own preferences for methods, texts and references, this book should find a useful place in the teaching of navigation.

C. M. HUFFER

Navigational Handbook with Tables. By T. F. Hickerson. Chapel Hill, N. C. Published by the Author, 1944. 1+77 pages. $1.50.

The principal part of this "Handbook" is a 45-page expansion of the Ageton Table (H. O. 211) of logarithmic cosecants and secants, in which angles are given to each 0.2 minute of arc instead of the usual 0.5 minute. This should make interpolation unnecessary, or at least simpler, in "solving the usual problems of navigation." The number of pages required has been cut in half by a slight reorganization of the column headings. This change necessitates modification of the rules for determination of the quadrant of Z, the new rules being printed in bold-face type on the pages of work forms instead of on each page of the table as in H. O. 211.

The book also contains a condensed table of meridional parts, an extensive table for "Conversion of Time into Arc and Vice Versa," and the usual tables of corrections to observed altitudes.

Probably the most useful part of the book is the discussion of the uses of the tables as found in the first 29 pages. Here the author solves 21 examples in detail, deriving the necessary formulas and in the various cases presenting 12 systematic work forms, some of which are new. In addition, he gives 28 problems, with answers.

Notation in the main is standard except for the use of positive and negative signs instead of "names" (north and south), which should appeal to mathematicians if not to practical navigators.

The typography is clean and well spaced; but the binding is inadequate and the book can not be expected to hold together long under normal usage.

R. C. HUFFER

Elements of Trigonometry. Plane and Spherical with Applications. By L. M. Kells, W. F. Kern, J. R. Bland, and J. B. Orleans. New York and London, McGraw-Hill Book Company, Inc., 1943. 10+363 pages. $1.80.

This volume, which includes both plane and spherical trigonometry, has 194 pages on plane trigonometry, plus 54 pages in two chapters on logarithms and the slide rule, and 85 pages on spherical trigonometry. It is, according to the authors, an attempt to tone down the text by Kells, Kern and Bland to the level of understanding of third and fourth year high school students. Insofar as the attractiveness of treatment, readable language and good illustrations are concerned, they have done well.

The early introduction to the solution of the right triangle and application to problems of surveying, plane sailing, and vectors, helps to impress upon the student the practical importance of the subject. The very large number of exercises and problems, while offering a challenge to the superior student, imposes a real responsibility upon the teacher, for a judicious selection of problems to be deferred, as the authors recommend, for later practice and review.

By placing the discussion of logarithms in a separate chapter near the end of the book the right triangle presents a direct, unified topic. It is probable that the study of this unit would have to be interrupted in order to insure the availability of this tool, whose application comes rather early.

In these days, when more people are becoming familiar with the slide rule, the presence of a chapter on its use, and problems designed for slide rule solution is valuable.

One of the outstanding features of the text is the large number of practical and well illustrated problems in applied fields, notably in navigation.

While, supposedly, high school students will be chiefly interested in the applications to problems in mensuration, yet the authors have included more exercises in identities than are to be found in many college textbooks.

One might take issue with their definitions of principal values for inverse functions of negative quantities, which are at variance with usual practice; *e.g.*, $\csc^{-1}(-a)$ is a negative angle with terminal line in the third quadrant while $\sin^{-1}(-1/a)$ is a negative angle terminating in the fourth quadrant.

The textual material would certainly be adequate for most college courses in trigonometry. The extensive offering of exercises and problems (well over 1000 in 57 sets) calls for careful selection for a one semester course.

The 85 pages devoted to spherical trigonometry would, as the authors note, form a good separate course. If a teacher wished to include any substantial amount of this in the one semester course in trigonometry, a somewhat drastic cutting of the matter in plane trigonometry would be necessary.

The usual development of formulas for spherical trigonometry is beautifully supplemented by discussion of and problems in navigation, as one might readily expect, from the background of the authors.

Only two typographical errors were noted, and those were not serious.

H. P. PETTIT

General Mathematics in American Colleges. By Kenneth Brown. New York, Bureau of Publications, Teachers College, Columbia University, 1943. 1+167 pages. $2.35.

The development of "general mathematics" courses in American colleges, stemming from the pioneer urgings of John Perry (England), Felix Klein (Germany), and E. H. Moore (The United States), should be of great interest to teachers of mathematics. Especially after a forty-year climb does the present status of the movement warrant study. That the stature of general mathematics is by now large is readily inferred from the list of text-books on the subject imposingly arrayed in the excellent bibliography in the volume under review. That its study has occupied a large number of investigators bent on improving the teaching of mathematics is apparent in the list of magazine articles given in the same bibliography. It is time for a book like the one here reviewed, offering both an historical account of the trend and an evaluation of its achievements.

By general mathematics the author means a "modern non-compartmental" course in freshman mathematics. Such a course may be offered to students interested in "tool" mathematics and further work in the subject, or to students who wish a "cultural" course presenting in itself alone the mathematical knowledge and appreciation which is proper for an intelligent citizen. The author's investigation of such courses (based on questionnaires sent to students and instructors in many colleges) indicates that those organized directly for only one of the above types of student achieve a certain amount of success, but single courses aimed in both directions at once fail to achieve results in either direction. Extracts from student opinion concerning these courses reveal little thought beyond the student-bromidic. Samples of procedure in classes visited by the author disclose teaching methods that are both good and (in one case) appallingly and humorously bad. Such evidence can, of course, be held flimsy enough to vitiate the author's conclusions.

The author is even more thorough in his examination of text-books. And the evidence, by quotation from the printed words (mostly prefaces) of the authors of these books, is more substantial though somewhat theoretical: what an author wants to achieve and what he actually achieves are often different. Books are classified into three groups: (i) Preparatory, (ii) Cultural, (iii) Cultural-preparatory, with obvious reference to the student-types and courses mentioned previously. They are studied as to objectives, content, emphasis on mathematical rigor, type and number of exercises, style, prerequisites assumed, and testings of the material by actual class-room use. Opinions of (anonymous) instructors concerning tests (anonymous) are rather pointlessly given. The author makes no attempt to label a given book as good, bad, or indifferent—wisely, of course, as professional courtesy might not always have stood the strain. But he concludes that books in class (iii) are generally unsatisfactory, a not unreasonable conclusion since the two parts of the double objective are so incompatible; those in class (i), the most stabilized class of all, give much satisfaction; and those in the newer class (ii) seem to show large and steady improvement. These estimates

of texts likewise describe his final evaluation of the success of the types of courses in general mathematics; in addition, he seems to feel that there is a fruitful future for courses and books of the cultural type.

These things are presented in a style which is serious, earnest, and generally precise, though sometimes repetitive and a trifle wearisome. But exposition-by-quotation, so extensively used, is a dangerous game: Examining one recent cultural text-book on the subject of exercises, the author (page 90) condemns with this quotation, "The exercises have been relegated to the appendix so that continuity of reading may not be interrupted." Standing alone, this sounds a death knell. Yet investigation of the appendix reveals that, among historical and critical notes, there are copious lists of exercises nicely cross-referenced with the appropriate parts of the text; these number well over a thousand; they are carefully graded; some demand researches on the part of the student; many contain judicious hints of procedure for the non-technical student for whom they are designed. Moreover, examination of the book itself discloses that the author leans heavily on exposition by means of exercises, so that he cannot place "little emphasis on formal exercises." This reviewer is convinced that the author of this text must writhe at such misrepresentation—and he should know, for reviewer and author are the same person!

GAYLORD MERRIMAN

NEW BOOKS RECEIVED

Mathematics. Second Edition. By J. W. Breneman. New York and London, McGraw-Hill Book Company, Inc., 1944. 12+224 pages. $1.75.

A Guide to Tables of Bessel Functions. By Harry Bateman and R. C. Archibald. (Vol. 1, Number 7 of Mathematical Tables and other Aids to Computation.) Washington, D. C., Division of Physical Sciences, The National Research Council, 1944. 104 pages. $1.75.

Notre Dame Mathematical Lectures. Numbers 3 and 4. University Press, Notre Dame (Lithoprinted), 1944. Number 3: Algebra of Analysis. By Karl Menger. 50 pages. $1.00. Number 4: Alignment Charts. By L. R. Ford; The Teaching of the Calculus of Probability. By A. H. Copeland; On the Theory of Complex Functions. By Emil Artin. 70 pages. $1.25.

Reports of a Mathematical Colloquium. Second Series, Double Issue 5–6. Edited by Karl Menger. University Press, Notre Dame, 1944. 80 pages. $1.00.

Nautical Mathematics and Marine Navigation. By S. A. Walling, J. C. Hill and C. J. Rees. Cambridge University Press; New York, The Macmillan Company, 1944. 9+221 pages. $2.00.

Vital Mathematics. By E. B. Allen, Dis Maly, and S. H. Starkey, Jr., New York, The Macmillan Company, 1944. 7+456+22 (answers) pages. $1.80.

Alignment Charts, Construction and Use. By M. Kraitchik. New York, Van Nostrand Company, 1944. 94 pages. $2.50.

Engineering Mathematics. By H. Sohon. New York, Van Nostrand Company, 1944. 6+278 pages. $3.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 661. *Proposed by Howard Eves, Syracuse University*

A plane p is projected from a point L onto a plane p'. Find those points on p for which all angles on p having such a point for vertex are invariant under the projection.

E 662. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A number is represented by a in the scale of α and by b in the scale of β $(\beta<\alpha)$. Regarding both a and b as written in the scale of α, we write the difference $b-a=c$. Show how to determine the greatest possible value of a for given values of α, β, c; *e.g.*, when $\alpha=10$, $\beta=7$, and $c=3501$.

E 663. *Proposed by Irving Kaplansky, Columbia University*

If $2^n+1=p^r$, where p is a prime, prove that r is a power of 2 (including the possibility $r=2^0=1$).

E 664. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that if $x<1$,

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\int_1^{\infty}t^ne^{-t}dt=\frac{e^{x-1}}{1-x}\cdot$$

E 665. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C be a closed convex plane curve with continuous radius of curvature R. Let R_M be the greatest value of R. Given $\lambda\geqq R_M$, show that the area F_λ covered by the centers of circles of radius λ which contain C in their interior is given by

$$F_\lambda=F-L\lambda+\pi\lambda^2,$$

where L and F are the length and area of C.

E 637 [1944, 472]. Hint: Use a plane parallel to one face.

SOLUTIONS

A Tetrahedron whose Opposite Dihedral Angles Are Supplementary

E 617 [1944, 231]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

From a given tetrahedron we derive another by taking as vertices the points of contact of the insphere with the faces. Show that the dihedral angles at pairs of opposite edges of the first tetrahedron are supplementary if, and only if, the second tetrahedron is trirectangular.

Solution by L. M. Kelly, U. S. Coast Guard Academy. (The numbers in parentheses refer to Arts. of Altshiller Court, *Modern Pure Solid Geometry.*) Let $A_1A_2A_3A_4$ be the original tetrahedron, $B_1B_2B_3B_4$ the derived tetrahedron, O and R the incenter and inradius of the former (or the circumcenter and circumradius of the latter). We first observe that the dihedral angles at the edges A_iA_j and A_kA_l are supplementary if and only if the plane angles B_iOB_j and B_kOB_l are supplementary. Thus the problem is reduced to showing that a tetrahedron is trirectangular if, and only if, the angles subtended by pairs of opposite edges at the circumcenter are supplementary.

Suppose first that $B_1B_2B_3B_4$ is given to be trirectangular. Then its vertices belong to a rectangular parallelepiped with center O, and it follows at once from congruent triangles that the desired angles are supplementary.

Now suppose that the angles (such as B_1OB_2 and B_3OB_4) are given to be supplementary. Application of the cosine law easily shows that the sum of the squares of any two opposite edges equals $4R^2$. We conclude (185) that the bimedians are equal, whence (by the converse of 210, which is easily seen to be true), the tetrahedron is orthocentric. From the distance formula

$$OG^2 = R^2 - \sum B_iB_j^2/16,$$

where G is the centroid of $B_1B_2B_3B_4$, we deduce $OG=R/2$. Hence the orthocenter H, being the reflection of O in G, must lie on the circumsphere. The polar sphere has center H and radius $\sqrt{HO^2-R^2}=0$. But this radius is also equal to $\sqrt{HB_i\cdot HH_i}$, where H_i is the foot of the altitude from B_i. (See 795 and 823.) Thus H must coincide with either a vertex or the foot of an altitude. In either case the tetrahedron is evidently trirectangular.

Parallel Curves

E 630 [1944, 348 and 405]. *Proposed by R. A. Rosenbaum, U.S.N.R.*

For any point P of a given closed convex curve C, let P' be that point on the exterior normal to C at P for which $PP'=k$, a constant. The locus of P' is a curve C', parallel to C. Let s, s' be the respective lengths of C, C', and A, A' the areas within these curves. Show that

$$s' = s + 2\pi k,$$

$$A' = A + sk + \pi k^2.$$

Solution by Howard Eves, Syracuse University. Neglecting differentials of higher order, we easily see (from a figure showing two adjacent normals) that

$$ds' = ds + kd\phi,$$
$$d(A' - A) = \tfrac{1}{2}k(ds + ds'),$$

where $d\phi$ is the angle between the two adjacent normals. Integrating for a complete circuit, these equations become

$$s' = s + 2\pi k,$$
$$A' - A = \tfrac{1}{2}k(s + s') = \tfrac{1}{2}k(2s + 2\pi k) = sk + \pi k^2.$$

Also solved by Chandler Davis, J. T. Webster, and R. H. Wilson, Jr.

A Cube of the form *abab*

E 632 [1944, 405]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

What is the smallest radix which admits a perfect cube of the form *abab*?

Solution by Colin Blyth, Queen's University. If *abab* is a cube in the scale of r, we have $k^3 = ar^3 + br^2 + ar + b = (r^2+1)(ar+b)$. Here r^2+1 is the larger factor, since

$$ar + b \leqq r(r-1) + r - 1 = r^2 - 1.$$

Therefore r^2+1 must be divisible by the square of some prime. Testing in turn $r = 2, 3, \cdots$, we find 7 to be the smallest value satisfying this necessary condition. If $r = 7$, we have

$$k^3 = 2 \cdot 5^2(7a + b).$$

Thus k must be divisible by both 2 and 5, say $k = 10k_1$. Then

$$7a + b = 20k_1^3.$$

Since $7a + b < 49$, we can only have $k_1 = 1$, $a = 2$, $b = 6$. Thus the radix 7 admits the cube $2626 = 13^3$ (denary $1000 = 10^3$), and is the smallest radix which admits a cube of the form *abab*.

Also solved by D. H. Browne, W. E. Buker, N. J. Fine, Daniel Finkel, Irving Kaplansky, J. B. Kelly, Walter Penney, E. D. Schell, E. P. Starke, F. E. Wood, and the proposer.

It was pointed out that the next possible radix is 38 (with $a = 11$, $b = 7$).

The *n*th Involute of a Circle

E 635 [1944, 405]. *Proposed by R. A. Rosenbaum, U.S.N.R.*

Derive parametric equations for the involute of the involute . . . (n times) of a circle (with the same starting point for each process of unwinding).

Solution by J. B. Kelly, Langley Field, Va. Taking the radius of the circle to be 1, and its center at the origin, we shall prove by induction that the nth involute has the parametric equations $x = x_n(\theta)$, $y = y_n(\theta)$, where

$$x_n(\theta) = 1 + \int_0^\theta \frac{\theta^n}{n!} \cos\left(\theta - \frac{(n-1)\pi}{2}\right) d\theta,$$

$$y_n(\theta) = \int_0^\theta \frac{\theta^n}{n!} \sin\left(\theta - \frac{(n-1)\pi}{2}\right) d\theta.$$

Since $x_0(\theta) = \cos\theta$ and $y_0(\theta) = \sin\theta$, this is true when $n = 0$. So we shall assume the result with $n = m$, and deduce the case when $n = m+1$.

The involute of the curve $x = u(\theta)$, $y = v(\theta)$ has the parametric equations

$$x = u(\theta) - s(\theta)\cos\phi(\theta), \qquad y = v(\theta) - s(\theta)\sin\phi(\theta),$$

where $\phi(\theta)$ is the angle of slope of the curve, and $s(\theta)$ is the length measured from the point where the unwinding commences, in our case (1, 0). Setting $u(\theta) = x_m(\theta)$, $v(\theta) = y_m(\theta)$, we have

$$s(\theta) = \int_0^\theta \sqrt{\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2}\, d\theta = \int_0^\theta \frac{\theta^m}{m!}\, d\theta = \frac{\theta^{m+1}}{(m+1)!},$$

$$\phi(\theta) = \arctan\left(\frac{dv}{d\theta}\Big/\frac{du}{d\theta}\right) = \theta - \frac{(m-1)\pi}{2}.$$

Hence the involute of the curve $x = x_m(\theta)$, $y = y_m(\theta)$ is given by

$$x = 1 + \int_0^\theta \frac{\theta^m}{m!} \cos\left(\theta - \frac{(m-1)\pi}{2}\right) d\theta - \frac{\theta^{m+1}}{(m+1)!} \cos\left(\theta - \frac{(m-1)\pi}{2}\right)$$

$$= 1 + \int_0^\theta \frac{\theta^{m+1}}{(m+1)!} \cos\left(\theta - \frac{m\pi}{2}\right) d\theta = x_{m+1}(\theta),$$

$$y = \int_0^\theta \frac{\theta^m}{m!} \sin\left(\theta - \frac{(m-1)\pi}{2}\right) d\theta - \frac{\theta^{m+1}}{(m+1)!} \sin\left(\theta - \frac{(m-1)\pi}{2}\right)$$

$$= \int_0^\theta \frac{\theta^{m+1}}{(m+1)!} \sin\left(\theta - \frac{m\pi}{2}\right) d\theta = y_{m+1}(\theta).$$

This completes the induction.

Also solved by the proposer, in the form

$$x = A\cos\theta + B\sin\theta, \qquad y = A\sin\theta - B\cos\theta,$$

$$A = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^{p/2}\frac{\theta^p}{p!},$$

$$B = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots + (-1)^{(q+1)/2}\frac{\theta^q}{q!},$$

$$p = 2\left[\frac{n}{2}\right], \qquad q = 2\left[\frac{n+1}{2}\right] - 1.$$

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4151. *Proposed by B. M. Stewart, Michigan State College*

Let O be a point at which a given curve has a second derivative; let the tangent and normal at O serve respectively as x, y axes, the equation of the curve becoming $y=f(x)$; and let P; (x, y) be a point on the curve, say with positive x. Denote the arc length OP by s and locate the point S: $(s, 0)$. The line SP intersects the y-axis in the point B: $(0, b)$. If R indicates $[1+(y')^2]^{3/2}/y''$, show that the limiting position of the point B as P approaches O is such that $\lim b=3 \lim R$.

This is a generalization of a problem in the calculus of Granville, Smith, and Longley, 1934, p. 177, ex. 20.

4152. *Proposed by W. J. Taylor, Washington, D. C.*

Prove the following trigonometric expansion for the binomial coefficient

$$\frac{N!}{\left(\dfrac{N+x}{2}\right)!\left(\dfrac{N-x}{2}\right)!} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m\pi x}{N}, \qquad -N < x < N.$$

4153. *Proposed by V. Thébault, Tennie, Sarthe, France*

A sphere (S), radius r, rolls on the plane of the face BCD of the tetrahedron $ABCD$ so that its center S lies on a fixed sphere concentric with (O) the circumsphere of $ABCD$. Show that: (1) The tetrahedron $BCDS$ is inscribed in a fixed sphere (S'). (2) The sphere tangent interiorly to (S) and passing through the points B, C, D envelopes a fixed sphere concentric with (O). If OS' is prolonged by $S'A'=r/2$, the sphere with center A' and passing through B, C, D is orthogonal to (S).

SOLUTIONS

A Nearly Periodic Product

4100 [1943, 569]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $N=1\ 2\ 3\ \cdots\ n$ be a number in the system of base $n+1$ where N is written with the consecutive increasing digits omitting 0. The product $P=N\cdot L$ is formed where $L=\alpha\beta$ is a number of two digits such that $\gamma=\alpha+\beta$ is a number less

than n and such that γ and n have δ as the greatest common divisor. Show that

$$P = N\cdot L = ab \cdots pqab \cdots pq \cdots ab \cdots p(n-\gamma)q,$$

the $\delta-1$ periods $ab \cdots pq$ being formed by n/δ distinct digits and the number of $(n/\delta)+1$ digits on the right contains the digit $(n-\gamma)$ between the digits p and q.

Dedicated to E. P. Starke.

Solution by E. P. Starke, Rutgers University. This problem is similar to 3851 [1940, 56] and may be proved by a slight modification of the earlier method.

(1) We note first the $(n+1)$-digit number

$$R = n\cdot N = (n+1)\cdot N - N = 111 \cdots 11101.$$

Then the proposed number $P = N\cdot L = R\cdot L/n$. We need also the $(n+2)$-digit number

$$M = R\cdot L = \alpha\gamma\gamma\gamma \cdots \gamma\beta\alpha\beta.$$

The problem is to discuss possible quotients P of the numbers M divided by n. It seems slightly simpler to treat instead the $(n+2)$-digit number

$$K = M + n\cdot L = M + \alpha\beta 0 - \alpha\beta = \alpha\gamma\gamma\gamma \cdots \gamma\gamma\beta 0,$$

of which all digits are γ except the first and the last two.

(2) Any number is congruent, mod n, to the sum of its digits. (Thus M and K are multiplies of n.)

(3) In carrying out the division of K by n, the ith and jth remainders, r_i and r_j, $i<j<n$, are alike if and only if the $(j-i+1)$-digit number $r_i\gamma\gamma\gamma \cdots \gamma(\gamma-r_j)$* is divisible by n. By (2) it is congruent, mod n, to $\gamma(j-i)$ which is divisible by n/δ. Hence the number K/n consists of δ periods of n/δ digits each and a final zero.

(4) For the divisions of the 2-digit numbers $\theta\gamma$, $\theta = 0, 1, 2, \cdots, (n-1)$ by n, different remainders imply different quotients. Thus the digits within any one period of K/n are distinct. Further, the penultimate digit of K/n must be $q = (n+1-\beta)$, since the corresponding digit of K is β.

(5) Finally $P = (K/n) - L$ has the same digits as K/n except that the last two digits are $(n-\alpha-\beta)q$ instead of $q0$. This completes the proof.

Editorial Note. The proposer remarked that this completes his theorem in *Mathesis*, 1937, pp. 7–10 where γ and n are relatively prime. He gave the example $n+1=10$, $L=24$, $\delta=3$, $n/\delta=3$

$$P = N\cdot L = 123456789 \times 24 = 2962962936.$$

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* If $r_j > \gamma$, the last two digits are $(\gamma-1)(\gamma+n+1-r_j)$.

Number Theory Determinants

4101 [1943, 637]. *Proposed by R. C. Buck, Harvard University*

Show that

$$D(n)=\begin{vmatrix} (1,1)^\lambda, & (1,2)^\lambda, & \cdots, & (1,n)^\lambda \\ (2,1)^\lambda, & (2,2)^\lambda, & \cdots, & (2,n)^\lambda \\ \vdots & \vdots & & \vdots \\ (n,1)^\lambda, & (n,2)^\lambda, & \cdots, & (n,n)^\lambda \end{vmatrix}$$

$$= (n!)^\lambda\left(1-\frac{1}{2^\lambda}\right)^{[n/2]}\left(1-\frac{1}{3^\lambda}\right)^{[n/3]}\left(1-\frac{1}{5^\lambda}\right)^{[n/3]}\cdots$$

where (i, j) means the greatest common divisor of the integers i, j.

Solution by H. S. Zuckerman, University of Washington. We can generalize the problem by considering

$$(1)\qquad D_n = |f((i,j))|,$$

where $f(x)$ is defined for all positive integral values of x. To evaluate $D(n)$ we define $\psi(k)$ by the equations

$$(2)\qquad f(l) = \sum_{k|l}\psi(k),$$

and we define a_{kl} to be 1 if l divides k, zero otherwise. We then have

$$\sum_l a_{rl}a_{sl}\psi(l) = \sum_{l|(r,s)}\psi(l) = f((r,s))$$

and we can write $D(n)=|a_{rl}|\,|a_{sl}\psi(l)|$ where the determinants on the right are of order n. Since $a_{rl}=0$ if $r<l$ and $a_{ll}=1$ we have $|a_{rl}|=1$ and $|a_{sl}\psi(l)|=\prod_{l=1}^n\psi(l)$ and hence $D(n)=\prod_{l=1}^n\psi(l)$.

Now inverting (2) by the Möbius inversion formula we have

$$\psi(l) = \sum_{k|l}\mu(k)f\left(\frac{l}{k}\right),$$

and hence the formula

$$(3)\qquad D(n) = \prod_{l=1}^n\left\{\sum_{k|l}\mu(k)f\left(\frac{l}{k}\right)\right\}.$$

For the case $f(x)=x^\lambda$ we have

$$\psi(l) = \sum_{k|l}\mu(k)\left(\frac{l}{k}\right)^\lambda = l^\lambda\sum_{k|l}\frac{\mu(k)}{k^\lambda} = l^\lambda\prod_{p|l}\left(1-\frac{1}{p^\lambda}\right),$$

and hence

$$D(n) = \prod_{l=1}^{n} l^{\lambda} \prod_{p|l}\left(1 - \frac{1}{p^{\lambda}}\right) = (n!)^{\lambda} \prod_{p\leqq n} \prod_{1\leqq l\leqq n, p|l}\left(1 - \frac{1}{p^{\lambda}}\right)$$

$$= (n!)^{\lambda} \prod_{p}\left(1 - \frac{1}{p^{\lambda}}\right)^{[n/p]}$$

which is the required result.

Formula (3) can be used to obtain interesting evaluations of (1) in other cases. For example if $f(x) = \delta(x)$, the sum of the divisors of x, it is easily seen that we have

$$\sum_{k|l} \mu(k)\delta\left(\frac{l}{k}\right) = l,$$

and hence the evaluation $D(n) = n!$

It is also possible to invert the process and determine $f(x)$ to obtain a desired $D(n)$. From (3) we have

$$\sum_{k|l} \mu(k) f\left(\frac{l}{k}\right) = \frac{D(n)}{D(n-1)}$$

Inverting this by the Möbius formula we obtain

$$f(n) = \sum_{k|n} \frac{D(k)}{D(k-1)}\cdot$$

As an example, if we wish to obtain $D(n) = a^n$ we take

$$f(n) = \sum_{k|n} a = a\tau(n)$$

where $\tau(n)$ is the number of divisors of n.

Solved also by C. D. Olds, Robert Steinberg, and the proposer.

Editorial Note. The proposer begins his solution with the definition of $J_{\lambda}(n)$, using the Möbius inversion, and the function $\delta(i/k)$ equivalent to the a_{ik} above. He remarks that Dickson's History of the Theory of Numbers, vol. 1, p. 123 gives variations, generalizations, and indications of proof of related theorems. Olds also made a similar remark and called attention to the great number of variations cited. The function $J_{\lambda}(n)$ occurs in the solutions of 4002 [1942, 621] and 4010 [1942, 696].

Digits in a Special Product

4103 [1943, 638]. *Proposed Victor Thébault, Tennie, Sarthe, France*

In the system of base $n+1$ the product $P = N\cdot L$ is formed where the number N of $n-1$ digits in descending order is $n(n-2)(n-3) \cdots 321$, and L is less than n and prime to n. If we have $L < n/2$, then the product P has n distinct digits chosen in suitable order from $0, 1, 2, \cdots, n$, and the missing digit is $n-L$.

If $L>n/2$, the digit $n-1$ appears twice in P, and the missing digits are n and $(n-1-L)$, the remaining digits being distinct.

Dedicated to E. P. Starke.

Solution by E. P. Starke, Rutgers University. We may put $N=R-M$, where R is the n-digit number $11000\cdots 0$ and M is the $(n-1)$-digit number $123\cdots(n-2)n$. Note that $n\cdot M$ is the n-digit number $111\cdots 1$, so that

$$P = L\cdot N = L\cdot R - L\cdot M$$

is the n-digit number $L\cdot R=LL000\cdots 0$ minus the $(n-1)$-digit number $L\cdot M=LLL\cdots L/n$.

A discussion of $L\cdot M$ analogous to the solution of 3851 [1940, 56] follows.

(1) Any number is congruent, mod n, to the sum of its digits.

(2) If $n\cdot L\cdot M=LLL\cdots L$ is divided by n, no two of the partial divisions can produce the same remainder. For if the ith and jth remainders, r_i and $r_j(i<j<n)$ could be equal, the $(j-i+1)$-digit number $r_iLL\cdots L(L-r_j)$* would be divisible by n whereas by (1) it is congruent mod n to $L(j-i)$ with $j-i<n$ and L relatively prime to n. Similarly no remainder r_i can be zero since the number $LL\cdots L$ of less than n digits is not divisible by n.

(3) For the divisions of the two-digit numbers $\theta L(\theta=0, 1, 2, \cdots, n-1)$ by n, different remainders imply different quotients. Thus the $n-1$ digits of $L\cdot M$ are all distinct. The quotient 0 does not occur. The last quotient is $(n-L+1)$. The first partial division, $(n+1)L+L=nL+2L$ by n, indicates that the first digit of $L\cdot M$ is (a) L when $2L<n$ and (*b*) $(L+1)$ when $2L>n$.

(4) Besides 0, let d be the digit missing from $L\cdot M$. Then $L\cdot M+d\equiv\sum_{k=1}^{n}k$, but $M+(n-1)\equiv\sum_{k=1}^{n}k$, whence $(L-1)\sum k+d+L\equiv 0$. From $\sum k=n(n+1)/2$ we see that $\sum k\equiv 0$ if n is odd; while for n even, L (prime to n) is odd and $L-1$ is even. Thus always $(L-1)\sum k\equiv 0$ and then $d+L\equiv 0$ and $d=n-L$. To summarize: $L\cdot M$ has the form

$$\text{(a)}\quad Lg_2g_3\cdots g_{n-2}(n-L+1), \qquad \text{(b)}\ (L+1)g_2g_3\cdots g_{n-2}(n-L+1),$$

according as (a) $L<n/2$ or (b) $L>n/2$, where g_i, $i=2, 3, \cdots, n-2$, are the remaining digits. The digits are distinct and 0 and $n-L$ are missing.

(5) To subtract $L\cdot M$ from $L\cdot R$ we first change the form of $L\cdot R$ by "borrowing" and have in case (a),

$$\begin{array}{lllll}
L\cdot R = (L-1)(L+n) & & & n\ n\cdots n(n+1) & \\
L\cdot M = & L & & g_2g_3\cdots g_{n-2}(n-L+1) & \\
\hline
P = (L-1) & n(n-g_2)(n-g_3)\cdots(n-g_{n-2}) & & L & .
\end{array}$$

Since 0, $n-L$, L, $n-L+1$ are not included among the g_i, we see that n, L, $n-L$, $L-1$ are the digits not included among the $(n-g_i)$, but $L-1$, n, and L occur at the ends. Thus the digits of P are all distinct and only $n-L$ is missing.

* If $r_j>L$, the last two digits are $(L-1)(L+n+1-r_j)$.

Case (b) is similar:

$$\begin{array}{rllllll}
L\cdot R = & (L-1)(L+n) & n & n & \cdots & n & (n+1) \\
L\cdot M = & & (L+1) & g_2 & g_3 & \cdots & g_{n-2}(n-L+1) \\
\hline
P = & (L-1)(n-1)(n-g_2)(n-g_3)\cdots(n-g_{n-2}) & & & & & L \quad .
\end{array}$$

Since 0, $n-L$, $L+1$, $n-L+1$ are not among the g_i, then n, L, $n-L-1$, $L-1$ are missing from the $(n-g_i)$. Including $L-1$, $n-1$ and L, we see that in P the digit $n-1$ will occur twice and that n and $n-L-1$ will be missing.

Editorial Note. The proposer gave for $n+1=10$

$$P = 97654321 \times 4 = 390617284,$$
$$P = 97654321 \times 7 = 683580247.$$

The Möbius Function

4104 [1944, 49]. *Revised. Proposed by E. T. Bell, California Institute of Technology*

Two symmetric functions, $M(x_1, \cdots, x_n)$, $S(x_1, \cdots, x_n)$, of n non-negative integers $x_1, \cdots, x_n$ are defined as follows

$$M(x_1, \cdots, x_n) \equiv M'(x_1)\cdots M'(x_n),$$

in which $M'(x)=1, -1, 0$, according as $x=0$, $x=1$, $x>1$; if $S_j(x_1, \cdots, x_n)$ is the jth elementary symmetric function of $x_1, \cdots, x_n$,

$$S(x_1, \cdots, x_n) \equiv 1 + \sum_{j=1}^{n} jS_j(x_1, \cdots, x_n).$$

Prove that $\sum M(x_1-b_1, \cdots, x_n-b_n)S(b_1, \cdots, b_n)=$ the number of integers >0 in the set $x_1, x_2, \cdots, x_n$, and $=1$, if $x_1=x_2=\cdots=x_n=0$, where the summation refers to all integers b_i such that

$$0 \leqq b_i \leqq x_i, \qquad i = 1, 2, \cdots, n.$$

Solution by N. J. Fine, Indianapolis, Ind. Let $p_1, \cdots, p_n$ be any set of distinct primes, and let $N=p_1^{x_1}\cdots p_n^{x_n}$. By definition, $M(x_1, \cdots, x_n)=\mu(N)$. For any divisor m of N, define $f(m)$ as the number of distinct prime factors of m, unless $m=1$; let $f(1)=1$. Further, if d is a divisor of N, $d=p_1^{b_1}\cdots p_n^{b_n}$, define $g(d)=\sum_{m|d}f(m)$. In this sum, group all those m with exactly j distinct factors; the number of such m is clearly $S_j(b_1, \cdots, b_n)$. Adding 1 for $m=1$, we have

$$g(d) = 1 + \sum_{j=1}^{n} jS_j(b_1, \cdots, b_n) = S(b_1, \cdots, b_n).$$

Now, by the Möbius inversion formula,

$$f(N) = \sum_{d|N} \mu(N/d)g(d) = \sum M(x_1-b_1, \cdots, x_n-b_n)S(b_1, \cdots, b_n).$$

Therefore the required sum is equal to the number of *positive* integers in the set $x_1, \cdots, x_n$, or unity if all are zero.

Solved also by the proposer.

Tetrahedron and Monge Point

4106 [1944, 49]. *Proposed by N. A. Court, University of Oklahoma*

If the Monge point of a tetrahedron lies on the circumsphere, show that (a) The line joining the circumcenter to the centroid of a face is equal to half the corresponding median of the tetrahedron; (b) Each median subtends a right angle at the Monge point. Conversely.

Solution by Howard Eves, Syracuse University. Let the tetrahedron be $ABCD$ with circumcenter O, centroid G, and Monge point M. Let G_a be the centroid of face BCD. Then A, O, G, M, G_a are coplanar, $OM=OA$, $OG=GM$, $AG=3GG_a$. Let K be the centroid of triangle AOM.

(a) OG_aMK is a parallelogram because OM and KG_a bisect each other. Therefore $OG_a=KM=KA=\frac{1}{2}AG_a$.

(b) G_aM is parallel to OK, and therefore perpendicular to AM.

Conversely, if G_aM is perpendicular to AM, then OK is perpendicular to AM and triangle AOM is isosceles, with $OM=OA$. Thus M is on the circumsphere.

We have actually established a stronger converse than is implied by the wording of the problem. We have shown that if *one* median of a tetrahedron subtends a right angle at the Monge point then all the medians do likewise, and the Monge point lies on the circumsphere.

Solved also by L. M. Kelly, P. D. Thomas and the proposer.

Spheres and Powers

4109 [1944, 96]. *Proposed by N. A. Court, University of Oklahoma*

Prove that the sum of the n^2 powers of n given points with respect to the n spheres having for diameters the n segments joining the given points to a variable point in space is constant.

Solution by Louis Brand, University of Cincinnati. LEMMA: *The power of a point P with respect to a sphere having AB as diameter is $\overrightarrow{PA}\cdot\overrightarrow{PB}$.* For if C is the center of the sphere, $\overrightarrow{PA}\cdot\overrightarrow{PB}$ can be written

$$(\overrightarrow{PC}+\overrightarrow{CA})\cdot(\overrightarrow{PC}+\overrightarrow{CB}) = (\overrightarrow{PC}+\overrightarrow{CA})\cdot(\overrightarrow{PC}-\overrightarrow{CA}) = (PC)^2-(CA)^2.$$

Let $P_1, P_2, \cdots, P_n$ be the n given points, P the variable point; and let p_{ij} denote the power of P_i with respect to the sphere on P_jP as diameter. Then, from the lemma above,

$$p_{ij}+p_{ji} = \overrightarrow{P_iP_j}\cdot\overrightarrow{P_iP}+\overrightarrow{P_jP_i}\cdot\overrightarrow{P_jP} = \overrightarrow{P_iP_j}\cdot\overrightarrow{P_iP_j};$$

hence the sum of the n^2 powers p_{ij} is equal to the sum of the squares of the $\frac{1}{2}n(n-1)$ distances between the points P_i.

This sum can be expressed in another form. For if $\mathbf{r}_i$ and $\mathbf{r}^*$ denote position vectors of the points P_i and of their mean center P^*, and $\mathbf{r} = \overrightarrow{OP}$,

$$\begin{aligned}\sum_{ij} p_{ij} &= \sum_{i=1}^{n}\sum_{j=1}^{n} (\mathbf{r}_j - \mathbf{r}_i)\cdot(\mathbf{r} - \mathbf{r}_i) \\ &= n\sum_{i=1}^{n} (\mathbf{r}^* - \mathbf{r}_i)\cdot(\mathbf{r} - \mathbf{r}_i) \\ &= n\left(n\mathbf{r}^*\cdot\mathbf{r} - n\mathbf{r}^*\cdot\mathbf{r}^* - n\mathbf{r}^*\cdot\mathbf{r} + \sum_1^n \mathbf{r}_i\cdot\mathbf{r}_i\right) \\ &= n\sum_1^n \mathbf{r}_i^2 - n^2(\mathbf{r}^*)^2.\end{aligned}$$

If the origin is chosen at P^*,

$$\sum_{ij} p_{ij} = n\sum_1^n \mathbf{r}_i^2.$$

Solved also by H. Eves, Irving Kaplansky, L. M. Kelly, C. E. Springer, Robert Steinberg, P. D. Thomas, J. A. Zilber, and the proposer.

Eves remarked that the theorem and proof are valid for any euclidean space.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. C. L. Clark of the University of Virginia has been appointed to an assistant professorship at Oregon State College.

Assistant Professor W. M. Davis of Cornell College, Mt. Vernon, Iowa, has been promoted to a professorship.

Adjunct Professor W. S. Krogdahl of the University of South Carolina has been promoted to an associate professorship.

Professor C. J. Rees of the University of Delaware has been granted leave of absence for civilian service with the Fourteenth Air Force in China. In his absence Assistant Professor B. C. Webber is acting chairman of the mathematics department.

Professor H. A. DoBell of the New York State College for Teachers at Albany died December 8, 1944.

Professor W. P. Ott of the University of Alabama died December 25, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

NEW SELECTIVE SERVICE REGULATIONS

Upon December 12, 1944, the Selective Service System released a statement pertaining to a stricter application of Local Board Memorandum No. 115 on occupational classification. The first part of this statement is reproduced below. One sentence has been emphasized by the use of italics because of the great amount of discussion which it has caused.

"Since May 1944,* the provisions of Local Board Memorandum No. 115 have been strictly applied for registrants ages 18 through 25 and liberally applied for registrants ages 26 through 37 in order to fill calls substantially from younger men capable of the highest degree of efficiency under combat conditions. Estimates of available men indicate that the armed forces calls after February 1, 1945 cannot be filled substantially from men ages 18 through 25. The larger number of American Divisons now actually engaged in combat, and the continuing pressure against the enemy has increased the requirements for physically fit soldiers and sailors. For these reasons, it will be necessary to induct increasing numbers of men from the older age group.

"The larger number of American Divisions now actually engaged in combat, and the continuing pressure against the enemy has also greatly expanded the requirements for military supplies of all sorts. This has resulted in a sharp upwards revision of many war production schedules, requiring a substantial number of additional workers in direct war production activities.

"Because of these requirements, it is imperative that local boards fill calls for the armed forces, and that they be filled by the reclassification, as it becomes necessary, of men in the older age groups who do not meet a stricter application of the provisions of Local Board Memorandum No. 115 in the light of the immediate urgencies for men in the armed forces and the civilian war effort.

"In applying the tests for occupational deferment for registrants ages 26 through 37, greater consideration will be given to registrants now engaged, or who become engaged, in war production or in support of the war effort, than to those engaged in activities not supporting the immediate prosecution of the war. *Registrants of lesser skills may be more important in war production or activities directly supporting the war effort than those of greater skills in other activities not in direct support of the war effort.*

* Note the article, "Deferment of Mathematicians Ages 26 through 37," in this MONTHLY for August–September, 1944.

"Under the stricter interpretation of Local Board Memorandum No. 115 for a registrant age 26 through 37, local boards will consider the continuance of occupational deferment on the basis of general information available, specific information in the registrant's file, representations by other Federal Government agencies, and the local board's own knowledge of the relative importance of civilian activities and the labor supply conditions existing in the area in which the registrant is working."

NOTES ON THE NAVY V-12 PROGRAM

The following quotation is from Navy V-12 Bulletin, No. 280, issued December 23, 1944.

"Screening procedures used in preceding terms for assigning subject trainees to upper-level curricula will not be repeated during the current term. A new plan for the assignment of these students to appropriate upper-level courses will be announced at an early date. The Commanding Officers of Navy V-12 Units are directed to inform trainees and academic authorities that the new plan will provide for the continuation in the V-12 Program of all trainees now on board, under the same general conditions as heretofore. No trainee now under instruction will be separated from the Program for any reason other than academic failure, lack of officer-like qualities, breach of discipline, or lack of physical qualifications. It is anticipated that the new plan, when announced, will give subject students greater rather than less freedom in the selection of academic majors and the election of courses of study beyond the second term."—L. E. Denfeld, The Assistant Chief of Naval Personnel.

Recent communications from the Navy Department indicate that the Navy intends to continue the V-12 Program after July 1, 1945. Institutional participation in the Program after that date will undoubtedly depend on several factors, including the result of Congressional debate upon a proposal to extend the Naval ROTC Program. The following excerpts from Navy V-12 Bulletin, No. 278, tell of some plans which have been developed for the V-12 term, starting July 1, 1945.

"This Bulletin outlines procedures to be followed in the selection and transfer to the Navy V-12 Program of applicants to be assigned to college training on July 1, 1945. It is anticipated that those men selected will be ultimately transferred to the Naval Reserve Officer Training Corps.... As it is anticipated that the entire available quota for this sixth increment of the Navy V-12 Program will consist of approximately 2000 enlisted men, it is imperative that only outstanding applicants who are fully qualified in all respects be recommended for such training by Commanding Officers. The primary purpose of the Program is to give prospective Naval Officers appropriate training at the college level in those fields of study most useful to the Navy in accordance with its needs. Length of training offered to each successful applicant will depend on (a) his

previous college education, if any, (b) the type of specialized courses for which he is qualified and to which in the discretion of the Navy he is assigned, and (c) his continued demonstration of satisfactory scholarship and officer-like qualities. Applicants will be permitted to express a preference for the type of duty (Deck, Engineering and Supply Corps only) toward which they wish their training to point.... Commanding Officers are directed to consider for transfer to the Navy V-12 Program only those applicants who (a) apply voluntarily for such training, (b) understand fully that an extended period of college academic training is involved, (c) meet all prescribed requirements, and (d) are considered to be definitely outstanding for training as Officer Candidates.

"Each applicant for this program must meet the following (academic) requirements: (1) Be a high school graduate or have been in attendance at or accepted for admission by an accredited college or university. High school or college transcript must show successful completion of courses in elementary algebra and plane geometry; additional courses in mathematics and physics are desirable. (2) Have passed the O'Rourke General Classification Test (given prior to 15 June 1943) with a score of 88 or above; or the new General Classification Test, Forms 1, 2, 3, or 1s (given subsequent to June 15, 1943) with a score of 60 or above. (In the case of an applicant for whom no General Classification Test score is available, it is recommended that an appropriate written and/or oral test be given to determine whether he is properly qualified to pursue successfully a college curriculum generally considered to be more exacting and more difficult than a normal course at a liberal arts college, and to obviate the selection of men who would subsequently have to be returned to general duty for failure to meet minimum educational requirements.)... Enlisted men who have had more than two years of college cannot be recommended for this program; men who have successfully completed two or more years of college work are eligible for the Reserve Midshipman Program, Class V-7."

ACTIVITIES OF THE COMMITTEE ON EXAMINATIONS

W. T. Reid, Northwestern University

In June, 1944, Professor M. H. Stone, Chairman of the War Policy Committee of the A. M. S. and the M. A. A., appointed a Subcommittee on Examinations to advise the Examinations Staff of the United States Armed Forces Institute in connection with examinations in mathematics at the college level to be sponsored by the U. S. A. F. I. This Subcommittee was instituted at the request of Professor R. W. Tyler, Director of the Examinations Staff for the U. S. A. F. I., following initial contact between Professor Tyler and the War Training Programs Subcommittee of the War Policy Committee. The following individuals were appointed to this Subcommittee: Professors Ralph Beatley (Harvard); L. L. Dines (Carnegie Institute of Technology); W. L. Hart (Minnesota); C. C. MacDuffee (Wisconsin); W. T. Reid (Northwestern; Chicago, at time of appointment), Chairman.

At a meeting of the Subcommittee held early in July 1944, it was decided to concentrate on perfecting the U. S. A. F. I. subject examinations for College Algebra, Plane Trigonometry, Analytic Geometry, and Calculus. The Subcommittee devoted its efforts to this task from that time until November 30, 1944, when the examinations construction project of the U. S. A. F. I. was terminated by the War Department.

The purpose of these subject examinations is to provide an instrument for evaluating the work of men in courses taken under the auspices of the U. S. A. F. I. A large number of men have received some mathematical training while in the Armed Forces, and these examinations are planned to help meet the need for some means of providing fair credit and proper placement for these men when they return to collegiate institutions. The examinations are of the multiple choice type; the revised forms prepared by this Subcommittee have been planned for examination periods three hours in length. The selection of material covered by these examinations was based on results of an extensive sampling of the opinions of college departments of mathematics in regard to the content of courses in the considered subjects. These opinions were secured, at the suggestion of Professor Tyler, by circulating in June, 1944, a questionnaire prepared by Professor Reid, with the aid of Professor Hart.

Each of these examinations has been prepared in two forms. One of these forms, referred to as Form A, is the confidential form administered only under safeguards provided by the U. S. A. F. I. The War Department has made arrangements for the Form A of the examinations to be given under the following three conditions: firstly, by the Armed Forces Institute for men and women in service; secondly, by the Veterans Administration for those who seek vocational and educational guidance from one of the Veterans Advisement Centers; thirdly, by the Veterans Testing Service which is being set up on the campus of the University of Chicago by the American Council on Education. Copies of the other form, Form B, of these examinations may be purchased from the American Council on Education by faculty members of educational institutions; orders for these tests should be addressed to Cooperative Test Service, 15 Amsterdam Avenue, New York, N.Y. Some institutions may desire to use results on Form B of the examinations in granting credit for work done with the U. S. A. F. I. If an institution desires to use Form A of one of these examinations in accrediting the work of a given individual, however, application should be made to the Veterans Testing Service, The University of Chicago, 6010 Dorchester Avenue, Chicago 37, Ill. In response to such an application a copy of the examination will be issued to a duly appointed representative of the institution for administration, and subsequent return of examination and answer sheet to Chicago for scoring and recording.

In view of the late date of preparation, the forms of these subject examinations prepared by this Subcommittee have not been standardized by the Armed Forces Institute. Both forms A and B of these examinations will be standardized in the near future, however, under a project that is being undertaken jointly

by the War Department and the Cooperative Test Service of the American Council on Education.

It is to be expected that various schools will desire to determine supplementary data on Form B of these tests for their own information and use. Such a procedure would be highly desirable, and it is to be hoped that representative colleges and universities will secure copies of Form B of these examinations for this purpose.

It is to be emphasized that the official responsibilities of this Subcommittee are limited to the above-mentioned subject examinations. The examinations in mathematics at the high school level were under the supervision of consultants nominated by officers of the National Council of Teachers of Mathematics. Outside the province of the Subcommittee are also certain end-of-course tests in mathematics at the college level prepared for the U. S. A. F. I.; these examinations are non-accreditation examinations for correspondence courses and class instruction on mathematics courses offered by the Institute.

The procedure of examination construction followed by the Subcommittee was as follows:

(a) One member of the committee assumed the responsibility of constructing the examination in a given course;

(b) subsequent to the construction of an examination, two other members served as official critics, going over thoroughly all questions and responses, and offering criticisms and suggestions;

(c) after the member responsible for the construction had made revisions in the light of the criticisms of the official critics, the examination was submitted to the group as a whole for further criticism, and, finally, for approval.

Following are descriptions of the individual examinations. The examinations strive to test the examinee's appreciation and understanding of the fundamental concepts, as well as his skill in techniques.

ALGEBRA—COLLEGE LEVEL

Sec. I. Fundamental Operations, Radicals, Exponents, and Linear Equations in One Unknown. 13 items.

Sec. II. Systems of Linear Equations in One and Two Unknowns; Quadratic Equations and Related Topics. 20 items.

Sec. III. Binomial Theorem, Progressions, Variations, and Complex Numbers. 14 items.

Sec. IV. Inequalities and the Theory of Equations. 14 items.

Sec. V. Determinants, Permutations, Combinations, and Probability. 9 items.

PLANE TRIGONOMETRY

Sec. I. Right Triangles and Applications. 10 items.

Sec. II. Logarithms and the use of Tables. 10 items.

Sec. III. Oblique Triangles and Applications. 9 items.
Sec. IV. General Angle, Variation of Functions, Graphs. 14 items.
Sec. V. Identities and Equations. 17 items.

ANALYTIC GEOMETRY

Sec. I. Points and Lines. 15 items.
Sec. II. General Loci and the Conic Sections. 28 items.
Sec. III. Transformation of Coordinates; Parametric Equations. 9 items.
Sec. IV. Polar Coordinates. 11 items.
Sec. V. Space Geometry. 12 items.

[If examinee is qualified to take only Secs. I–IV on Plane Analytic Geometry, it is recommended that examination time be reduced to 2 hours, 25 min.]

CALCULUS I. DIFFERENTIAL CALCULUS (NO INTEGRATION INCLUDED)

Sec. I. Functions; Definition and Interpretation of Derivatives. 11 items.
Sec. II. Formal Differentiation. 15 items.
Sec. III. Applications of Differentiation. 24 items.
Sec. IV. Infinite Series; Partial Derivatives. 16 items.

CALCULUS II. INTEGRAL CALCULUS (FOLLOWING INITIAL COURSE CONTAINING NO INTEGRATION)

Sec. I. Indefinite Integrals. 25 items.
Sec. II. Definite Integrals. 25 items.
Sec. III. Double and Triple Integrals. 10 items.

Section IV of the above described examination in Calculus I is issued as a supplement, thus permitting interchange with an alternate Sec. IV, containing 16 problems on integration. With this interchange, there is provided the following examination:

CALCULUS I. DIFFERENTIAL CALCULUS (INCLUDING SIMPLE INTEGRATION)

Sec. I. Functions; Definition and Interpretation of Derivatives. 11 items.
Sec. II. Formal Differentiation. 15 items.
Sec. III. Applications of Differentiation. 24 items.
Sec. IV. Integration. 16 items.

The provision of this alternate form of the examination in the first course in Calculus seemed desirable in view of the varied manner in which Calculus is taught throughout the country. An accompanying examination for the one in Calculus I, including simple integration, might be obtained by deleting sixteen problems from Secs. I and II of the above described examination in Calculus II, and adjoining the supplement on Infinite Series and Partial Derivatives. If a man has taken enough work under the auspices of the U.S.A.F.I. to present himself for examination in the two courses in Calculus described above, however, it is advised that he take Calculus I, as first described, and Calculus II, although the institution to which he goes may present the topics in calculus in a different order.

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-eighth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado State College of Education, Greeley, Colorado, on April 14 and 15, 1944. There were three sessions, the final session being a joint meeting with the Mathematics Section of the Eastern Division of the Colorado Education Association. Professor A. E. Mallory, Chairman of the Section, presided at each of the sessions.

The attendance was thirty-two, including the following twelve members of the Association: C. F. Barr, A. G. Clark, J. C. Fitterer, Leota C. Hayward, A. J. Kempner, Claribel Kendall, W. J. LeVeque, A. J. Lewis, A. E. Mallory, Greta Neubauer, A. W. Recht, E. C. Varnum.

At the business meeting the following officers were elected for the coming year: Chairman, Jack Britton, University of Colorado; Vice-Chairman, C. F. Barr, University of Wyoming.

The following papers were presented:

1. *Mathematics in the A. S. T. P.*, by Professor A. G. Clark, Colorado State College of Agriculture and Mechanic Arts.

2. *Vibrating membranes*, by August Newlander, University of Denver, introduced by the Secretary.

It was pointed out that the vibrations of a circular membrane are in many respects similar to the vibrations of a string. The displacement of a particular point of the membrane can be obtained by solving a partial differential equation of the second order. The solution of the differential equation can be expressed by means of an infinite series involving sines, cosines, and Bessel functions. The determination of certain constants dependent upon the boundary conditions involves the use of Fourier series and the Fourier-Bessel expansion of a function. After all constants have been determined, the result is an expression giving the displacement of any point of the membrane in terms of its position and the time after releasing the membrane from rest.

3. *Reduction of inverse tangents to integral arguments*, by Professor E. C. Varnum, University of Wyoming.

By a study of the operation $(a-b)/(1-ab)$ the speaker developed formulas by which inverse tangents of rational arguments may be reduced to those having integral arguments, the latter having been well tabulated in recent projects.

4. *A new definition of the Gamma function*, by Mrs. Margaret Matchett, University of Denver, introduced by the Secretary.

The speaker remarked that the Gamma function is uniquely defined by its

functional equation and the condition of logarithmic convexity. It was also stated that this definition yields an explicit expression for the Gamma function as an infinite product.

5. *Suggested changes in the content of high school mathematics*, by Ruth Hoffman, Denver Public Schools, introduced by the Secretary.

6. *Trends in grade placement of arithmetic fundamentals*, by L. B. Garner, Cameron School, Greeley, Colorado, introduced by Professor Mallory.

7. *I know better than I teach, now*, by Professor A. W. Recht, University of Denver.

In this address it was suggested that every teacher constantly fails to reach standards of teaching which he knows to be better. Twelve points for good teaching were submitted with the suggestion that they be used for periodic check-ups. One of these points, that of keeping the student informed of his standing day by day, was explained in detail. A method was shown by which the daily running averages of students in a whole class could be written down in two or three minutes from one setting of a slide rule.

8. *Early computation with Hindoo-Arabic numbers*, by Professor A. E. Mallory, Colorado State College of Education.

A. J. LEWIS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, Brooklyn, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

THE CARUS MONOGRAPHS

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935; Fourth Impression, 1944.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930; Third Impression, 1943.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929; Third Impression, 1936; Fourth Impression, 1943.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSORS DAVID EUGENE SMITH and JEKUTHIEL GINSBURG. (First Impression, 1934.)

No. 6. *Fourier Series and Orthogonal Polynomials,* by PROFESSOR DUNHAM JACKSON. (First Impression, 1941.)

No. 7. *Vectors and Matrices,* by PROFESSOR C. C. MACDUFFEE. (First Impression 1943.)

Price $1.25 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary, MCGRAW HALL, Cornell University, ITHACA, N.Y.

Additional copies for members, and copies for non-members, may be purchased from the Open Court Publishing Co., La Salle, Illinois, at the regular price of $2.00 per copy.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in November 1944 to Professor R. H. Cameron for his paper, "Some introductory exercises in the manipulation of Fourier transforms," published in the *National Mathematics Magazine,* vol. 15 (1941), pp. 331-356.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

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EUCLIDEAN METRIC INVARIANTS OF CONICS BY TENSOR ALGEBRA

T. C. DOYLE, Stanford University

1. Introduction. This paper originated in a desire to furnish an elementary illustration of the scope and methods of tensor algebra to non-specialists in the field. Because of the familiarity of the transformation theory of the conics, this together with the formation of their euclidean metric invariants by tensor algebra has been chosen as the illustrative example. This paper will also serve as a link between the Aronhold-Clebsch symbolism of classical algebraic invariant theory and the modern tensor notation.

2. Coordinate transformations. The vanishing of the quadratic form

$$f_0 = (ax)^2 = (a_0x^0 + a_1x^1 + a_2x^2)^2 = g_{00}x^0x^0 + g_{11}x^1x^1 + g_{22}x^2x^2 \\ + 2g_{01}x^0x^1 + 2g_{02}x^0x^2 + 2g_{12}x^1x^2$$

constitutes the equation in homogeneous coordinates x^A, $A = 0, 1, 2$, of a general conic section. The expression $(ax)^2$ is purely symbolic, the a_A having no numerical significance when written alone but assuming the meaning $a_Aa_B = g_{AB}$ when written as a quadratic product. This notation was adopted by Aronhold and Clebsch, and most of the literature pertaining to algebraic invariant theory prior to the advent of tensor algebra employed this symbolism.

The projective geometry of the conic C, $f_0 = 0$, consists of those properties of C which appear the same to an observer in any one of the totality of coordinate systems $\bar{x}^A$ related to the system x^A by the projective transformations

$$x^A = \sum_0^2 a^A_Q\bar{x}^Q = a^A_Q\bar{x}^Q, \qquad |a^A_B| \neq 0,$$

and their inverses

$$\bar{x}^A = A^A_Qx^Q, \qquad a^A_QA^Q_B = \delta^A_B, \qquad \delta^A_B = 0, \quad A \neq B, \qquad \delta^A_B = 1, \qquad A = B.$$

Let $L = u_Qx^Q$ be a linear form whose vanishing gives the equation of the line L. Under $x \to \bar{x}$,

$$L = u_Qa^Q_A\bar{x}^A = \bar{u}_A\bar{x}^A = \bar{L},$$

so that $\bar{u}_A = u_Qa^Q_A$. Transforming f_0 in a similar way we obtain the transformation law of the coefficients g_{AB} of C,

$$x^A = a^A_Q\bar{x}^Q, \qquad \bar{u}_A = u_Qa^Q_A, \qquad \bar{g}_{AB} = g_{QR}a^Q_Aa^R_B,$$

$$\bar{x}^A = A^A_Qx^Q, \qquad u_A = \bar{u}_QA^Q_A, \qquad g_{AB} = \bar{g}_{QR}A^Q_AA^R_B.$$

We define as a *contravariant vector* λ^A, a *covariant vector* μ_A, and a *covariant tensor* μ_{AB} any quantity transforming cogrediently with x^A, u_A and g_{AB} respectively.

By an extension of this idea we are led to tensors of any rank. For example, if both ϵ^{ABC} and ϵ_{ABC} have the values $+1$ and -1 in *all coordinate systems* according as the indices form an even or odd permutation of the natural order 0, 1, 2, and the value 0 otherwise, then

$$|a|\epsilon^{ABC} = \bar{\epsilon}^{QRS} a_Q^A a_R^B a_S^C, \qquad \bar{\epsilon}_{ABC} = |a|^{-1} \epsilon_{QRS} a_A^Q a_B^R a_C^S,$$

and ϵ^{ABC} and ϵ_{ABC} are the components of contravariant and covariant tensors of rank three. But furthermore the components of these tensors are the *same in all coordinate systems* and for this reason they are called *invariant* tensors. The presence of the determinental factor $|a|$ is signified by calling ϵ^{ABC} and ϵ_{ABC} *relative* tensors of *weights* $+1$ and -1 respectively.

The contraction of a covariant vector μ_A with a contravariant vector λ^A gives

$$\mu_Q\lambda^Q = \bar{\mu}_R A_S^R a_Q^S \bar{\lambda}^Q = \bar{\mu}_Q \bar{\lambda}^Q$$

so that an invariant form results from this process. All invariants of tensor algebra may be formed by this method of contracting a covariant with a contravariant index.

3. Projective invariants of the conic. We display in this section the projective invariants of a point x^A, a line u_A, and the conic g_{AB}. These will arise as the simultaneous invariants of the three ground forms

$$f_0 = (aX)^2 = (bX)^2 = (cX)^2, \quad f_1 = (uX) = u_Q X^Q, \quad F_1 = (xU) = x^Q U_Q, \tag{3.1}$$

where for the moment X^A and U_A represent general point and line coordinates.

By the first fundamental theorem of the symbolic method, Clebsch[1] has shown that all the projective invariants of the forms (3.1) which are integral and rational in the coefficients of the forms are expressible in terms of the symbolic factors (ax), (ux), (abc), and (abu). The latter are formed by all possible contractions of a covariant with a contravariant linear symbol and by all possible determinants of exclusively covariant or exclusively contravariant linear symbols. It is well known that there are but four basic invariants of the system (3.1), namely

$$\begin{aligned} &D = (abc)^2 = 6|g_{AB}|, \qquad f_0 = (ax)^2 = g_{QR}x^Q x^R, \qquad J_0 = (ux) = u_Q x^Q, \\ &f'' = (abu)^2 = |g_{AB}| g^{QR} u_Q u_R, \qquad (g_{BQ} g^{QA} = \delta_B^A), \end{aligned} \tag{3.2}$$

and that these constitute a *complete basic system of invariants* of the forms (3.1) in the sense that any rational integral invariant of (3.1) is expressible in terms of these four.

The invariance of $|g_{AB}|$ follows at once from $\bar{g}_{AB} = g_{QR} a_A^Q a_B^R$ on forming the determinant of both sides, $|\bar{g}_{AB}| = |a|^2 |g_{AB}|$. This same invariant is also expressible as a result of solving the last of (3.2) for g^{AB},

$$2g^{AB} = \left| g_{AB} \right|^{-1} \epsilon^{AQR} \epsilon^{BST} g_{QS} g_{RT}$$

and contracting with g_{AB} to obtain

$$6 \left| g_{AB} \right| = \epsilon^{AQR} \epsilon^{BST} g_{AB} g_{QS} g_{RT}.$$

4. The euclidean metric group. The projective group of transformations a^A_B may be restricted to become the euclidean metric group E of rotations and translations. This is done by insisting that the circular points $(0; 1, \pm i)$ defined as the singular line conic

$$\delta^{AB} u_A u_B = u_1^2 + u_2^2 = (0u_0 + u_1 + iu_2)(0u_0 + u_1 - iu_2) = 0$$

have the same coordinates to within a factor of proportionality in all systems, and that the coordinates of any point $(x^0; x^1, x^2)$ not on the line at infinity, L_∞, with the equation $\delta^0_Q x^Q = x^0 = 0$, have the coordinates $(1; x^1, x^2)$.

The resulting transformation equations are $x^A = a^A_Q \bar{x}^Q$, where

$$\left\| \begin{matrix} a^0_0 & a^0_1 & a^0_2 \\ a^1_0 & a^1_1 & a^1_2 \\ a^2_0 & a^2_1 & a^2_2 \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 & 0 \\ a & \cos\theta & -\sin\theta \\ b & \sin\theta & \cos\theta \end{matrix} \right\|.$$

Thus E is determined as a subgroup of the projective group by insisting upon the formal invariance of $\phi = (lX) = X^0$ and $\Phi = (\tau U)^2 = U_1^2 + U_2^2$ in all coordinate systems. The transformation equations of g_{AB} under the two subgroups of E are

$$\begin{array}{lll} & \text{(translation)} & \text{(rotation)} \\ & \bar{g}_{00} = g_{00} + g_{rs} a^r_0 a^s_0 & \bar{g}_{00} = g_{00} \\ (4.1) & \bar{g}_{0i} = g_{0i} + g_{ri} a^r_0 & \bar{g}_{0i} = g_{0s} a^s_i \\ & \bar{g}_{ij} = g_{ij} & \bar{g}_{ij} = g_{rs} a^r_i a^s_j. \end{array}$$

5. The euclidean metric ground forms. E is determined as a subgroup of the projective group by stipulating that the forms $\phi = X^0$ and $\Phi = U_1^2 + U_2^2$ remain unaltered. By an application of the adjunction principle of algebraic invariant theory a complete basic system of invariants of a point x^A, a line u_A, and a conic g_{AB} under E will be found by adjoining the two forms ϕ and Φ to the forms (3.1),

$$(5.1) \qquad f_0 = (aX)^2, \quad f_1 = (uX), \quad \phi = (lX), \quad F_1 = (xU), \quad \Phi = (\tau U)^2 = (\nu U)^2,$$

and *asking for a complete basic system of invariants of these forms under the projective group.*

By the same application of the first fundamental theorem of the symbolic method as introduced in section 3, all the projective invariants of the augmented system of forms (5.1) may be built from the symbolic factors

$$(ax), \quad (ux), \quad (lx), \quad (a\tau), \quad (u\tau), \quad (abc), \quad (abu), \quad (abl), \quad (aul), \quad (x\tau\nu).$$

But τ and ν may occur only in quadratic combinations of the type

$$(a\tau)(b\tau) = a_1b_1 + a_2b_2 = (a \mid b)$$

and

$$(x\tau\nu)(a\tau)(b\nu) = (lx)(abl);$$

and hence all the projective invariants of (5.1) may be built from the symbolic factors

(5.2) (ax), (ux), (lx), $(a \mid b)$, $(a \mid a)$, $(u \mid u)$, (abc), (abu), (abl), (aul).

It is clear that any projective invariant of (5.1) will be *a fortiori* an invariant of the subgroup E, but it is not obvious that conversely any invariant of (3.1) under E is likewise a *projective* invariant of the augmented system (5.1). R. Weitzenböck[2] has proved that this converse theorem holds and hence that any invariant of (3.1) under E may be built from the symbolic factors (5.2).

6. Tensor algebra and its geometrical significance. The coefficients of the forms (5.1), g_{AB}, u_A, δ^0_A, x^A and δ^{AB}, plus the alternating invariant tensor ϵ^{ABC} of section 2 will form the building blocks of the tensor formulation of the invariants analogous to the symbolic factors (5.2). That these tensors suffice for the formation of all the projective invariants of (5.1) is a basic theorem due to C. M. Cramlet.[3] In the tensor algebra of invariants it plays the rôle of the first fundamental theorem of the symbolic method cited in section 3. We now make certain observations which will be needed later.

a. The transformation law of the line coordinates u_A decomposes into $\bar{u}_0 = u_0 + u_r a^r_0$ and $\bar{u}_i = u_r a^r_i$ so that the vanishing of the coordinates u_i has a significance independent of the coordinate system, that is, the direction u_i is then that of L_∞.

b. $g^{AQ}u_Q$ is the pole of u_A as to C. If this pole is likewise on u_A then $g^{AQ}u_Qu_A = 0$ and u_A is a tangent of C.

c. Dually, $g_{AQ}x^Q$ is the polar of x^A as to C. Let y^A be on this polar, then $g_{AQ}x^Qy^A = 0$. But $g_{AQ}y^Q$ is the polar of y^A and hence if y^A is on the polar of x^A, so likewise is x^A on the polar of y^A.

d. $\delta^0_Q\epsilon^{QBC} = \epsilon^{0BC}$ is an invariant tensor giving the contravariant coordinates of the line δ^0_A, namely L_∞. $\epsilon^{0QR}u_Qv_R = \epsilon^{0qr}u_qv_r = u_1v_2 - u_2v_1 = 0$ is the condition for intersection of the lines u_A and v_A on L_∞, that is, for parallelism of the directions u_i and v_i. Since only the components ϵ^{0ik} are operative in contractions, the simpler notation $\epsilon^{ik} = \epsilon^{0ik}$ will be employed in such contractions.

e. $\epsilon^{0AQ}u_Q$ is the intersection point of the line u_A with L_∞.

f. $\delta^{RS}u_Rv_S = \delta^{rs}u_rv_s = u_1v_1 + u_2v_2 = 0$ is the condition for orthogonality of the directions u_i and v_i. If $v_A = u_A$ the line u_A is self-orthogonal and the directions u_i form two parallel pencils on the circular points $(0; 1, \pm i)$. Again, since only the components δ^{ij} are effective in contraction, only these components will be indicated in contractions.

g. A pencil of parallel lines with the direction u_i will intersect L_∞ in the point $\epsilon^{0As}u_s$. The direction v_j of the polar of this point is given by $v_j = g_{jr}\epsilon^{rs}u_s$. The directions u_j and v_j are said to be *conjugate* with respect to C. When C is a central conic $|g_{lm}| \neq 0$, the conjugate relationship is reciprocal and this reciprocity may be brought into evidence by introducing the factor $(-|g_{lm}|)^{-1/2}$ so that

$$v_j = (-\mid g_{lm} \mid)^{-1/2} g_{jr}\epsilon^{rs}u_s$$

and

$$v_j = (-\mid g_{lm} \mid)^{-1} g_{jr}\epsilon^{rs}g_{st}\epsilon^{ti}v_i,$$

since

$$(-\mid g_{lm} \mid)^{-1} g_{jr}\epsilon^{rs} g_{st}\epsilon^{ti} = \delta^i_j.$$

The condition that the direction conjugate to u_i be at the same time orthogonal to u_i is

$$\delta^{jr}g_{rs}\epsilon^{sk}u_ju_k = 0. \tag{6.1}$$

h. $\delta^{AQ}u_Q$ is a point on L_∞ since $\delta^{0A} = 0$. If v_A is any line on this point, $v_A\delta^{AQ}u_Q = 0$ and the directions v_i and u_i are orthogonal. Hence $\delta^{AQ}u_Q$ is the point on L_∞ determined by the direction orthogonal to u_i.

i. $\epsilon^{ABQ}u_Q$ are the contravariant components of the line u_A.

j. $y^A = \epsilon^{ABS}u_Rv_S$ is the point of intersection of the lines u_A and v_A, for $u_Ay^A = v_Ay^A = 0$.

7. The euclidean metric invariants of the conic. The complete basic system of invariants of the forms (5.1) in terms of the symbolic factors (5.2) has been found by R. Weitzenböck.[4] We tabulate these in both the symbolic notation used by Weitzenböck and the tensor notation of this paper. Any rational integral invariant of g_{AB}, x^A, and u_A under E is expressible in terms of these 18 basic invariants.

The verification that no one of the 18 basic invariants, say S_1, is expressible rationally and integrally in terms of the remaining 17 invariants of the system may be made in the following way. From a consideration of the degrees 2, 1, 1 of S_1 in g_{AB}, x^A, and u_A respectively it is apparent that the condition for dependency of S_1 is that there exist constants $k_0, \cdots, k_4$, wherein $k_0 \neq 0$ and at least one of the remaining k's is not zero, for which

$$k_0S_1 + k_1L\Omega + k_2CJ_0 + J_1(k_3V_2 + k_4T_2) = 0$$

holds *identically* in g_{AB}, x^A and u_A. But for the selection

$$u_0 = u_1 = x^1 = x^2 = g_{11} = g_{22} = g_{01} = 0$$

it is clear that $J_0 = J_1 = \Omega = 0$, while $S_1 = -g_{02}g_{12}u_2x^0 \neq 0$ and hence $k_0 = 0$ and S_1 is independent.

Name	Type	Symbolic form	Tensor form	Degree in g_{AB}	x^A	u_A
D		$(abc)^2$	$6\|g_{AB}\|$	3	0	0
J_1	g_{AB} only	$(a\|a)$	$\delta^{rs}g_{rs}$	1	0	0
C		$(abl)^2$	$2\|g_{ij}\|$	2	0	0
f_0		$(ax)^2$	$g_{QR}x^Qx^R$	1	2	0
L	g_{AB} and x^A	(lx)	x^0	0	1	0
W		$(xa)(a\|b)(bx)$	$g_{Qr}\delta^{rs}g_{sR}x^Qx^R$	2	2	0
S_2		$(abl)(ax)(b\|c)(cx)$	$g_{As}\epsilon^{sr}g_{rt}\delta^{tu}g_{uB}x^Ax^B$	3	2	0
f''		$(abu)^2$	$2\|g_{AB}\|g^{QR}u_Qu_R$	2	0	2
Φ		$(u\|u)$	$\delta^{rs}u_ru_s$	0	0	2
V_1	g_{AB} and u_A	$(a\|u)^2$	$g_{rs}\delta^{ri}\delta^{sj}u_iu_j$	1	0	2
T_1		$(aul)(a\|u)$	$g_{rs}\delta^{ri}\epsilon^{sj}u_iu_j$	1	0	2
Ω		$(abu)(abl)$	$g_{Au}\epsilon^{uv}g_{vB}\epsilon^{ABQ}u_Q$	2	0	1
J_0		(ux)	u_Qx^Q	0	1	1
V_2		$(xa)(a\|u)$	$g_{Aq}\delta^{qr}u_rx^A$	1	1	1
T_2	g_{AB}, x^A and u_A	$(aul)(ax)$	$g_{Aq}\epsilon^{qr}u_rx^A$	1	1	1
R_1		$(abu)(ax)(b\|u)$	$g_{AQ}g_{Br}\delta^{rs}u_s\epsilon^{QBC}u_Cx^A$	2	1	2
R_2		$(abu)(ax)(b\|c)(cx)$	$g_{Ar}\delta^{rs}g_{sQ}\epsilon^{QRC}u_Rg_{CB}x^Ax^B$	3	2	1
S_1		$(abl)(ax)(b\|u)$	$g_{Ar}\epsilon^{rs}g_{st}\delta^{tu}u_ux^A$	2	1	1

8. Reduction to canonical forms. Before interpreting geometrically the vanishing of the basic invariants the equation of the proper conics for which $|g_{AB}|\neq 0$ will be reduced to its canonical forms. L_∞ intersects C in the pair of points given by $g_{rs}x^rx^s=0$. Whence $C=2|g_{ij}|<0$, $C=0$ and $C>0$ this pair will be real and distinct, real and equal, and conjugate complex so that C will be a hyperbola, parabola, or an ellipse respectively. If L_∞ is to cut C in the circular points $(0;\ 1,\ \pm i)$ then $g_{12}=0$ and $g_{11}-g_{22}=0$. Thus, for real coefficients, the condition that an ellipse be a circle is that

$$J_1^2 - 2C = (g_{11}-g_{22})^2 + 4g_{12}^2 = 0.$$

We first reduce to canonical form the equation of the central conics for which $|g_{jk}|\neq 0$. Under the translation $x^i=\bar{x}^i+a_0^i$ it may be seen from (4.1) that $\bar{g}_{0i}=g_{0i}+g_{qi}a_0^q$. Thus the axes may be translated to a new origin at the point a_0^A for which $\bar{g}_{0i}=0$. Now the center y^A of C is characterized by the condition that the polar of the center is L_∞, $g_{QA}y^Q=k\delta_A^0$, so that $g_{0i}+g_{qi}y^q=0$ and the new origin at a_0^A is the center of C. By a normalization of the homogeneous coefficients we may choose $k=-1$. In the new system the center y^A is $(1;\ 0,\ 0)$ so that $g_{0i}=0$ and $g_{00}=-1$. Under any subsequent rotation without translation it may also be seen from (4.1) that the relations $g_{0i}=0$ and $g_{00}=-1$ are not disturbed.

A further reduction of the equation of the central conic may be effected by a rotation of axes about its center to bring them into coincidence with the unique

pair of orthogonal conjugate diameters whose directions must be the roots of the quadratic equation $\delta^{jr}g_{rs}\epsilon^{sk}u_ju_k=0$ as given by (6.1). But with the coordinate axes now in these directions the equation must admit the solutions (0; 1, 0) and (0; 0, 1) so that $g_{12}=0$. We thus arrive at the familiar canonical form,

$$-(x^0)^2+g_{11}(x^1)^2+g_{22}(x^2)^2=0,$$

which is an ellipse or an hyperbola according as $|g_{jk}|=g_{11}g_{22}>0$ or <0 respectively.

In the case of the parabola for which $|g_{jk}|=0$, let λ^A be a point on L_∞ with the polar $g_{Ar}\lambda^r$. If this polar is to be L_∞ itself, $g_{ir}\lambda^r=0$, so that the non-trivial solution of this pair of homogeneous equations gives the point of tangency of C with L_∞. If x^A be any point not on L_∞, its polar $g_{AQ}x^Q$ will have the direction $g_{i0}+g_{ir}x^r$. The parallel pencil on λ^A will have the direction $\epsilon_{ir}\lambda^r$ and the condition that the polar of x^A be orthogonal to this direction is

$$\delta^{ij}\epsilon_{ir}\lambda^r(g_{j0}+g_{js}x^s)=0.$$

For variable x^i this is the equation of the axis of the parabola. If we choose the coordinate system to make this axis $x^2=0$, then $\lambda^1\neq 0$, $\lambda^2=0$ and $g_{ir}\lambda^r=0$ gives $g_{11}=0$, $g_{12}=0$, and the equation of the axis becomes $-\lambda^1(g_{02}+g_{22}x^2)=0$, from whence $g_{02}=0$ and $g_{22}\neq 0$. The axis will cut C in the points given by $x^0(2g_{01}x^1+g_{00}x^0)=0$, and on choosing the origin (1; 0, 0) to be at one of these intersections, $g_{00}=0$ and $g_{01}\neq 0$. Normalization by division by g_{22} gives the canonical form $(x^2)^2+2g_{01}x^0x^1=0$.

9. Geometrical interpretation of the invariants. In the same paper in which he tabulated the 18 invariants of a conic, a point, and a line, Weitzenböck likewise partially interpreted geometrically the vanishing of each. To point out the ease with which the geometric meaning of an invariant equation may be read from its tensor formulation we shall interpret in much greater detail than did Weitzenböck the vanishing of the two most interesting invariants, R_1 and R_2. Section 6 should be consulted as the key to these interpretations.

$R_1=(\epsilon^{QBC}u_C)(g_{QA}g_{Br})(x^A\cdot\delta^{rs}u_s)=0$ for given u_A reads: (contravariant line u_A) (has a covariant pole) (which is collinear with x^A and the point on L_∞ in the direction normal to u_i). Thus $R_1=0$ gives the line through the pole of u_A which is orthogonal to u_A. R_1 vanishes identically in x^A for u_A the L_∞, since the line joining the center of C with any point x^A in the plane is orthogonal to L_∞.

$R_1=(x^Ag_{AQ}\epsilon^{QBC})(u_C\cdot g_{Br}\delta^{rs}u_s)=0$ for given x^A reads: (contravariant polar of x^A) (is incident with intersection Q of line u_A and diameter of C conjugate to the direction normal to u_i). Thus to construct the line conic $R_1=0$ corresponding to a given point x^A, choose Q any point on the polar of x^A and pass a diameter D through Q. Then the line u_A passes through Q orthogonal to the direction conjugate to D.

For the central conic,

$$-(x^0)^2+g_{11}(x^1)^2+g_{22}(x^2)^2=0, \qquad 1/g_{11}-1/g_{22}=c^2\geqq 0,$$

and for the parabola $(x^2)^2+2g_{01}x^0x^1=0$,

$$R_1 = g_{11}g_{22}(-\ x^2u_0u_1+x^1u_0u_2+c^2x^0u_1u_2)=0,$$
$$R_1 = g_{01}(x^0u_0u_2+x^2u_1^2-(g_{01}x^0+x^1)u_1u_2)=0.$$

If the conic is a circle, $R_1=0$ implies $(-x^2u_1+x^1u_2)u_0=0$ and the line conic degenerates into the pencil of diameters and the family of lines orthogonal to the diameter through x^A.

If the point x^A is on the major (transverse) axis, $x^2=0$, the line conic degenerates into the family of lines orthogonal to the major (transverse) axis and into a pencil of lines on the point $(1;c^2/x^1, 0)$. Thus $R_1=0$ defines on the major (transverse) axis an involution of points whose double points are the two foci. Similarly if x^A be taken on the minor (conjugate) axis, $x^1=0$, there is defined an involution of points $(1;0,x^2)\rightarrow(1;0,-c^2/x^2)$ whose double points are the pair of imaginary foci.

In the case of the parabola, if x^A is on L_∞, the line conic degenerates into the family $u_1=0$ of lines parallel to the axis and into the family of lines perpendicular to the direction $(0;x^1,x^2)$. If x^A is on the axis $x^2=0$ the line conic also degenerates, this time into the family of lines $u_2=0$ orthogonal to the axis and into the pencil on the point $(1;\ -g_{01}-x^1, 0)$. Thus $R_1=0$ sets up an involution on the axis of the parabola whose double points are the focus and the infinite point on the axis.

$R_2=(\epsilon^{QRC}u_R)(g_{Qs}g_{BC})(x^B)(\delta^{rs}g_{rA}x^A)=0$ for given u_A reads: (the contravariant line u_A) (has a covariant pole) (collinear with x^B) (and with the point on L_∞ perpendicular to the polar of x^A). Thus to construct the point x^A, let P be the pole of u_A and draw any line L through P. Draw the diameter of C conjugate to the direction orthogonal to L. This diameter will intersect L in the desired point x^A.

For the central conic and the parabola

$$R_2=(g_{11}g_{22})^2(c^2u_0x^1x^2+g_{11}^{-2}u_1x^0x^2-g_{22}^{-2}u_2x^0x^1)=0,$$
$$R_2=-\ g_{01}(g_{01}^2u_2x^0x^0+u_1x^1x^2-(g_{01}u_1+u_0)x^0x^2)=0.$$

If C is a circle, the conic $R_2=0$ degenerates into L_∞ and the diameter perpendicular to u_i.

If the line u_A is perpendicular to the major (transverse) axis, so that $u_2=0$, then $(c^2u_0x^1+g_{11}^{-2}u_1x^0)x^2=0$ and $R_2=0$ degenerates into the major (transverse) axis and a line perpendicular to this axis so that to the line $(u_0;\ u_1,\ 0)$, or $x^1=-u_0/u_1=X$, there corresponds the line $\bar{x}^1=(1/X)(a^2/e^2)$, $a^2=1/g_{11}$ and $e^2=c^2/a^2$. The double lines of the involution are the two directices $x^1=\pm a/e$ of the conic. Similarly if the line u_A is perpendicular to the minor (conjugate) axis there is defined an involution of lines orthogonal to this axis whose double lines are the pair of imaginary directrices.

In the case of the parabola, if the line u_A is taken normal to the axis so that $u_2=0$, the conic

$$R_2 = - g_{01}(u_1x^1 - (g_{01}u_1 + u_0)x^0)x^2 = 0$$

degenerates into the axis and a line orthogonal to it. Thus $R_2=0$ sets up an involution among the normals to the axis in which to the line $(u_0;\ u_1,\ 0)$, or $x^1=-u_0/u_1=X$, there corresponds the line $x^1=g_{01}-X$, the double lines of the involution being the directrix and L_∞. If the line u_A is parallel to the axis, $u_1=0$, then $(g_{01}^2u_2x^0-u_0x^2)x^0=0$ and $R_2=0$ degenerates into L_∞ and a line parallel to the axis. Thus $R_2=0$ sets up an involution of lines parallel to the axis whereby the line $(u_0;\ 0,\ u_2)$, or $x^2=-u_0/u_2=Y$, has as its corresponding line $x^2=(-1/Y)g_{01}^2$ so that the double lines of the involution are $x^2=\pm ig_{01}$ which are the two polars of the circular points $(0; 1,\ \pm i)$.

$R_2=(g_{CB}x^B)(x^Ag_{Ar}\delta^{rs}g_{sQ})(\epsilon^{QRC})(u_R)=0$ for given x^A reads: (the polar of x^A) (and the diameter conjugate to the direction orthogonal to the polar of x^A) (intersect in a point) (which is the center of the pencil of lines u_A). For a central conic, R_2 vanishes identically in u_A when x^A is at the center. For a parabola, R_2 vanishes identically either when x^A is the point of contact of C with L_∞ or when x^A is the intersection with L_∞ of the tangent at the vertex.

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That famous motto. Plato's writings do not convince any mathematician that their author was strongly addicted to geometry. . . . We know that he encouraged mathematics. . . . But if—which nobody believes—the μηδεὶς ἀγεωμέτρητος εἰσίτω [Let no man ignorant of geometry enter] of Tzetzes had been written over his gate, it would no more have indicated the geometry within than a warning not to forget to bring a packet of sandwiches would now give promise of a good dinner.—Augustus De Morgan (1858).—*Contributed.*

A little known property of the normal probability curve. It is a little known but easily proved fact that the inflectional tangents of the normal probability curve strike the base line two sigma units each side of the axis of symmetry. This property of course holds equally well when the x- and y-units are unequal. Consequently, lines joining these base points with the inflection points should appear to be tangent to the curve. Checks of the diagrams in ten common texts fail to conform to this test.—*W. F. Cheney, Jr.*

LUCAS'S TESTS FOR MERSENNE NUMBERS

IRVING KAPLANSKY, New York City

1. The theorem. The purpose of this note is to give a brief self-contained proof of the following theorem, which includes as special cases the famous Lucas tests for primality of Mersenne numbers.

THEOREM. *Let $q=2^k-1$ and suppose integers c, d are known such that d and c^2-d are quadratic non-residues of q, if q is prime. Define $W_2=2(c^2+d)/(c^2-d)$, $W_{i+1}=W_i^2-2$. Then q is prime if and only if** $q \mid W_k$.

Note. The conditions of the theorem are a little peculiar in that we are called upon to announce in advance that d and c^2-d would be non-residues of q, if q were prime; we then use this information to test the primality of q. However because of the special form of $q=2^k-1$, we can be certain that some numbers are non-residues if q is prime, *e.g.*, -1, -2, 3, 6, and products of these by squares. By trial we can then find a variety of suitable values of c and d. We list below all distinct values of W_2 obtainable with c^2 and $|d|$ less than 100.

c	d	W_2
1	-2	$-2/3$
1	3	-4
3	6	10
5	-2	46/27
9	6	58/25
5	27	-52

These six values of W_2 provide us with as many "universal" tests for Mersenne numbers, $W_2=-4$ being the one discovered by Lucas. If however the form of q is further prescribed, there may be still other valid choices of c and d. For example, if k is of the form $4n-1$, then 5 is a non-residue of q, and we may take $d=5$, $c=1$, $W_2=-3$. This is the other of Lucas's two tests and is the one he used to prove that $2^{127}-1$ is prime.

2. Preliminary lemmas. Our proof of the above theorem is modelled on one given by D. H. Lehmer† for the case $W_2=4$.

Define $U_n=(a^n-b^n)/(a-b)$ with $a=c+\sqrt{d}$, $b=c-\sqrt{d}$, and let $V_n=a^n+b^n$ $=U_{2n}/U_n$. In the following lemmas it will always be tacitly assumed that the primes under discussion do not divide $2abcd$.

We begin by noting two readily verified identities:

* If W_k is not an integer, we mean that q divides the numerator of W_k when the latter is in its lowest terms. In a particular case we could in any event always begin by making W_2 an integer mod q.

† Journal London Math. Soc. 10, 1935, 162–165. The author is indebted to Professor Lehmer for his suggestions concerning this paper.

$$U_{m+n} = U_n U_{m+1} - abU_m U_{n-1}, \tag{1}$$

$$U_n^2 - U_{n-1}U_{n+1} = (ab)^{n-1}. \tag{2}$$

LEMMA 1. *A prime cannot divide two successive U's.*

Proof. Take $n=2$ in (1):

$$U_{m+2} = (a + b)U_{m+1} - abU_m. \tag{3}$$

It follows that a prime which divides two successive U's will divide all of them, including $U_1 = 1$.

LEMMA 2. *A prime cannot divide both U_n and V_n.*

Proof. From (1) with $m=n$ we obtain

$$\begin{aligned} V_n &= U_{n+1} - abU_{n-1} \\ &= (a + b)U_n - 2abU_{n-1} \quad \text{by (3).} \end{aligned}$$

A prime which divides both U_n and V_n will therefore also divide U_{n-1}.

LEMMA 3. *The integers r for which a prime p divides U_r are all multiples of a single integer.*

Proof. It will suffice to show that $p \mid U_m$ and $p \mid U_n$ imply that $p \mid U_{n\pm m}$ $(n>m)$. That $p \mid U_{n+m}$ is evident from (1). Now replace n by $n-m$ in (1):

$$U_n = U_{n-m}U_{m+1} - abU_m U_{n-m-1}.$$

Then $p \mid U_{n-m}U_{m+1}$. But p cannot divide both U_m and U_{m+1}; hence $p \mid U_{n-m}$.

The next two lemmas form the foundation on which the Lucas theory is built. For completeness we include the familiar proofs.*

LEMMA 4. *If p is prime, $U_p \equiv (d \mid p) \pmod{p}$.*

Proof. From the definition of U_n we readily find

$$U_p = pc^{p-1} + \binom{p}{3}c^{p-3}d + \binom{p}{5}c^{p-5}d^2 + \cdots + d^{(p-1)/2}.$$

All the binomial coefficients except the last vanish mod p. Hence

$$U_p \equiv d^{(p-1)/2} \equiv (d \mid p) \pmod{p}.$$

LEMMA 5. *If p is prime, $U_{p-(d|p)} \equiv 0 \pmod{p}$.*

Proof.

$$U_{p+1} = (p + 1)c^p + \binom{p+1}{3}c^{p-2}d + \cdots + (p + 1)cd^{(p-1)/2}.$$

* See for example Hardy and Wright, An Introduction to the Theory of Numbers, pp. 147–149.

Mod p, the only binomial coefficients which survive are the first and last. Also $c^p \equiv c \pmod{p}$ by Fermat's "little theorem." Hence

$$(4)\qquad \begin{aligned} U_{p+1} &\equiv c(1 + d^{(p-1)/2}) \pmod{p} \\ &\equiv 0 \pmod{p} \quad \text{if and only if} \quad (d \mid p) = -1. \end{aligned}$$

Now set $n=p$ in (2). We find, using Lemma 4 and Fermat's theorem,

$$(5)\qquad U_{p-1}U_{p+1} \equiv 0 \pmod{p}.$$

Lemma 5 follows at once from (4) and (5).

LEMMA 6. *If p is prime, and $(ab|p)=(d|p)=-1$, then $p| V_t$, where $t=(p+1)/2$.*

Proof. By Lemma 5, $p| U_{2t}$. Next take $n=t$, $m=t-1$ in (1).

$$U_p = U_t^2 - abU_{t-1}^2.$$

If $p| U_t$ we have

$$U_p \equiv (d \mid p) \equiv -1 \equiv -abU_{t-1}^2 \pmod{p},$$

in contradiction of $(ab|p)=-1$. Hence $p\nmid U_t$, $p| V_t=U_{2t}/U_t$.

LEMMA 7. *If $q| V_t$ with t a power of 2, and $q\leqq 4t^2-4t$, then q is prime.*

Proof. If q is not prime, select a prime divisor $p\leqq\sqrt{q}$. Then $p| V_t$ and a fortiori $p| U_{2t}$. The smallest integer r for which $p| U_r$ is then, by Lemma 3, a power of 2. But by Lemma 5, $r\leqq p+1$, and $p+1\leqq\sqrt{q}+1<2t$. Hence $r\leqq t$ and p divides both U_t and V_t in contradiction of Lemma 2.

3. Proof of the theorem. The completion of the proof is now immediate. We note that $W_2=(a^2+b^2)/ab=V_2/ab$ and an induction readily verifies that

$$W_k = V_{2^{k-1}}/(ab)^{2^{k-2}}.$$

Now suppose $q| W_k$, then $q| V_{2^{k-1}}$ a fortiori, and by Lemma 7 q is prime. Conversely if q is prime, the hypotheses of Lemma 6 are fulfilled and $q| V_{2^{k-1}}$. Since q is prime to $ab=c^2-d$, we have $q| W_k$.

Saccheri's dilemma. Did [Saccheri] really believe that his beautiful Non-Euclidean geometry was nothing but a reductio-ad-absurdum? In order to guess Saccheri's own mind we have to remember, that in the close and constant intercourse with other mathematicians of his Order, his genius may have been intimidated by repeated discussions, and that before he could publish anything, the manuscript had to pass the censorship of at least two of the Jesuits.—G. B. Halsted (1910).

—*Contributed.*

Send suitable items to Professor E. T. Bell, California Institute of Technology, Pasadena 4, California.

A PROPERTY OF APPELL SETS*

C. J. THORNE, Louisiana State University

1. Introduction. An Appell set of polynomials $\{P_n(x)\}$, $n=0, 1, \cdots$ ($P_n(x)$ of degree n) is a set for which

$$P_n'(x) = P_{n-1}(x), \qquad n = 1, 2, \cdots. \tag{1}$$

An equivalent definition is the existence of a formal power series $A(t)=\sum a_n t^n$ ($a_0 \neq 0$) such that (formally)

$$A(t)e^{tx} = \sum P_n(x)t^n. \tag{2}$$

All summations are from 0 to ∞ unless otherwise stated. $A(t)$ is called the generating function of the polynomials. Each such $A(t)$ defines an Appell set. Some well known sets are $(x-a)^n/n!$, the Hermitian polynomials, the Bernoulli polynomials, and the Euler polynomials.

Sheffer† studied such sets. He proved that an Appell set satisfies a linear differential equation of the form

$$\sum L_r(x)y^{[r]}(x) = \lambda y(x), \tag{3}$$

if and only if $A(t)=e^{Q(t)}$. Here summation is 0 to n, $L_r(x)$ is a polynomial in x of degree not exceeding r, λ is a parameter, $[r]$ means the rth derivative, and $Q(t)$ is a polynomial in t. Moreover, except for $Q(t)=$constant, the minimum order of such differential equations is the same as the degree of $Q(t)$. Webster‡ has proven that any Appell set which is at the same time an orthogonal set in the usual sense is reducible to the Hermitian set by a linear transformation.

The purpose of this paper is to give a characterization of Appell polynomials in terms of a Stieltjes integral. Appell polynomials are then more useful in applications of the functional method.§

2. A new characterization. Let $\alpha(x)$ be a function of bounded variation on‖ (a, b) for which the integrals

$$\mu_n = \int_a^b x^n d\alpha(x), \qquad n = 0, 1, \cdots \tag{4}$$

all exist. Further assume that

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* The present paper grew out of work the author did while a fellow at Brown University during the summer of 1942.

† Summary paper, I. M. Sheffer, Some applications of certain polynomial classes. Am. Math. Soc. Bulletin, December 1941, 885–898.

‡ M. S. Webster, Orthogonal polynomials with orthogonal derivatives. Am. Math. Soc. Bulletin, December 1938, 880–888.

§ C. J. Thorne and J. V. Atanasoff, A functional method for the solution of thin plate problems applied to a square clamped plate with a central point load. Iowa State College Journal of Science, vol. XIV, 333–343.

‖ This can be an infinite interval.

$$\mu_0 \neq 0. \tag{5}$$

Then

THEOREM I. *There exists a unique sequence of polynomials* $\{\phi_n(x)\}$, $n=0, 1, \cdots$, ϕ_n *being of degree* n, *such that*

$$\int_a^b \phi_n^{[r]}(x)d\alpha(x) = \delta_n^r = \begin{cases} 0, & n \neq r, \\ 1, & n = r. \end{cases} \tag{6}$$

Proof. $\phi_n(x)$ will be of the form

$$\phi_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0. \tag{7}$$

Relations (6) applied to (7) yield the results

$$\begin{aligned} &n!a_n\mu_0 = 1, \\ &\frac{n!}{1!}a_n\mu_1 + (n-1)!a_{n-1}\mu_0 = 0, \\ &\frac{n!}{2!}a_n\mu_2 + \frac{(n-1)!}{1!}a_{n-1}\mu_1 + (n-2)!a_{n-2}\mu_0 = 0, \\ &\quad\vdots \\ &a_n\mu_n + a_{n-1}\mu_{n-1} + a_{n-2}\mu_{n-2} + \cdots + a_0\mu_0 = 0. \end{aligned} \tag{8}$$

The system (8) has a unique solution if

$$n!(n-1)!(n-2)!\cdots 1\mu_0^{n+1} \neq 0. \tag{9}$$

Since $\mu_0 \neq 0$, we have the usual unique solution to (8):

$$a_{n-r+1} = \frac{\begin{vmatrix} \mu_0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{\mu_1}{1!} & \mu_0 & 0 & \cdots & 0 & 0 \\ \frac{\mu_2}{2!} & \frac{\mu_1}{1!} & \mu_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\mu_{r-1}}{(r-1)!} & \frac{\mu_{r-2}}{(r-2)!} & \frac{\mu_{r-3}}{(r-3)!} & \cdots & \frac{\mu_1}{1!} & 0 \end{vmatrix}}{(n-r+1)!\mu_0^r}. \tag{10}$$

THEOREM II. *The set* $\{\phi_n(x)\}$ *is an Appell set.*

Proof. In (6) change n to $n+1$ and r to $r+1$ and let $\psi_n(x) = \phi_{n+1}^{[1]}(x)$. Then

$$\int_a^b \psi_n^{[r]}(x)d\alpha(x) = \delta_{n+1}^{r+1} = \delta_n^r, \tag{11}$$

whence by uniqueness, $\psi_n = \phi_n$, *i.e.* $\phi_{n+1}^{[1]}(x) = \phi_n(x)$.

COROLLARY. *The generating function $A(t)$ for $\{\phi_n(x)\}$ is given by*

$$A(t) = \left[\int_a^b e^{tx} d\alpha(x)\right]^{-1} = \left[\sum \frac{t^m}{m!}\mu_m\right]^{-1} \tag{12}$$

Proof. From definition (2),

$$A(t)e^{tx} = \sum t^n \phi_n(x). \tag{13}$$

By property (6),

$$A(t)t^r \int_a^b e^{tx} d\alpha(x) = t^r.$$

Hence

$$A(t) = \left[\int_a^b e^{tx} d\alpha(x)\right]^{-1} = \left[\sum \frac{t^m}{m!}\mu_m\right]^{-1}.$$

THEOREM III. *Every Appell set has the property* (6). *More precisely: To every Appell set $\{P_n(x)\}$ corresponds a function $\alpha(x)$ of bounded variation in* $(0, \infty)$ *such that*

(a) *the integrals* (4) *all exist on the range* $(0, \infty)$;

(b) $\mu_0 \neq 0$;

(c) $\int_0^\infty P_n^{[r]}(x) d\alpha(x) = \delta_n^r$.

Proof. Let $A(t) = \sum a_n t^n$ $(a_0 \neq 0)$ be the generating function for $\{P_n(x)\}$, and define $\{\mu_n\}$ by the formal relation $A(t) = [\sum t^m/m!\, \mu_m]^{-1}$. Boas* has shown that a function $\alpha(x)$ of bounded variation on $(0, \infty)$ exists (not however uniquely) such that

$$\mu_n = \int_0^\infty x^n d\alpha(x), \qquad n = 0, 1, \cdots.$$

Hence

$$A(t) = \left[\int_0^\infty e^{tx} d\alpha(x)\right]^{-1}, \tag{12'}$$

and by the corollary to Theorem II, relations (6) are seen to hold. Since $a_0 \neq 0$, therefore $\mu_0 \neq 0$.

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* Widder, The Laplace Transform, p. 139.

THE NEED FOR COOPERATIVE ACTION IN MATHEMATICAL EDUCATION

RALEIGH SCHORLING, University of Michigan

In dealing with the problem of this symposium I shall, with the exception of a single suggestion, limit myself to the needs of teachers of mathematics in our elementary and secondary schools. Surely, I need not, to this group, emphasize the fact that this level of training is of paramount importance. If our institutions are to be not only surely preserved but also constantly improved, if the right kind of foundation for more advanced work is to be laid, the national organizations dealing with advanced mathematical training need to give careful attention to the mathematics of the earlier years.

How shall we improve the mathematical offering of the lower schools? Obviously, a necessary step, though not the complete solution, is to design better undergraduate mathematical curricula for the prospective teachers of mathematics for our elementary and secondary schools. And, how can we do this job on which we have all so grievously failed?

Fortunately we were given excellent advice forty-two years ago by Professor E. H. Moore in his presidential address before the American Mathematical Society. May I read you four quotations in which I need not change a single word in order to make them apply to today's crucial problems in mathematical education?

> "I wish . . . to express the hope that we may secure the active cooperation of many colleagues in the domains of science and of administration, so that the first carefully chosen steps of a really important advance movement may be taken in the near future."
>
> "Undoubtedly in many parts of the country improvements in organization and methods of instruction in mathematics have been made these last years. All persons who are, or may become, actively interested in this movement of reform should in some way unite themselves, in order that the plans and the experience, whether of success or failure, of one may be immediately made available in the guidance of his colleagues. . . ."
>
> "Teaching must become more of a profession. And this implies not only that the teacher must be better trained for this career but also in his career he be given greater freedom with greater responsibility. To this end closer relations should be established between the teachers of the colleges and those of the secondary schools; standing provisions should be made for conferences as to improvement of the secondary school curricula . . . , and the leading secondary school teachers should be steadily encouraged to devise and try out plans looking in any way toward improvement."
>
> "Furthermore, there is evident need of a national organization having its center of gravity in the whole body of science instructors in the secondary

schools; and those of us interested in these questions will naturally relate ourselves also to this organization."*

Many college teachers of mathematics feel, and justly so, that they have an important role in the education of teachers for the elementary and secondary schools. In national organizations dealing with higher education, there are individuals vitally concerned with the work in the lower schools. In fact, some of our national societies have standing committees on the education of teachers. There is, however, no machinery, no way by which the constructive thinking of the leaders in the various national organizations can be mobilized and be brought to bear continuously on common problems relating to pedagogy and to the undergraduate curricula for prospective teachers.

The teacher of mathematics in elementary and secondary schools *needs* your guidance and support. A few years ago mathematics in these grades was subjected to drastic criticisms. Incidentally, some were no doubt valid. The common practice of schools to allow the omission of all mathematics from the programs of pupils beyond the eighth grade was undoubtedly an over-simplification—at any rate it was devastating to the education of many students. Though the training program of the schools of the armed forces did not uncover this serious situation, it did publicize the shortage in mathematics.

At the moment, mathematics in the high school is enormously popular. There have been vast increases in the numbers that elect the various high school courses. There are good reasons for believing that this gratifying situation will continue for some time in the postwar period. The fact that the mental climate with respect to high school mathematics has changed is not due to the efforts of the American Mathematical Association and the National Council of Teachers of Mathematics. Time and circumstances have played in our favor, but they may not do so a second time. We need to keep a discerning eye on the situation that may exist five years after the war ends. Unless we provide sensible programs (note the plural) in mathematics for all pupils of the secondary schools, each appropriate to the ability and need to be served, the devastating fire of criticism may again be directed at mathematics. At any rate, it should be.

We need your help on a number of perplexing problems which in the long run affect your main interest. As illustrations, let us consider the following:

1. *The traditional sequential courses in high school mathematics will need to be further revised.* A fraction of the school's student body—unfortunately no one knows how large—will need a broader and deeper foundation in mathematics than educators and even teachers of mathematics have realized. No one should assume that all is well with the traditional courses. In most schools they are woefully out-of-date as regards both subject matter and method. Far too many men with good native ability came to the technical jobs of the army and navy with no clear understanding of such important concepts as vector, tolerance, interpolation, representative fraction, scale drawing, tangent of an angle, mi-

* Presidential address delivered before The American Mathematical Society at its ninth annual meeting, December 29, 1902. Science, N. S. Vol. XVII, pp. 401–416, March 13, 1903.

crometer, vernier, gage block, metric system, practical constructions, logarithms, and slide rule. Yet these are among the very things that would have added vitality and meaning to the abstract symbolism and theory which their former mathematics teachers struggled to teach them.

The traditional sequential courses should be reserved for the pupils who can profit by such courses and who want to take them. The door to the four-year sequence should be wide open, but no one should be pushed through it. Failure should, for the most part, be avoided by proper guidance to other appropriate mathematics courses. The unique values of the traditional courses cannot be achieved by a constant gearing down, nor by the inclusion of popular material in a futile effort to meet the mathematical needs of pupils who should not have elected the courses in the first place. The first step in the postwar period is to scrutinize the traditional courses in the light of recommendations that have long been widely approved.

2. *The necessary competency in dealing with whole numbers, common fractions, decimals, and percent must be attained.* The widely publicized letter by Admiral Nimitz* dramatized to the nation the alleged shortage in arithmetic. It would be futile to point out that this drastic indictment was based on wholly inadequate data collected by outmoded tests, for the reason that the main observation of low competency in arithmetic is amply supported by data that have been available to school people for more than a decade.

The simplicity of mathematical skills needed by the masses in the armed forces is even greater than was generally assumed, but these skills, nevertheless, are very difficult for many persons to achieve. Provision for growth in the mastery of arithmetic should be continuous throughout the elementary and secondary schools. Achievement of ability in mathematical reasoning and in sensible use of mathematical concepts requires much time and continuous practice. Courses of study of the elementary school should be reconsidered, and adequate time should be given to the teaching of arithmetic. Moreover, we have not given enough attention to arithmetic in teaching the senior high school courses. In fact, we have too often allowed the fundamental skills to deteriorate.

In the past two decades, professional courses for teachers dealing with the teaching of arithmetic practically disappeared from our teachers colleges, with the result that most beginning teachers do not know how to teach arithmetic. In fact, some actually fear it and escape the task by excessive attention to other activities in which they feel more confident. To make matters worse, many are tempted to teach the little they do teach by the incidental method—presumably the ideal method, but also the most difficult which only a few out of a hundred teachers could manage even if they had good training. The teaching of arithmetic can be and must be improved. Here is a problem that is not trivial, and one on which we need help.

3. *The secondary school should guarantee functional competence in mathematics to all who can possibly achieve it.* Competence in mathematics in many respects

* This MONTHLY, vol. 49, 1942, pp. 212–214.

parallels literacy in communication. The legal specification of literacy is fulfilled by the ability to write and to speak one's own language. In pioneer days, no doubt, this ability was adequate. Today the United States Army uses the phrase "functional literacy" to imply fourth-grade ability. Though no one knows exactly what functional literacy means, except by arbitrary definition, it is nevertheless clear that modern technology has stepped up the minimum requirements of literacy in communication.

In great-grandfather's day, when life was relatively simple, the ability to compute accurately when dealing with whole numbers, common fractions, decimals, and percent was adequate for the common affairs of life. There are good reasons for believing that the minimum requirement in mathematics for effective citizenship is moving up and is already *higher than mere control of the four fundamentals of arithmetic.*

The war has demonstrated in a dramatic way what is nonetheless true in peace time, but not so readily discernible; namely, that a boy lacking certain mathematical competencies is a pathetic victim of a ruthless system. In the armed forces men are so badly needed that they are expected to learn the impossible overnight, and we pour vast sums into their training; in peace times they shift from job to job and finally drift into the ranks of the unemployed.

An understanding and appreciation of the important elements of our culture, one of the admittedly valid aims of education, is contingent upon possession of certain basic concepts and principles from the areas of mathematics, biology, physics, chemistry, geology, and other sciences. These concepts and principles are the instruments without which man cannot think intelligently about machines, housing, business, transportation, communication, public and personal health, investments, insurance, or scientific inquiry. An understanding of these aspects of modern life demands a common language of quantitative symbols, fundamental operations, and basic ideas—perhaps some sixty in number—of arithmetic, of algebra, and of geometry.

Thus, functional competence in the mathematics of common affairs seems as crucial as functional literacy in communication. Unfortunately, it is not generally recognized that the minimum mathematical training needed for citizenship is higher than the level commonly achieved by the product of our schools. From the point of view of the educated person, the minimum mathematical equipment needed in common affairs is absurdly simple. But the task of insuring mathematical competence to all people is enormously difficult—one that has never been done in any community in all history. The job facing mathematics teachers is almost certain to be more difficult and more important in the years ahead. It is a challenge that mathematics teachers should accept. Here is a third problem on which we need your guidance.

4. *The mathematics departments of our secondary schools should provide appropriate courses for the neglected groups of students—very large ones—whose needs cannot possibly be met in traditional courses.* The secondary schools have had too many dissatisfied customers who, in depression days, accumulated as unemployed youths and in C.C.C. camps. The military training program has taken

these boys and has taught them a simple technical science and related mathematics. Several hundred thousand boys undoubtedly rendered outstanding service in saving our institutions by the technical science and mathematics they learned *after* leaving school. Moreover, a program correlating and emphasizing industrial arts, science, and mathematics is probably a good approach to the *general education* of many pupils who are never going to be satisfied with a purely academic program. Our secondary school has never been able to come to grips with the problem of mass education—one obstacle being that those concerned with higher education have not fully realized the dual responsibility of the high schools, first, for giving good training to future leaders, and second, for providing a general education appropriate for the major fraction of the high school population. This can be done. In fact, it must be done if in the long run our institutions are to survive. But it is enormously difficult without the full support of higher institutions.

We must provide a more realistic curriculum for the large number of persons who will continue to be absorbed by industry, trade, farm, and business fairly early in life. In particular, we must give more attention to the needs of industry which the traditional mathematics teacher has neglected far too long. Since there are in our high schools these large groups of students whose needs cannot possibly be met in traditional mathematical courses, the sensible thing to do is to provide good courses with very different goals and different experiences for groups with different needs. Furthermore, we must somehow do this in a manner that does not stigmatize any group. This task will be done better if the Mathematical Association of America provides help and guidance.

5. *The undergraduate curricula of prospective teachers of mathematics need to be improved.* The crucial manpower shortage as regards mathematics in the early months of the war suggests that the standard training of mathematics teachers may be inadequate. It seems that too many teachers lack familiarity with a vast range of uses of mathematics such as those, for example, in aviation, industry, trade, business statistics, and in general the simpler applications of a practical physics. It will be difficult for some to teach, not to mention design, courses having the new emphasis that the postwar era appears to demand. Perhaps the main difficulty is, as Professor Moore suggested, that there is too large a gap between pure and applied mathematics. Several institutions are providing courses for prospective teachers that include extensive work experiences in industry and business. It may be that the undergraduate curricula for prospective teachers of high school mathematics need to include a course that correlates a block of simple practical applications drawn from industrial arts, physics, and mathematics. I have no conviction on these guesses. My task is completed when I urge you to consider this problem and to decide what responsibility you have in helping us find the right answer. The fact that your program committee selected this topic for this symposium is evidence that some members in our Association believe that this problem requires your attention.

The foregoing is perhaps sufficient to show that high school teachers of

mathematics need your help. The next question is. Do they want your cooperation? The answer is clear and unequivocal. The American Mathematical Association has in the past thirty years been of great help to high school mathematics teachers. We have for example profited greatly by the friendly guidance of Herbert E. Slaught, the wise strategy of J. W. Young, the driving enthusiasm of Earl R. Hedrick, and the challenging vision of E. H. Moore. However, the complexity of our problems may now require an even higher degree of cooperation. The Association may need to spend some money if it wants a job done right. Research funds of the type that initiated and supported the Committee of 1923 are now hard to get. Some of the most competent members of our Association may need to lay aside for a time pure research and devote their energy to practical problems of the lower schools. Do not overlook the fact that J. W. Young gave three of his best professional years in order that he might render a service that thoughtful mathematics teachers of high school will deeply appreciate for generations to come.

If effective action can best be reached by the road of cooperation, then it seems desirable to select the groups that offer a hopeful beginning. I should like to outline a situation which invites active collaboration.

There is the Commission on Postwar Plans of the National Council of the Teachers of Mathematics. This Commission published its first report in the 1944 May number of the *Mathematics Teacher*. This Commission is a hard-working group of vigorous personalities who in normal times do not see eye to eye on many issues, but who have been driven by the exigencies of these grave times to splendid teamwork. The Commission has a number of projects under way which I believe are of vital interest to this Association. For example, the Commission recently proposed to the Board of Directors of the National Council of Teachers of Mathematics, that the Council be reorganized so as to consist of three sections, the first section to deal with the mathematics of the elementary school, the second, with the mathematics of the high school, and the third, with the mathematics of the junior college. You will agree, I think, that the mathematics of the junior college, which grew up like Topsy, is in need of a careful study. This undertaking might well be the joint effort of the Association and the Council.

I have been instructed, by unanimous vote, to invite the Association to work with the Commission on all matters that relate to the mathematics of the lower schools and, in particular, to join us in the study of the junior college offering in mathematics.

The second possibility for cooperation is provided by the Cooperative Committee on Science Teaching which represents five national organizations including the Mathematical Association of America. The Committee was created in 1941 and has been supported by funds from one of the foundations. It works on educational problems which no single scientific group can solve by working alone. Among these problems has been that of the certification and college preparation of science teachers for high schools. Another problem attacked by the Committee was that of using high school science and mathematics to meet

manpower needs during the war. A further function of this committee has been to serve as a forum in which representatives of the scientific societies have been able to state the views of their own groups and to learn the convictions of other groups on the teaching of science and mathematics at the elementary and secondary levels.

It is now proposed that the work of the Committee be reviewed and that it be reconstituted by the parent organizations. It is hoped that the Committee will be reorganized to include the following nine groups: physics, mathematics, chemistry, biology, research in science teaching, engineering, geology, astronomy, and physiography. Since the Committee has, at long last, made a good beginning in the manner suggested in one of the quotations from Professor E.H. Moore, I should like to urge that the Mathematical Association of America continue to participate in the work of that Committee.

The third opportunity for effective joint action is offered in cooperation with the United States Office of Education. Recently, Commissioner John W. Studebaker invited the National Council of Teachers of Mathematics to summon a small committee to consider ways and means by which the United States Office of Education might render more and better service to teachers of mathematics. We prepared a six-page report in which we listed things that the Office of Education might do for teachers of mathematics on three levels, in elementary, in secondary, and in higher institutions, to improve the teaching of mathematics. We were guided by the criteria that the Office of Education should not undertake anything unless (1) it is a problem national in scope, and (2) it cannot be undertaken by an individual, by a state, or an educational institution without excessive cost, duplication of effort, or the chance of getting incomplete and unreliable reports. Our report is now before the budget committee with the full support of Commissioner Studebaker. Newspapers have in recent days announced President Roosevelt's endorsement of the reorganization of the Office of Education that includes the services suggested in our report. The proposal is at the moment in a promising stage. However, anyone wise in the ways of Washington, especially of Congress, will not be too hopeful. Assuming that the proposal is rejected this year, one may venture the prediction that this project will go forward in the not too distant future, if it is given adequate support. With the exception of publishing the report by the National Committee on Mathematical requirements in 1923 and the Report on Preinduction Courses in Mathematics in 1943, the United States Office of Education has done little that has been of value to the cause of mathematics, although there are many services that long ago should have been provided for teachers of mathematics. It is to be hoped that the Mathematical Association of America may give this new proposal its hearty support.

With this long preface, I express the hope that the Mathematical Association of America may take the following steps:

1. To create a standing committee to give special attention to all vital ques-

tions relating to the mathematics of grades 1–14 inclusive, and to undergraduate curricula for the prospective teachers of mathematics for these grades.

2. To elect a member to the Commission on Postwar Plans of the National Council of the Teachers of Mathematics. (No one knows how long the Commission will need to operate, but the Mathematical Association of America can, of course, end its support at any time without notice.)

3. To participate in the work of the Cooperative Committee on Science Teaching. (A memorandum entitled "Proposal for the Continuance and Reconstitution of the Cooperative Committee on Science Teaching" has been submitted to the officers of the Mathematical Association of America.)

4. To vote its hearty approval of the proposal submitted by a committee of the National Council of Teachers of Mathematics to the U. S. Office of Education. (A document entitled "Functions of the Office of Education with Respect to the Teaching of Mathematics" has been submitted to the officers of the Mathematical Association of America.)

I now close with a fifth quotation from Professor Moore which, with only one word changed, seems appropriate:

> "I have accordingly felt at liberty to bring to the attention of the *Association* these matters of pedagogy of elementary mathematics, and I do so with the firm conviction that it would be possible for the *Association*, by giving still more attention to these matters, to further most effectively the highest interests of mathematics in this country."

A testimonial to Euclid. He studied and nearly mastered the six books of Euclid since he was a member of Congress.

He began a course of rigid mental discipline with the intent to improve his faculties, especially his powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow-lawyers, half a dozen in a room, filled the air with interminable snoring.—Abraham Lincoln (Short Autobiography, 1860).

Another testimonial to Euclid. Geometry is nothing if it be not rigorous. . . . The methods of Euclid are, by almost universal consent, unexceptionable in point of rigor.—H. J. S. Smith (1873).

A final testimonial to Euclid. It has been customary when Euclid, considered as a textbook, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test. . . . The value of his work as a masterpiece of logic has been very grossly exaggerated.—Bertrand Russell (1902).—*Contributed.*

A UNIFORM METHOD OF SOLVING CUBICS AND QUARTICS

A. PEN-TUNG SAH, National University of Amoy, Changting, China

1. Introduction. While the literal solutions of algebraic equations of degree less than five are well-known, there seems to be a lack of a general method of attack which will solve them in a uniform manner. In this note such a general method is given, basing its theory on the properties of a cyclic matrix which occurs frequently in the discussion of alternating current symmetrical machines.

2. Outline of the method. Although the method does not work, beyond a certain point, for quintics and equations of still higher degree, we shall outline the procedure in terms of a quintic and then exemplify it by cubics and quartics. Let the given quintic be:

$$x^5 - px^4 + qx^3 - rx^2 + sx - t = 0, \tag{1}$$

and the roots be $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$, $x_5 = e$. Then we may write (1) as a determinant with diagonal terms only, thus:

$$\begin{vmatrix} a-x & 0 & 0 & 0 & 0 \\ 0 & b-x & 0 & 0 & 0 \\ 0 & 0 & c-x & 0 & 0 \\ 0 & 0 & 0 & d-x & 0 \\ 0 & 0 & 0 & 0 & e-x \end{vmatrix} = 0. \tag{2}$$

The determinant (2) is the characteristic equation of the simple diagonal matrix:

$$\phi = \begin{Vmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{Vmatrix}. \tag{3}$$

Suppose this is transformed into a cyclic matrix ϕ' by using the following unitary matrix τ_5 and its inverse τ_5^{-1} in a unitary transformation:

$$\tau_5 = \frac{1}{\sqrt{5}} \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & n & n^2 & n^3 & n^4 \\ 1 & n^2 & n^4 & n & n^3 \\ 1 & n^3 & n & n^4 & n^2 \\ 1 & n^4 & n^3 & n^2 & n \end{Vmatrix}, \qquad \tau_5^{-1} = \frac{1}{\sqrt{5}} \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & n^4 & n^3 & n^2 & n \\ 1 & n^3 & n & n^4 & n^2 \\ 1 & n^2 & n^4 & n & n^3 \\ 1 & n & n^2 & n^3 & n^4 \end{Vmatrix}; \tag{4}$$

wherein n is one of the complex fifth roots of unity, *i.e.* $n^5 = 1$, so that

$$\phi' = \tau_5^{-1}\phi\tau_5 = \begin{Vmatrix} A & E & D & C & B \\ B & A & E & D & C \\ C & B & A & E & D \\ D & C & B & A & E \\ E & D & C & B & A \end{Vmatrix}, \tag{5}$$

wherein

$$\begin{aligned} 5A &= a + b + c + d + e, \\ 5B &= a + n^4b + n^3c + n^2d + ne, \\ 5C &= a + n^3b + nc + n^4d + n^2e, \\ 5D &= a + n^2b + n^4c + nd + n^3e, \\ 5E &= a + nb + n^2c + n^3d + n^4e; \end{aligned} \tag{6}$$

or inversely:

$$\begin{aligned} a &= A + B + C + D + E, \\ b &= A + nB + n^2C + n^3D + n^4E, \\ c &= A + n^2B + n^4C + nD + n^3E, \\ d &= A + n^3B + nC + n^4D + n^2E, \\ e &= A + n^4B + n^3C + n^2D + nE. \end{aligned} \tag{7}$$

The transformation being unitary, the characteristic equation of ϕ', namely

$$\begin{vmatrix} A-x & E & D & C & B \\ B & A-x & E & D & C \\ C & B & A-x & E & D \\ D & C & B & A-x & E \\ E & D & C & B & A-x \end{vmatrix} = 0 \tag{8}$$

will have the same roots a, b, c, d, e, as the original equation (1) or (2). Thus by equating the coefficients of the powers of x in (8) after expansion to the corresponding coefficients of the given equation (1), *i.e.*, p, q, r, s, t, we have changed the problem of finding the roots a, b, c, d, e of (1) into the solution of five simultaneous equations involving A, B, C, D, E as unknowns. If these five equations can be solved algebraically for A, B, C, D, E, then the required roots of the given equation (1) can be evaluated through the relations (7).

3. Application to a cubic. For a cubic, the cyclic matrix is

$$\alpha = \begin{Vmatrix} A & C & B \\ B & A & C \\ C & B & A \end{Vmatrix} \tag{9}$$

and the characteristic equation is

$$\begin{vmatrix} A-x & C & B \\ B & A-x & C \\ C & B & A-x \end{vmatrix} \equiv -(x^3 - px^2 + qx - r) = 0, \tag{10}$$

wherein the given coefficients p, q, r are to be equated to the three scalar invariants of the matrix (9), namely the sum of the leading diagonals, the sum of the leading diagonal first minors and the determinant of the matrix, *i.e.*,

$$p = 3A, \tag{10a}$$

$$q = 3(A^2 - BC), \tag{10b}$$

$$r = A^3 + B^3 + C^3 - 3ABC. \tag{10c}$$

Although (10c) is still a cubic, it can be solved simultaneously with (10a) and (10b), requiring only a quadratic resolvent. Thus eliminating A from (10b) and (10c) in succession, we find

$$-3BC = q - 3A^2 = q - p^2/3 = Q, \text{ say}, \tag{11a}$$

$$B^3 + C^3 = r - A^3 + 3ABC = r - pq/3 + 2p^3/27 = R, \text{ say}. \tag{11b}$$

Expressing B in terms of $1/C$ from (11a) and then substituting it into (11b), the following quadratic in $y = B^3$ (or C^3) is obtained:

$$y^2 - Ry - Q^3/27 = 0, \tag{12}$$

giving as roots or values of B^3 and C^3 the following:

$$B^3 = \frac{R}{2} + \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}}, \tag{13a}$$

$$C^3 = \frac{R}{2} - \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}}. \tag{13b}$$

Using any pair of corresponding values for B and C given by the above in terms of p, q, r and noting that $A = p/3$, we get, according to (7), the required roots:

$$\begin{aligned} x_1 &= a = A + B + C, \\ x_2 &= b = A + \omega B + \omega^2 C, \\ x_3 &= c = A + \omega^2 B + \omega C. \end{aligned} \tag{14}$$

wherein ω and ω^2 are the two complex roots of unity; namely $\omega = -\frac{1}{2} + i\sqrt{3/2}$, $\omega^2 = -\frac{1}{2} - i\sqrt{3/2}$; $i = \sqrt{-1}$.
These will be recognized as Cardan's formulas.

4. Application to a quartic. To apply the above procedure to a quartic, consider the cyclic matrix:

(15)
$$\beta = \begin{Vmatrix} A & D & C & B \\ B & A & D & C \\ C & B & A & D \\ D & C & B & A \end{Vmatrix},$$

whose characteristic equation is:

(16)
$$\begin{vmatrix} A-x & D & C & B \\ B & A-x & D & C \\ C & B & A-x & D \\ D & C & B & A-x \end{vmatrix} \equiv -(x^4 - px^3 + qx^2 - rx + s) = 0,$$

with the coefficients p, q, r, s given by the four scalar invariants of β as follows:

(17a) $$p = 4A,$$

(17b) $$q = 6A^2 - 2(2BD + C^2),$$

(17c) $$r = 4A^3 - 4A(2BD + C^2) + 4C(B^2 + D^2),$$

(17d) $$s = A^4 - 2A^2(2BD + C^2) + 4AC(B^2 + D^2) - B^4 + C^4 - D^4 + 2B^2D^2 - 4BC^2D.$$

Although (17d) is still an equation of the fourth degree, in A, B, C, D, it can be solved simultaneously with the others in (17) by a straightforward algebraic reduction, resulting in a cubic resolvent. Thus from (17a) and (17b) by eliminating A, we find:

(18a) $$-2(2BD + C^2) = q - 6A^2 = q - 3p^2/8 = 4Q, \text{ say,}$$

and substituting this into (17c), we get

(18b) $$\begin{aligned} 4C(B^2 + D^2) &= r - 4A^3 + 4A(2BD + C^2) \\ &= r + 8A^3 - 2Aq = r + p^3/8 - pq/2 = 8R, \text{ say.} \end{aligned}$$

Substituting (18a) and (18b) into (17d), we get

(19) $$-B^4 + C^4 - D^4 + 2B^2D^2 - 4BC^2D = s - 3A^4 + A^2q - Ar = 16S, \text{ say.}$$

Also from (18a),

(20a) $$-2BD = 2Q + C^2, \quad \text{or} \quad 4B^2D^2 = (2Q + C^2)^2,$$

and from (18b),

(20b) $$B^2 + D^2 = 2R/C, \quad \text{or} \quad B^4 + 2B^2D^2 + D^4 = (2R/C)^2,$$

yielding on subtraction:

(20c) $$B^4 - 2B^2D^2 + D^4 = (2R/C)^2 - (2Q + C^2)^2.$$

Substituting (20a) and (20c) into (19), we get:

$$C^4 + 2C^2(2Q + C^2) = 16S + (2R/C)^2 - (2Q + C^2)^2,$$

which, on multiplying by $1/C^2$ and collecting terms, gives

$$4C^6 + 8QC^4 + (4Q^2 - 16S)C^2 - 4R^2 = 0.$$

With $z = C^2$, this becomes a cubic in z, as follows:

$$z^3 + 2Qz^2 + (Q^2 - 4S)z - R^2 = 0. \tag{21}$$

This will be recognized as the Descartes' resolvent for a quartic. Having found C as the principal square root of the resolvent cubic (21), the values of $(B+D)$ and $(B-D)$, and hence B and D, may be evaluated from (20a) and (20b) in terms of A, Q, R, S and then in terms of p, q, r, s. Finally the roots of the quartic (16) are by (7) as follows:

$$\begin{aligned} x_1 &= a = A + B + C + D, \\ x_2 &= b = A + iB - C - iD, \\ x_3 &= c = A - B + C - D, \\ x_4 &= d = A - iB - C + iD. \end{aligned} \tag{22}$$

wherein $i = \sqrt{-1}$ as usual.

5. Concluding remarks. The procedure fails in the case of quintics or equations having a degree higher than the fourth, as it should, since it is not possible to solve such an equation by algebraic methods. The reason for the failure of the present method is that after the unitary transformation and elimination, we are still left with an equation which is of a higher degree than the fifth and which is not amenable to algebraic solutions. However, it would be of interest to study carefully the set of simultaneous equations after having made the unitary transformation to see if some constraining condition or conditions among the coefficients of the given equation might not be found which would be sufficient to permit the present procedure to succeed.

It would hardly be necessary to mention that the method as above given is not limited to real values of the coefficients p, q, r, s, nor to real roots a, b, c, d. Finally, it may be noted that the method also works in case of the quadratic, but the reduction is so simple that the case looks rather trivial.

Lorelei. So flowered the beauteous body of a new geometry [Saccheri's], mermaid-like, the latter portions somewhat fishy, but oh! the elegant torso.—G. B. Halsted (1920).

A difference of opinion. Geometry is not an experimental science.—Poincaré (1899).
We may in fact regard geometry as a branch of physics.—Einstein (1920).
Never in the world were two opinions alike.—Montaigne. —*Contributed.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE SECOND DERIVATIVE

A. B. FARNELL, U. S. Military Academy

When the notion of the second derivative is first encountered in the calculus, the examples and exercises are usually of such a character that the second derivative can be simplified by making use of the original function. As far as is known, no general principle has been enunciated for this occurrence.

THEOREM. *If $F(x, y)$ is a homogeneous function of order n in x and y, then $F(x, y)$ is a factor of the second derivative of y with respect to x of the equation $F(x, y)=c$.*

By Euler's theorem on homogeneous functions, we have

$$xF_x(x, y) + yF_y(x, y) = nF(x, y),$$

or more shortly,

$$xF_x + yF_y = nF, \tag{1}$$

where F_x denotes the partial derivative of $F(x, y)$ with respect to x. Differentiating the identity (1) partially with respect to x, and then also with respect to y, we obtain the identities:

$$xF_{xx} + yF_{xy} = (n-1)F_x, \tag{2}$$

$$xF_{xy} + yF_{yy} = (n-1)F_y. \tag{3}$$

Elimination of the coefficients x and y in (1), (2), and (3), yields the identity

$$F_x^2F_{yy} - 2F_xF_yF_{xy} + F_y^2F_{xx} = nF(F_{xx}F_{yy} - F_{xy}^2)/(n-1), \tag{4}$$

assuming $n\neq 1$, and $F_x\neq 0$. (For either of these cases the theorem is evidently true, although trivial.) From the equation $F(x, y)=c$, we have

$$y'' = -\frac{F_x^2F_{yy} - 2F_xF_yF_{xy} + F_y^2F_{xx}}{F_y^3};$$

and using (4), this becomes

$$y'' = -\frac{nF(F_{xx}F_{yy} - F_{xy}^2)}{(n-1)F_y^3}, \tag{5}$$

as desired.

CHECKING THE SAS CASE IN TRIGONOMETRY

HOWARD EVES, Syracuse University

The favored logarithmic solution of the SAS (two sides and the included angle) case in trigonometry utilizes the law of tangents. If the two given sides are taken as a and b and the given included angle as C, then the formulas successively employed are

$$A + B = 180° - C, \qquad \tan \tfrac{1}{2}(A - B) = [(a - b)/(a + b)] \tan \tfrac{1}{2}(A + B),$$

$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B), \qquad B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B),$$

$$c = a \sin C/\sin A. \tag{1}$$

Most textbooks of the subject then offer the formula

$$c = b \sin C/\sin B \tag{2}$$

as a check. It is, of course, granted that this formula does not check c itself, but presumably does check A, B, and $\log c$. Therefore, if caution is used in finding c from its logarithm, the whole problem is believed to be thoroughly checked.

It is the purpose of this note to show that it is very easy (and, as a matter of fact, relatively common among trigonometry students) to get incorrect values for A, B, and c and still have the check "check."

To this end suppose that at the very start of the solution a mistake was made in obtaining $A+B$, but that from there on everything was done correctly. Let us designate the parts (now incorrect) thus found by A', B', c'. Then

$$\begin{aligned}
\frac{\sin A'}{\sin B'} &= \frac{\sin [\frac{1}{2}(A' + B') + \frac{1}{2}(A' - B')]}{\sin [\frac{1}{2}(A' + B') - \frac{1}{2}(A' - B')]} \\
&= \frac{\sin \frac{1}{2}(A' + B') \cos \frac{1}{2}(A' - B') + \cos \frac{1}{2}(A' + B') \sin \frac{1}{2}(A' - B')}{\sin \frac{1}{2}(A' + B') \cos \frac{1}{2}(A' - B') - \cos \frac{1}{2}(A' + B') \sin \frac{1}{2}(A' - B')} \\
&= \frac{\tan \frac{1}{2}(A' + B') + \tan \frac{1}{2}(A' - B')}{\tan \frac{1}{2}(A' + B') - \tan \frac{1}{2}(A' - B')} \\
&= \frac{\tan \frac{1}{2}(A' + B') + [(a - b)/(a + b)] \tan \frac{1}{2}(A' + B')}{\tan \frac{1}{2}(A' + B') - [(a - b)/(a + b)] \tan \frac{1}{2}(A' + B')} \\
&= \frac{1 + (a - b)/(a + b)}{1 - (a - b)/(a + b)} \\
&= a/b.
\end{aligned}$$

Therefore, by the law of sines,

$$\sin A'/\sin B' = \sin A/\sin B.$$

A glance at formulas (1) and (2) now shows that the check will "check" for our

incorrect results A', B', and c', for all we are doing is replacing the denominators in (1) and (2) by numbers proportional to them.

As before mentioned, a mistake in finding $A+B$ is common enough among trigonometry students to consider this usual check as unsatisfactory. The remedy, however, is simple. In addition to carrying out the usual check, also check on the sum of the final angles A, B, C. This augmented check, along with care exercised in finding c from log c, may be considered sufficiently good. Of course a longer, but sound, check can be given by using one of the Mollweide relations.

A NOTE ON AREAS

R. A. JOHNSON, Brooklyn College

It is well known (*cf.*, for example, Goursat-Hedrick, *Mathematical Analysis*, v. 1, §94) that if x and y are differentiable functions of a parameter t, the integral

$$I = \frac{1}{2}\int_{t_1}^{t_2}(xdy - ydx) \tag{1}$$

represents the vectorial area bounded by the curve $x=x(t)$, $y=y(t)$, and the lines OP_1, OP_2, where P_1, P_2 are the points of the curve corresponding to t_1 and t_2. It is a simple matter to transform this integral into the usual expression for area in polar coordinates.

In one of my classes we arrived at the following striking transformation of the integral, which (though I have made no extensive search) I think I have not seen in print.*

If the parameter t is the ratio y/x, the integral (1) *takes the form*

$$I = \frac{1}{2}\int_{t_1}^{t_2} x^2 dt.$$

For at once, if $y/x=t$, $xdy-ydx=x^2dt$.

As an illustration, take the folium of Descartes, whose parametric equations

$$x = 3at(1+t^3)^{-1}. \qquad y = 3at^2(1+t^3)^{-1}$$

are known to every undergraduate. We have then for the area bounded by the loop of the curve

$$A = \frac{9}{2}a^2\int_0^\infty (1+t^3)^{-2}t^2dt = \frac{3}{2}a^2.$$

The labor involved here compares favorably with that of the usual process of expressing $\int ydx$ in terms of t.

* While this note is in press I learn that the result is given by de la Vallée Poussin, Cours d'analyse infinitesimale, 1938 edition, vol. 1, p. 320, ex. 7. I still think that it deserves to be better known.

The parameter $t=y/x$ is frequently used. It rationalizes any curve whose equation has the form

$$p_n(x, y) - p_{n-1}(x, y) = 0,$$

where p_r is a homogeneous polynomial of degree r; that is, any curve having a singularity of multiplicity $n-1$ at the origin.

Incidentally, will some reader contribute a historical note on the universal acceptance of the symbol

$$\int Pdx + Qdy$$

without parentheses?

NOTE ON THE SOLUTION OF DIOPHANTINE EQUATIONS

NATHAN MANTEL, Wright Field, Ohio

Given an equation of the type $ax-by=c$, where a, b, and c are integers, and a and b have no factors in common, we are to find the least positive integral solutions for x and y. A general procedure has already been developed for this problem. The procedure shown below is felt to be a good deal more efficient than the established procedure. As an example let us take the following equation

$$4397x - 3112y = 2569.$$

We may now state the problem as that of trying to find a multiple of 4397 which leaves a remainder of 2569 when divided by 3112.

We have

$$4397 = 3112 + 1285 \tag{1}$$

$$-4397 = -2(3112) + 1827. \tag{2}$$

Since $1827-1285=542$, equation (2) minus equation (1) gives

$$-2(4397) = -3(3112) + 542. \tag{3}$$

The procedure consists in applying the division algorithm to the remainders to obtain a suitable multiple of 4397 to add to a multiple already found. Thus $1285-2(542)=201$ and equation (1) minus 2 times equation (3) gives

$$5(4397) = 7(3112) + 201. \tag{4}$$

Again $1827-9(201)=18$, and equation (2) minus 9 times equation (4) gives

$$-46(4397) = -65(3112) + 18. \tag{5}$$

Finally $201-11(18)=3$, and equation (4) minus 11 times equation (5) gives

$$511(4397) = 722(3112) + 3. \tag{6}$$

The division algorithm is also applied to 2569 and those of the above remainders found suitable to use. Thus $2569=1285+1284$, $1284=6(201)+78$, $78=4(18)+6$, and $6=2(3)$; or $2569=5(542)-141$, $-141=-8(18)+3$. In the first instance $2569=1285+6(201)+4(18)+2(3)$, and 1, 6, 4, 2 are the proper multipliers of equations (1), (4), (5), (6), respectively, which give us the value of x. Thus $x=1+6(5)+4(-46)+2(511)=869$. In the second instance $2569=5(542)-8(18)+1(3)$ and $x=5(-2)+(-8)(-46)+1(511)$. Had the value of x been negative or greater than 3112, the least positive value of x could have been obtained by adding or subtracting an appropriate multiple of 3112. The value of y can be obtained by substitution or simultaneously with x.

The following is the appearance of the work sheet for this problem:

$$4397x - 3112y = 2569$$

4397	0			
3112	2569			
1285				
1827		2569		
542		1285	1	1
1084		1284		
201		1206	30	42
1809		78		
18		72	−184	−260
198		6		
3		6	1022	1444
			$x=$ 869	$y=$1227

	x		y
1	1	1285	1
	−1	1827	−2
	−2	542	−3
6	5	201	7
4	−46	18	−65
2	511	3	722

This entire procedure is a matter of about 3 minutes. Experience has shown that a solution for one of the unknowns can be obtained within 15 minutes in cases where a and b are 7 digit integers.

Interesting integers. Professor G. H. Hardy, the English mathematician, had been riding in taxi-cab No. 1729, and remarked to Ramanujan, the gifted Indian, that the number seemed rather a dull one. "No," was the reply, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways."

One might conjecture that there is an interesting fact concerning each of the positive integers. Here is a "proof by induction" that such is the case. Certainly 1, which is a factor of each positive integer, qualifies, as do 2, the smallest prime; 3, the smallest odd prime; 4, Bieberbach's number; *etc.* Suppose the set S of positive integers concerning each of which there is no interesting fact is not vacuous, and let k be the smallest member of S. But this is a most interesting fact concerning k! Hence S has no smallest member and therefore is vacuous. Is the proof valid? —*E. F. Beckenbach.*

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

REMARKS ON A VARIATION OF NEWTON'S METHOD

1. Schwerdtfeger's recurring sequences. In a letter to the editor, H. Schwerdtfeger of the University of Adelaide, South Australia, reports a method for obtaining an approximate numerical solution of an analytic equation, which he has used since 1942 in his engineering mathematics classes. He and D. R. Blaskett have reported to the Australian Mathematical Society on the connection between their method and the power series for the inverse of an analytic function in the neighborhood of a root. The method, although superficially different, is essentially similar to that discussed by the editor in *A Variation of Newton's Method.* [This MONTHLY, vol. 51, No. 1 (1944), pp. 36–38.]

To find a simple real root α of an equation $\phi(x)=0$, defined by an analytic function $\phi(x)$ such that $\phi'(\alpha)\neq 0$, Schwerdtfeger rewrites the equation in the form $x=f(x)$, and sets up a recurring sequence of numbers a_n defined by a recursion formula

$$(1.1) \qquad a_{n+1} = f(a_n), \qquad n = 0, 1, 2, \cdots, \alpha = f(\alpha),$$

where a_0 is a sufficiently close approximation to the required root α so that

$$(1.2) \qquad |f'(x)| < 1 - \epsilon \quad \text{for} \quad |x-\alpha| \leqq |a_0 - \alpha|.$$

Although this condition insures the convergence of the sequence $\{a_n\}$ to the root α, more rapid convergence is obtained by adding the condition $f'(\alpha)=0$.

To obtain a function $f(x)$ such that $f(\alpha)=\alpha$, $f'(\alpha)=0$, Schwerdtfeger first proves the existence of two functions $A(x)$ and $B(x)$, analytic at $x=\alpha$, such that

$$(1.3) \qquad A(x)\phi(x) - B(x)\phi'(x) = 1.$$

He then defines the function

$$(1.4) \qquad f(x) = x + B(x)\phi(x) = x - [\phi(x)/\phi'(x)][1 - A(x)\phi(x)],$$

which has the required properties. The graph $y=f(x)$ crosses the line $y=x$ with zero slope at the point $x=\alpha$, so the sequence (1.1) converges rapidly to α. Still more rapid convergence is obtained by placing additional conditions on the function $A(x)$ of (1.3), so that $f''(\alpha)=0$, and perhaps also $f'''(\alpha)=0$, *etc.*

2. Bailey's rational approximation to the mth root of a positive number. As a special case of this method, Schwerdtfeger refers to a formula given without proof by V. A. Bailey [*Prodigious calculation*, Australian Journal of Science,

vol. 3, No. 4, 1941, pp. 78–80] for approximating rationally to a root α of the equation

$$(2.1)\qquad \phi(x) \equiv x^m - c = 0, \qquad c = \alpha^m > 0.$$

Bailey's function,

$$(2.2)\qquad f(x) = x\frac{(m-1)x^m + (m+1)c}{(m+1)x^m + (m-1)c},$$

is such that

$$(2.3)\qquad f'(x) = (m^2-1)\left[\frac{x^m - c}{m(x^m+c) + (x^m-c)}\right]^2.$$

Hence $f(\alpha) = \alpha$, $f'(\alpha) = f''(\alpha) = 0$. Successive values a_n obtained by this formula converge rapidly to the required root. For instance, to solve $\phi(x) \equiv x^3 - 2 = 0$, take $a_0 = 1$. Then

$$(2.4)\qquad f(x) = x\left(\frac{2x^3+8}{4x^3+4}\right), \qquad a_1 = f(a_0) = \frac{5}{4}, \qquad a_2 = f(a_1) = \frac{1905}{1512}.$$

The approximation a_2 is accurate to six significant figures, whereas a_3 would be accurate to eighteen significant figures.

It is interesting to note that Bailey's function may be obtained by setting $y = x^m - c$ in the approximating formula given in the editor's paper *A Variation of Newton's Method*, namely,

$$(2.5)\qquad f(x) = x + \frac{-y}{y' + (y''/2)(-y/y')}.$$

3. Estimates of the error. Let us denote the relative error in the nth approximation to α by E_n, so that

$$(3.1)\qquad E_n = \frac{a_n}{\alpha} - 1, \qquad E_{n+1} = \frac{f(a_n)}{\alpha} - 1.$$

Then if in (2.5) we expand the function y in a Taylor series about the point $x = \alpha$, we have

$$(3.2)\qquad y = c_1(x-\alpha) + c_2(x-\alpha)^2 + c_3(x-\alpha)^3 + \cdots,$$

$$(3.3)\qquad \frac{f(x)}{\alpha} - 1 = \left(\frac{x}{\alpha} - 1\right)^3\left(\frac{\alpha}{c_1}\right)^2(c_2^2 - c_1c_3) + \text{higher order terms.}$$

Hence

$$(3.4)\qquad E_{n+1} = E_n^3\left(\frac{\alpha}{c_1}\right)^2(c_2^2 - c_1c_3) + \text{higher order terms in } E_n.$$

This formula gives a good estimate of the relative errors in the successive approximations, and shows that with each new approximation the number of known correct significant figures in the root α is about tripled. In the special case of Bailey's function (2.2) the estimate of relative error is

$$E_{n+1} = E_n^3(m^2 - 1)/12 + \text{higher order terms in } E_n. \tag{3.5}$$

4. Calculation of roots by machine. Bailey's function (2.2) should prove extremely valuable for the accurate determination of mth roots by a calculating machine. From a table of mth powers an approximation x^m to the given number c is found, such that x approximates $\sqrt[m]{c}$ to three significant figures. A single division on the machine then suffices to determine the root to nine or ten significant figures. The estimate (3.5), computed by slide rule or by machine to two or three significant figures, serves to indicate more precisely the magnitude of the error of this second approximation.

CLUB REPORTS 1944–45

Advisers and officers of Mathematics Clubs are requested to send in their annual reports for 1944–45 to the editor as soon after the school year as possible. Reports should include the titles of talks followed by the names of the speakers, an account of noteworthy events or activities or prizes awarded, and should conclude with the names of the officers.

CLUB REPORT 1943–44

Mathematics Society, Brooklyn College

The Mathematics Society of Brooklyn College completed the Spring term of its 1944 program with pride in its fifty active members who worked on a term topic under the guidance of Dr. Kormes, the faculty adviser. The topic was *Fourier series and orthogonal functions*, with emphasis on the modern treatment based upon the concept of function-space. The following lectures were delivered:

General introduction, by Dr. Kormes
Examples of Fourier expansions, by S. Greif
Orthogonal functions, by J. Greenstadt
Fourier's theorem, by G. Washnitzer
Physical applications, by W. Hauser
Physical applications, by N. Potter
Orthogonal spaces, by S. Shecter.

Since many of the members will be in the armed services shortly, Dr. Kormes suggested that a committee be organized to communicate with these members so that the spirit of the Society may be continued. In this way the members in the armed forces will be kept in closer contact with their mathematics. The suggestion was immediately adopted by the Society.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Statistics with General Applications. By M. M. Blair. New York, Henry Holt and Company, Inc., 1944. 14+690 pages. \$3.50.

This book is designed to present a large body of statistical theory and practice in a practical and non-technical way. The author "has chosen and limited his materials for the purpose of meeting the needs of that large and ever-increasing body of students and businessmen who must have an elementary knowledge of statistics. . . . No effort is made to introduce the student to the derivation of formulas." The plan of the book seems to be to present complete worksheets for the solution of those problems usually included in elementary statistics. The examples chosen are chiefly from the fields of agriculture, business, education, and sociology.

The book is divided into five parts: 1. General Methods.—2. Large Sample Analysis.—3. Time Series Analysis.—4. Small Sample Analysis.—5. Curvilinear and Multiple Analysis.

It is deplorable that the manuscript was not edited before publication by a competent mathematician, if not by a mathematical statistician. The chapters are replete with misuse of mathematical notation and terminology, as well as actual errors. To illustrate, equal signs are frequently used as punctuation marks; a "power" is a positive integral exponent; an "exponent" is a positive fractional exponent; a "ratio" is a proportion, and a "rate" is a "mixed ratio"; any curve is a "line"; any integer is an "even" number; *etc.*

Some of the errors might be mentioned. The least squares line (page 243) is not necessarily "a line which on the average comes nearest to all the points or items of data." In Figure 53, the intercept a is drawn incorrectly. The denominator in Formula 72 should be $\sum p_0q_0$. One can not say (page 481) that "the chances are 20 to 1 that the two small samples were drawn from separate parent populations" simply because the null hypothesis gives those odds against obtaining the given difference. The calculation of S_y (page 554) assumes $\sum Z=0$, contrary to fact. The signs of a and b in the parabola $Y=a+bX+cX^2$ (page 556) depend on the position of the origin, not on the shape of the curve. A similar statement is true for the cubic parabola (page 565). Since the sum of the deviations from the arithmetic mean is zero, all the summations on page 582 vanish.

The book is further marred by many arithmetical errors in the examples, as well as by a large number of typographical errors. In the opinion of the reviewer, this text can not be recommended in its present form.

H. D. LARSEN

Methods of Advanced Calculus. By Philip Franklin. New York and London, McGraw-Hill Book Company, Inc., 1944. 12+486 pages. $4.50.

As is well-known, this text-book is the second written by Professor Franklin on the subject of Advanced Calculus, the first having been published in 1940 by John Wiley and Sons under the title of "A Treatise on Advanced Calculus." The two books are in some respects different as to content, but in all respects different as to point of view.

The reader, familiar with the first book, will recall that it was in the nature of an introduction to the theory of functions of a real and complex variable. As such, it began with a discussion of the real number system, the Bolzano-Weierstrass and Heine-Borel theorems, and limits and continuity, before considering the more usual topics of an advanced calculus course.

While the first book was thus a sort of "Cours d'Analyse," the new book, to quote from its introduction, "has two principal objectives: first, to refresh and improve the reader's technique in applying elementary calculus; second, to present those methods of advanced calculus which are most needed in applied mathematics." Thus, the new book has the purpose of serving the needs of prospective engineers, physicists, and others who may regard mathematics primarily as a tool. Accordingly, it attempts to include as many topics of possible practical interest as could be crowded into a space of somewhat less than five hundred $5\times 8''$ pages and treat these topics with special stress on the manipulative and numerical aspects.

The book opens with two chapters that include a definition of the elementary functions of a complex variable, a treatment of partial differentiation and implicit functions, a statement of Taylor's series for functions of one complex variable and of two real variables, and incidentally a discussion of Newton's method for the approximate solution of equations, as well as a discussion of determinants and linear equations (but not of matrices). The third chapter begins with a study of vector analysis, with applications to the theory of space curves and surfaces—a study which is completed in the eighth chapter with a treatment of the divergence theorem and Stokes', Green's and Gauss' theorems. Chapters IV to VII are concerned with formal and approximate integration, multiple integrals, the Gamma function and other definite integrals, elliptic integrals, and line integrals. In connection with the last topic, there is developed a bit of the theory of analytic functions of a complex variable—the Cauchy-Riemann differential equations, the Cauchy integral and the residue theorem with applications to the evaluation of definite integrals, and conformal mapping by means of the Schwarz transformation. Chapters IX to XI deal with ordinary and partial differential equations and boundary value problems involving expansions in terms of Legendre polynomials, Bessel's functions, and Fourier series. Finally, Chapter XII provides an introduction to the calculus of variations and Lagrange's equations.

Like the "Treatise," the new book gives general references and a list of about

100 exercises at the close of each chapter. Some of these exercises are simple applications of the text, but others have the admirable purpose of providing an opportunity for the student himself to develop some non-trivial material not given explicitly in the text. Answers are given to all the exercises—a feature which may be helpful to engineers and others attempting to read the book without the aid of an instructor.

In order to include so many topics, the new book intentionally gives some intuitive or heuristic "proofs" and omits others altogether. To save space, it occasionally also omits minor and sometimes important details. For example, on page 3, after defining e^z, sin z and cos z for complex z by means of their Maclaurin expansions, the author says: "Convergent series of this type may be multiplied and added together in the same way that polynomials are combined. It follows that the functions defined by the series satisfy the relations $e^{z_1+z_2}=e^{z_1}e^{z_2}$ as well as the addition theorems $\sin(z_1+z_2)= \cdots$." Also, on page 4, the book fails to give any definitions of limit and derivative of a function of a complex variable prior to the tabulation of the answers for the derivatives of elementary functions of a complex variable. Of course, these details can be supplied in class by the instructor, their omission causing a hardship principally to a student who wishes to read the book on his own.

The statements of the theorems in the book seem, on the whole, to be accurate. An exception is the following made on page 27 regarding the approximation r_1 to a root r of an algebraic equation $f(x)=0$. "Usually, if r_1 is correct to n decimals, $f(r_1)$ will be less than $1/10^n$." An example contradicting this statement is the equation $f(x)=400-600x+200x^2=0$, which has the exact root $r=2$, but, for the approximation $r_1=2.001$, we find $f(r_1)=0.6$.

Indeed, it is not the accuracy but the manner in which some of the theorems are stated that may be open to objection. For example, in chapter VIII, the details of the theorems are dispersed in the main body of the text, instead of being printed compactly in some distinctive type. While stating theorems in the more traditional manner would have detracted from the informality of the presentation, it would have enhanced the reference value of the text. This would be especially true if, in addition to being stated in italics, each unproved theorem had been accompanied by a specific page reference to some other book containing a proof.

Despite this objection, the new volume may be regarded as one of the best text-books now available for any advanced calculus course which is intended to be a terminal course in mathematics for engineers, physicists and the like. However, for prospective mathematicians and others for whom the advanced calculus is but a stepping stone to higher mathematics, there is no question that Franklin's first book, the "Treatise," still is a better choice.

MORRIS MARDEN

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 666. *Proposed by N. D. Lane, St. Andrew's College, Aurora, Ont.*

For what value of k is the curve $x^m y^n = k$ self-reciprocal with respect to the circle $x^2+y^2=1$?

E 667. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let P, Q, R, P', Q', R' be arbitrary points on the respective edges BC, CA, AB, DA, DB, DC of a given tetrahedron $ABCD$. Prove that four planes, parallel to the faces BCD, CDA, DAB, ABC, drawn through the centroids of the respective tetrahedra $AQRP'$, $BRPQ'$, $CPQR'$, $DP'Q'R'$, form a tetrahedron of constant volume.

E 668. *Proposed by Walter Penney, Navy Department, Washington, D. C.*

Prove that the equation $x^2-3y^2=17$ has no solution in integers.

E 669. *Proposed by J. H. Butchart, Grinnell College*

Let G be the centroid of n coplanar points P_i, Q any point in the same plane, and Δ_i the signed area of the triangle QGP_i. Show that

$$\sum \Delta_i = 0.$$

E 670. *Proposed by C. D. Olds, Purdue University*

Sum the series

$$\sum_{r=1}^{2n-1} (-1)^{r-1} r \Big/ \binom{2n}{r}.$$

SOLUTIONS

Curves of Constant Width

E 610 (*b*) [1944, 88 and 532]. *Proposed by Howard Eves, Syracuse University*

What is the least area that a closed curve of constant diameter d may have?

Remarks by R. P. Boas, Jr., Harvard University, and Henry Scheffé, Syracuse

University. The least area is $\frac{1}{2}(\pi-\sqrt{3})d$, attained only by the "Reuleaux triangle" formed by three arcs of circles of radius d with centers at the vertices of an equilateral triangle of side d. For a proof, and references to earlier proofs, see A. E. Mayer, *Der Inhalt der Gleichdicke*, Math. Annalen, vol. 110 (1934), pp. 97–127 (especially p. 115), or Bonnesen and Fenchel, *Theorie der konvexen Körper*, Berlin, 1934 (Ergebnisse series, no. 3). Nearby portions of the latter book discuss many interesting properties of curves of constant width and some of their generalizations (*e.g.*, solids of constant width).

Two Fours

E 631 [1944, 405]. *Proposed by J. A. Tierney, Norwich University, Northfield, Vt.*

Express the number 64, using two fours (and the operations of arithmetic).

I. *Solution by* 4 *members of the Mathematics Dept. of the Manhattan High School of Aviation Trades.*

$$64 = \sqrt{\sqrt{\sqrt{4^{4!}}}} = \sqrt{4^{\Gamma(4)}} = 4^{[\log 4!]} = \left\{\phi(4!)\right\}^{\sqrt{4}} = 4^{\sqrt{\pi(4!)}}$$

(where $\pi(x)$ is the number of primes not exceeding x), or 44 in the quindenary scale.

II. *Solution by D. H. Browne, Buffalo, N. Y.* $\phi(\phi(4^4))$.

III. *Solution by Howard Eves, Syracuse University.* $4^{\sqrt{4!}}$.

Also solved by W. E. Buker, L. H. Bunyan, L. R. Chase, A. L. Lanckton, H. D. Larsen, E. D. Schell, Sol Rubinon, Jeanette Van Os, Alan Wayne, R. H. Wilson, Jr., F. E. Wood, and Thomas Wyman.

An unpublished manuscript of the late W. W. Rouse Ball contains an expression for every positive integer up to 90 in terms of three fours. Two fours suffice in many cases.

Coaxal Spheres

E 633 [1944, 405]. *Proposed by N. A. Court, University of Oklahoma*

Given a point M and four spheres (A), (B), (C), (D) whose centers form a tetrahedron, let MEE' be the transversal from M to the two opposite edges BC, DA, and let spheres (E) and (E') be constructed coaxal with the pairs of spheres (B), (C), and (D), (A). We have analogous spheres (F) and (F'), (G) and (G'). Show that the sphere (M) coaxal with (E), (E') is likewise coaxal with (F), (F') and with (G), (G').

Solution by the Proposer. The sphere (E), being coaxal with the spheres (B), (C), is orthogonal to any sphere orthogonal to (B) and (C), and in particular to the orthogonal sphere (R) of the four given spheres (A), (B), (C), (D). The spheres (E'), (F), (F'), (G), (G') are orthogonal to (R) for similar reasons. The sphere (R), being orthogonal to (E), (E'), is also orthogonal to the sphere (M) coaxal with these spheres. The three spheres (F), (F'), (M) are orthogonal to the same sphere (R), and their three centers are collinear; hence these spheres are coaxal. Similarly for the spheres (G), (G'), (M).

Also solved by Howard Eves.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4154. *Proposed by H. F. Sandham, Trinity College, Dublin*

Find the envelope of the axes of conics inscribed in a quadrilateral.

4155. *Proposed by G. W. Wishard, Norwood, Ohio*

Find a formula for the sum of the series

$$1 + 2^2 + 3^3 + \cdots + n^n.$$

If such a formula is impossible find superior and inferior limits for the sum.

4156. *Proposed by Charles M. Stein, New York*

Show that if $f(u)$ is a polynomial in u and $D = d/dx$, then for any positive integer n

$$\left\{1 + \frac{[(k-1)n]!}{[kn]!}(y^n - x^n)D^n\right\} \cdots \left\{1 + \frac{1}{n!}(y^n - x^n)D^n\right\} f(x^n) = f(y^n).$$

If possible, generalize the above result to analytic functions $f(u)$ as in the special case $n=1$ in 3985 [1942, 345].

4157. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the base such that a number of eight digits of the form *ababcdcd* can be the square of a number of four digits *mnmn*, where the numbers of two digits *ab* and *cd*, or *ab* and *mn*, or *cd* and *mn* are consecutive.

SOLUTIONS

Geodesic Curvature

4105 [1944, 49]. *Proposed by J. H. Butchart, Grinnell College*

Let ρ, ρ_s, R be respectively the radius of curvature of a curve, the radius of curvature of the locus of the center of the osculating sphere, and the radius of this sphere. Then the radius of geodesic curvature ρ_g of the locus of the center of curvature with regard to the polar developable is given by $\rho_g = R^3/(2R^2 - \rho\rho_s)$.

Solution by C. E. Springer, University of Oklahoma. If the curve is represented by $x^i = x^i(s)$, $(i=1, 2, 3)$, then $R^2 = \rho^2 + \tau^2\rho'^2$, where τ is the radius of torsion of the curve, and the prime indicates differentiation with respect to the arc length s. Let α^i, β^i, γ^i be the direction cosines of the tangent, principal normal, and binormal to the curve. Then the radius of curvature ρ_s of the curve generated by the point $x^i + \rho\beta^i - \rho'\tau\gamma^i$ is readily found to be given by

$$\rho_s = \rho + \tau(\rho'\tau)'.$$

The element of arc on the polar developable

$$X^i = x^i + \rho\beta^i + t\gamma$$

is given by $d\sigma^2 = Edt^2 + 2Fdtds + Gds^2$, where

$$E = 1, \qquad F = -\rho/\tau, \qquad G = \frac{\rho^2}{\tau^2} + \left(\frac{t}{\tau} + \rho'\right)^2.$$

From the formula

$$\frac{1}{\rho_g} = -\frac{1}{\sqrt{EG - F^2}}\left(\frac{\partial}{\partial s}\frac{F}{\sqrt{G}} - \frac{\partial}{\partial t}\sqrt{G}\right),$$

(Eisenhart, *Differential Geometry*, 1909, p. 136), the expression for ρ_g with $t=0$ can be calculated. By means of the expressions for R and ρ_s given above, the formula for ρ_g reduces to $R^3/(2R^2 - \rho\rho_s)$.

Solved also by P. D. Thomas.

Editorial Note. The solution by Thomas is based on theorems in Eisenhart's *Differential Geometry*, pp. 14, 37, 149; and in Bell's *Coordinate Geometry of Three Dimensions*, pp. 301, 370, 371. The proposer did not give his proof but stated that it is based on developing the polar developable into a plane, the given curve reducing to a point.

The computations in the two solutions are quite lengthy and it appears simpler to use vectors. If P is a point on the given curve with the vector $\mathbf{r}$; $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ are unit vectors for the tangent, principal normal, and binormal; s is the arc length and prime marks denote derivatives with respect to s; then, if $\mathbf{c}$ denotes the vector of C the center of circular curvature, and $\mathbf{S}$ the vector of the center S of spherical curvature, we have

$$(1) \qquad \begin{aligned} &\mathbf{c} = \mathbf{r} + \rho\mathbf{n}, \qquad \mathbf{c}' = \rho\tau\mathbf{b} + \rho'\mathbf{n}, \\ &\mathbf{S} = \mathbf{c} + \sigma\rho'\mathbf{b}, \qquad \rho_S = \rho + \sigma(\sigma\rho')', \qquad R^2 = \rho^2 + (\sigma\rho')^2, \end{aligned}$$

where $\kappa = 1/\rho$, $\tau = 1/\sigma$ are respectively the curvature and torsion at P. Then, if the subscripts denote the corresponding quantities for the locus of C,

$$(2) \qquad s_1' = ds_1/ds = \tau R, \qquad R\mathbf{t}_1 = \rho\mathbf{b} + \sigma\rho'\mathbf{n}.$$

If ψ denotes the angle from the positive direction of $\mathbf{n}$ to $\mathbf{t}_1$, then

$$
(3)\qquad
\begin{aligned}
\mathbf{t}_1 &= \sin\psi\mathbf{b} + \cos\psi\mathbf{n}, \qquad \psi' = (\rho' R - \rho R')/R\sigma\rho',\\
\psi' + \tau &= (2R^2 - \rho\rho_S)/R^2\sigma,\\
\kappa_1\mathbf{n}_1 &= -\frac{\kappa\cos\psi}{\tau R}\mathbf{t} + \frac{\psi'+\tau}{\tau R}\,[\cos\psi\mathbf{b} - \sin\psi\mathbf{n}].
\end{aligned}
$$

The locus of C lies on the polar developable surface with the normal $\mathbf{t}$ at C, and hence the first term above is the component of the vector curvature $\kappa_1\mathbf{n}_1$ along the normal to the surface, the remaining part is therefore the curvature of the locus of C tangential to the surface, that is the vector geodesic curvature at C. The coefficient of $(\psi'+\tau)/\tau R$ is a unit vector in the direction CS_b, where S_b is the projection of S on the binormal at P and CS_b has the length R. Hence the geodesic curvature of the C locus is

$$\kappa_g = (2R^2 - \rho\rho_S)/R^3.$$

Thus, if $2R^2 > \rho\rho_s$, the locus of C turns from the geodesic tangent at C toward the region in which S_b lies.

A Periodic Product

4107 [1944, 50]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $N = 123 \cdots (n-3)(n-2)n$ be a number of $n-1$ digits in order of increasing magnitude in the system of base $n+1$. The product $P = N\cdot L$ is formed where $L = \alpha\beta$ consists of two digits whose sum $\alpha+\beta=\gamma$ is less than n and δ is the greatest common divisor of n and γ. Show that

$$P = N\cdot L = ab\cdots pqab\cdots pq\cdots ab\cdots pq,$$

where the δ periods $ab\cdots pq$ are formed by n/δ distinct digits.

Dedicated to E. P. Starke.

Solution by E. P. Starke, Rutgers University. (1) $n\cdot N$ is the n-digit number $11\cdots 1$, and $L\cdot n\cdot N$ is the $(n+1)$-digit number $M = \alpha\gamma\gamma\gamma\cdots\gamma\beta$. Hence the desired number $P = N\cdot L = M/n$. From the form of $M = \alpha\gamma\cdots$ it is evident that the first digit a of $P = M/n$ is α if $\alpha+\gamma<n$, but is $\alpha+1$ if $\alpha+\gamma \geqq n$. From the form of N and L, the last digit q of $P = N\cdot L$ must be $n+1-\beta$ unless $\beta=0$.

(2) Any number is congruent, mod n, to the sum of its digits.

(3) If M is divided by n, the remainders r_i and r_j $(i<j<n)$ of the ith and jth partial divisions are alike if and only if the $(j-i+1)$-digit number $r_i\gamma\gamma\gamma\cdots\gamma(\gamma-r_j)$* is divisible by n. By (2) it is congruent mod n to $\gamma(j-i)$ which is divisible by n if and only if $(j-i)$ is divisible by n/δ. Hence the number M/n consists of δ periods, $a\cdots q$, of n/δ digits each, a and q being given in (1).

(4) For the divisions of the 2-digit numbers $\theta\gamma$, $\theta=0, 1, \cdots, n-1$, by n, different remainders imply different quotients. Thus the digits within any one period of M/n are distinct. This completes the proof.

The connection of this problem with 3851 [1940, 56] and 4100 [1945, 163] should be noted. The methods of solution are entirely analogous.

* If $r_j>\gamma$, the last two digits are $(\gamma-1)(\gamma+n+1-r_j)$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor T. E. Mergendahl of Tufts College has been elected president of the Association of Teachers of Mathematics in New England.

Professor H. C. Carver and Assistant Professor Samuel Eilenberg of the University of Michigan have been granted leaves of absence, the former for work with the Army Air Forces and the latter to serve with the applied mathematics group at Columbia University.

Associate Professor Will Feller of Brown University has been appointed to a professorship at Cornell University.

Dr. N. A. Hall of Chance Vought Aircraft has been appointed mechanical engineer in charge of thermodynamic analysis for the research division of the United Aircraft Corporation in East Hartford, Connecticut.

Professor W. H. Hill of Kansas State Teachers College of Pittsburg, Kansas, has retired.

Assistant Professor J. E. Ikenberry of Franklin and Marshall College has been appointed professor of mathematics and head of the department at Madison College in Harrisonburg, Virginia.

Professor H. K. Justice of the University of Cincinnati has been granted a leave of absence to serve with the United States Government.

Assistant Professor Wilfred Kaplan of the University of Michigan, Assistant Professor Charles Loewner of the University of Louisville, and Associate Professor S. E. Warschawski of Washington University have been granted leaves of absence to serve with the applied mathematics group at Brown University.

Associate Professor G. E. Moore of the University of Illinois has been appointed assistant dean of liberal arts and sciences.

Associate Professor W. R. Murray of Franklin and Marshall College has been appointed chairman of the department.

Dr. C. D. Olds of Purdue University has been appointed to an assistant professorship at San Jose State College, San Jose, California.

Assistant Professor C. K. Payne of Washington Square College, New York University, has been promoted to an associate professorship.

Dr. L. M. Rauch has been appointed to an assistant professorship at Wesleyan University.

Dr. R. D. Specht of the University of Florida has been appointed to an assistant professorship at the University of Wisconsin.

Dr. R. E. Street of Dartmouth College has been appointed engineer in the General Electric Company at Schenectady.

The following appointments to instructorships have been announced:
Polytechnic Institute of Brooklyn: M. J. Forray
Southeastern Louisiana College (Hammond, Louisiana): Dr. Margaret R. Davis
University of Chicago (College): S. P. Hughart
University of Kentucky: Mrs. Betty D. Crawley
University of Nebraska: Dr. J. F. Heyda

Dean J. L. Gibson of the University of Utah died February 10, 1945. He was a charter member of the Association.

Professor Emeritus W. F. Long of Franklin and Marshall College died January 1, 1945.

M. M. S. Moriarty, formerly a teacher in Holyoke High School, Holyoke, Massachusetts, died April 30, 1944. He was a charter member of the Association.

Professor C. H. Rowe of the University of Dublin died in November 1944.

Professor Emeritus G. T. Sellew of Knox College died December 27, 1944. He was a charter member of the Association.

Dr. J. A. Swenson of Teachers College, Columbia University, died May 2, 1944.

Associate Professor A. L. Underhill of the University of Minnesota died January 18, 1945. He was a charter member of the Association.

P. H. Underwood, formerly a teacher in Ball High School, Galveston, Texas, died December 14, 1944. He was a charter member of the Association.

Professor N. R. Wilson of the University of Manitoba died December 27, 1944.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1945:

Brown University: From June 11 to September 1 the program of advanced instruction and research in mechanics will be in its fifth year. Since this program is supported by outside funds, because of its importance in building up mathematics in the field of applications to engineering, no tuition fees will be charged. The faculty will consist of Professors Stefan Bergman, Lipman Bers, G. E. Hay, Charles Loewner, Willy Prager, S. E. Warschawski, and two others still

to be chosen. A special feature outside the core of material in mechanics is a course in thermodynamics. In addition, the following courses are planned: introduction to partial differential equations, geometrical foundations of mechanics, theory of flight (engineering aerodynamics), elasticity, differential and integral equations of mathematical physics, advanced aerodynamics, advanced theory of vibrations, and special seminars and research in various phases in mechanics.

The Catholic University of America: From July 2 to August 9 the following advanced courses will be offered: By Professor Finan; theory of equations; theory of numbers. By Professor Ramler: advanced euclidean geometry; differential equations; analytic projective geometry. By Professor Rice: advanced calculus; mathematical statistics.

Columbia University: From July 2 to August 10 the following graduate courses will be offered: By Professor Kasner: introduction to modern mathematics; transformations and groups. By Professor Lorch: theory of numbers. By Professor Murray: differential equations. By Professor Ritt: theory of functions of a real variable.

Northwestern University: From June 23 to August 25 the following advanced courses will be offered: higher analysis; theory of statistics; introduction to the theory of numbers; geometry for teachers; the history and teaching of mathematics; theory of differential equations; seminar in analysis. These courses may be taken for six or nine weeks.

The Ohio State University: From June 19 to August 31 the following advanced courses will be offered: By Professor Bamforth: advanced euclidian geometry. By Professor Blumberg: introduction to modern mathematics; introduction to the theory of relativity. By Professor Jones: differential equations. By Professor Mickle: advanced calculus. By Professor Rado: theory of fields.

The State University of Iowa: From June 13 to August 8 the following graduate courses will be offered: By Professor Conkwright: theory of numbers. By Professor Chittenden: differential equations. By Professor Knowler: elements of statistics. By Professor Price: teaching of mathematics. By Professor Wylie: astronomy.

The University of Chicago: The first term of the summer quarter is from June 25 to August 3 and the second term from August 4 to September 15. The following advanced courses will be offered: By Professor Barnard: vector analysis. By Professor Hartung: two courses in the teaching of mathematics. By Professor Hestenes: calculus of variations. By Professor Graves: theory of functions of real variables. By Professor Logsdon: higher plane curves. By Professor Schilling: theory of numbers. These courses will meet four times per week in the first term and the first half of the second term.

The University of Colorado: From July 2 to August 22 a course in power series will be offered in addition to the usual elementary courses. Other advanced courses will be offered if there is sufficient demand.

The University of Michigan: From July 2 to August 24 the following courses will be offered in addition to the standard courses in differential equations, theory of equations and determinants, and advanced calculus: By Professor Anning: modern geometry. By Professor Craig: theory of statistics (second course). By Professor Coe: theoretical mechanics. By Professor Dwyer: theory of statistics (first course) and computational methods. By Professor Fischer: theory of probability and introduction to mathematics of life insurance. By Professor Hildebrandt: theory of functions of a real variable and partial differential equations. By Professor Karpinski: teaching of geometry and history of algebra. By Professor Rainich: algebraic theory and differential geometry. By Professor Wilder: introduction to the foundations of mathematics and combinatorial topology.

The University of Minnesota: From June 20 to July 28 the following advanced courses will be offered: By Professor Gibbens: tutorial course in advanced mathematics. By Professor Koehler: calculus of finite differences. By Professor Olmstead: limits and series. From July 30 to September 1 the following will be given: By Professor Carlson: tutorial course in advanced mathematics. By Professor Koehler: vector analysis.

The University of Pennsylvania: From July 5 to August 30 the following advanced courses will be offered: By Professor Caris: differential equations. By Professor Rademacher: partial differential equations. By Professor Whaples: vector analysis.

The University of Virginia: From July 2 to August 25 the following advanced courses will be offered: By Professor Hedlund: advanced calculus and applied mathematics. By Professor Whyburn: functions of real variables.

The University of Wisconsin: From May 28 to September 15, in addition to the usual elementary courses, the following advanced courses will be offered: By Professor Bruck: projective geometry. By Professor Everett: theory of equations. By Professor Langer: statics, differential equations. From June 23 to August 17 the following advanced courses will be offered: By Professor Evans: mathematics of educational statistics. By Professor Everett: theory of equations. By Professor Langer: mathematical applications. By Professor Trump: college geometry.

The University of Wyoming: From June 20 to July 24 the following advanced courses will be offered: By Professor Barr: differential equations, college geometry. By Professor Neubauer: history of mathematics. From July 25 to August 28: By Professor Barr: differential equations, advanced college algebra. By Professor Neubauer: fundamental concepts.

WAR INFORMATION

Edited by C. V. Newsom

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

NEW LIST OF ESSENTIAL ACTIVITIES

On January 15, 1945, the Selective Service System published a revised list of essential activities. In the introduction to the new Memorandum it is specified that "All technical, scientific and research personnel engaged in any of the activities in the list, whether or not the activity appears in bold type (indicating a critical activity) or regular type, are regarded as being engaged in critical activities." Quotations are given below from items 33 and 35 contained in the list.

33. *Educational services.*—Public and private industrial and agricultural vocational training; elementary, secondary, and preparatory schools; junior colleges, colleges, universities, and professional schools, educational and scientific research agencies; United States Maritime Service training program; Civil Aeronautics Administration Civilian Pilot Training Program; armed forces contract flying, ground and factory aviation schools; and the production of technical and vocational training films.

35. *Technical, scientific, and management services.* The supplying of technical, scientific and management services to establishments engaged in war production; union-management negotiation services; publication of technical and scientific books and journals.

FEDERATION OF WOMEN'S CLUBS METRIC RESOLUTION

At the annual convention of the General Federation of Women's Clubs held April 25–28, 1944, in St. Louis, the following resolution was introduced and adopted unanimously by the delegates. This organization represents 16,500 clubs and 2,500,000 individual members.

Whereas, the irregular, numerous, unwieldy, and complicated units of weights and measures used in the United States and Great Britain are a hindrance to the teaching of arithmetic, every day commercial transactions, and world trade, and

Whereas, the metric system of weights and measures has only three units; meter, liter, and gram, interrelated and decimally divided like our dollar, and

Whereas, the metric system is now used in the United States in science, some factories, jewelry and optical industries, all electrical and radio measurements, athletic events, some hospitals and government departments, and especially at present in the manufacture of ammunition, and

Whereas, the Council on Pharmacy and Chemistry of the American Medical Association has recently decided that henceforth it will use only the metric system, and

Whereas, the gradual introduction of the metric system in this country (exactly as it has been introduced in 55 other countries) is feasible, and

Whereas, the full adoption of the metric system by the United States would be of great benefit to this country in post-war reconstruction, in promoting international commercial relations, particularly with the countries of Latin America, Continental Europe and Asia, therefore be it

Resolved, that the General Federation of Women's Clubs in Convention assembled, April, 1944, endorses legislation in Congress for the nation-wide adoption of the metric system of weights and measures.

The foregoing resolution was drawn up and presented by Eleonore F. Hahn (Mrs. Otto Hahn), member of the Board of Directors of the General Federation of Women's Clubs.

NOTES ON THE NAVY V-12 PROGRAM

On February 17, 1945, President Roosevelt signed a bill authorizing an increase in the size of the Navy ROTC. At present there are twenty-seven NROTC organizations with an enrollment of approximately 6,500 men. The new measure increases the limitation on the size of the NROTC from 7,200 to 24,000 until one year after the cessation of hostilities; thereafter the upper limit will be 14,000. The number and location of the new ROTC units will be determined by a special committee of educators and Naval officers. It is expected that the additional units will be in operation by July 1, 1945.

The Navy also plans to increase the standards of the NROTC program to the level prevailing before the war. This means that the curriculum will be placed on an eight-term basis, thereby enabling students to complete their college education before being commissioned.

As a consequence of this reorganization of the Navy College Training Program, it is expected that a large proportion of the V-12 personnel will be transferred to the NROTC by July 1. Some special categories, however, will probably continue under the V-12 Program. Among these will be Marine Corps trainees, medical and theological students, physics majors, and aerology specialists.

THE MATHEMATICAL TABLES PROJECT

As a supplement to the article "The Mathematical Tables Project," which appeared in this MONTHLY for November, 1944, a complete list follows of all articles containing short tables prepared under the supervision of members of the Mathematical Tables Project up to January 1, 1945.

1. On the computation of the second differences of the $Si\ (x)$, $Ei\ (x)$, and $Ci\ (x)$ functions, by A. N. Lowan, Bull. Amer. Math. Soc., vol. 45 (1939), No. 8.

2. Errors in Hayashi's table of Bessel functions for complex arguments, by A. N. Lowan and Gertrude Blanch, Bull. Amer. Math. Soc., vol. 47 (1941), No. 4.

3. The internal temperature-density distribution of the sun, by G. Blanch, A. N. Lowan, R. E. Marshak, and H. A. Bethe, Astrophys. J., vol. 94 (1941), No. 1.

4. Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula, by A. N. Lowan, Norman Davids, and Arthur Levenson, Bull. Amer. Math. Soc., vol. 48 (1942), No. 10.

5. A table of coefficients for numerical differentiation, by A. N. Lowan, H. E. Salzer, and A. Hillman, Bull. Amer. Math. Soc., vol. 48 (1942), No. 12.

6. On the function $H(m, a, x) = \exp(-ix)\ F(m+1-ia, 2m+2; ix)$, by A. N. Lowan and William Horenstein, J. Math. and Phys., vol. 21 (1942), No. 4.

7. Table of integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$, by A. N. Lowan and Milton Abramowitz, J. Math. and Phys., vol. 22 (1943), No. 1.

8. Table of coefficients in numerical integration formulae, by A. N. Lowan and Hebert Salzer, J. Math. and Phys., vol. 22 (1943), No. 2.

9. Table of $Ji_0(x) = \int_x^\infty [J_0(t)/t]dt$ and related functions, by A. N. Lowan, G. Blanch, and M. Abramowitz, J. Math. and Phys., vol. 22 (1943), No. 2.

10. Table of Fourier coefficients, by A. N. Lowan and Jack Laderman, J. Math. and Phys., vol. 22 (1943), No. 3.

11. Coefficients for numerical differentiation with central differences, by H. E. Salzer, J. Math. and Phys., vol. 22 (1943), No. 3.

12. Seven-Point Lagrangian integration formulas, by G. Blanch and I. Rhodes, J. Math. and Phys., vol. 22 (1943), No. 4.

13. A short table of the first five zeros of the transcendental equation $J_0(x)Y_0(kx) - J_0(kx)Y_0(x) = 0$, by A. N. Lowan and A. Hillman, J. Math. and Phys., vol. 22 (1943), No. 4.

14. Table of coefficients for inverse interpolation with central differences, by H. E. Salzer, J. Math. and Phys., vol. 22 (1943), No. 4.

15. Table of $f_n(x) = [n!/(x/2)^n]J_n(x)$, by the Mathematical Tables Project, J. Math. and Phys., vol. 23 (1944), No. 1.

16. Table of coefficients for inverse interpolation with advancing differences, by H. E. Salzer, J. Math. and Phys., vol. 23 (1944), No. 2.

17. A new formula for inverse interpolation, by H. E. Salzer, Bull. Amer. Math. Soc., vol. 50 (1944), No. 8.

18. Coefficients for interpolation within a square grid in the complex plane, by A. N. Lowan and H. E. Salzer, J. Math. and Phys., vol. 23 (1944), No. 3.

19. Formulas for complex interpolation, by A. N. Lowan and H. E. Salzer, Quart. Appl. Math., vol. 2 (1944), No. 3.

20. Table of coefficients for differences in terms of the derivatives, by H. E. Salzer, J. Math. and Phys., vol. 23 (1944), No. 4.

21. On the distribution of errors in the nth tabular differences, by A. N. Lowan and J. Laderman, Ann. Math. Statist., vol. 10 (1939), No. 4.

22. Note on the computation of the differences of the $Si(x)$, $Ci(x)$, $Ei(x)$, and $Ei(-x)$ functions, by M. Abramowitz, Bull. Amer. Math. Soc., vol. 46 (1940), No. 4.

23. On the inversion of the q-series associated with Jacobian elliptic functions, by A. N. Lowan, G. Blanch, and W. Horenstein, Bull. Amer. Math. Soc., vol. 48 (1942), No. 10.

24. Roots of sin $z=z$, by A. P. Hillman and H. E. Salzer, Philos. Mag., vol. 34 (ser. 7), August, 1943.

25. Coefficients for numerical integration with central differences, by H. E. Salzer, Philos. Mag., vol. 35 (Ser. 7), April, 1944.

THE MATHEMATICAL ASSOCIATION OF AMERICA

REPORT OF THE TREASURER FOR THE YEAR 1944

The following report of the Secretary-Treasurer as Treasurer for the year 1944 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I. Total Funds of the Association on December 31, 1943

(See Treasurer's report, pp. 177–180 of the Monthly for March, 1944)

Current Fund (checking account)		$ 2,262.47
Savings Account, Cleveland Trust Company		1,721.97
Savings Account, Oberlin Savings Bank		811.20
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 7,016.73	
Chace Fund	8,974.07	
Houck Fund	8,436.24	
Chauvenet Fund	645.85	
Life Membership Fund	777.52	
General Fund	21,479.59	47,330.00
		$52,125.64

II. Current Fund Account for 1944

Receipts		Expenditures	
Balance, Jan. 1, 1944	$ 2,262.47	Monthly	
Individual dues	8,834.35	Publication	$ 4,922.05
Institutional dues	579.00	Reprints	51.44
Initiation fees	272.00	Editor-in-Chief's office	401.42
Subscriptions	2,726.54	Secretary-Treasurer's office	
Sale of back numbers Monthly	688.97	Clerical help	3,250.00
Advertisements	685.00	Postage	453.45
Sale of Archibald's Outline	57.00	Printing	247.29
Interest on General Fund	611.86	Office supplies	163.57
Interest on Carus Fund	199.91	Bank charges	79.33
Interest on Chace Fund	255.58	Exec. and Finance committees	116.68
Interest on Houck Fund	240.21	Regional Governors	161.56
Interest on Chauvenet Fund	18.22	Sections	89.35
Interest on Life Membership Fund	22.26	Subventions	
Interest from Hardy Fund	120.00	Amer. Math. Society	100.00
Sale of monographs (Carus)	1,136.99	Mathematical Reviews	500.00
Sale of Papyrus (Chace)	118.61	National Math. Magazine	200.00
From Oberlin Savings Bank	162.24	Back numbers Monthly	118.56
Miscellaneous sources	2.29	Register	578.56
Transferred from General Fund	1,941.78	Meetings	260.75
Transferred from Chauvenet Fund	31.78	Chauvenet prize (Chauv. Fund)	50.00
		Auditing	20.00
		Subscriptions to Annals	15.00
		B. F. Finkel (Hardy Fund)	120.00
		To General Fund	600.00
		To Ithaca Savings Bank	1,000.00
		Transferred to Carus Fund	1,336.90
		Transferred to Chace Fund	374.19
		Transferred to Houck Fund	240.21
		Transferred to Life Memb. Fund	22.26
		Balance, Dec. 31, 1944	5,474.49
	$20,947.06		$20,947.06

III. Savings Account, Cleveland Trust Company

Balance, Jan. 1, 1944	$1,721.97		
Interest	12.55	Transferred to Invested Funds	$1,734.52
	$1,734.52		$1,734.52

IV. Savings Account, Ithaca Savings Bank

From Current Fund	$1,000.00	Balance, Dec. 31, 1944	$1,000.00

V. Invested Funds, Cleveland Trust Company

Market value of securities, Dec. 31, 1943	$47,330.00	Decrease in value of securities	$ 1.75
From Savings Account	1,734.52	Market value of securities, Dec. 30, 1944	49,622.00
From Current Fund	600.00	Cash balance, Dec. 31, 1944	40.77
	$49,664.52		$49,664.52

List of Securities

	Par Value	Market Value Dec. 30, 1944
U. S. Savings Bonds	$1,700.00	$ 1,610.00
U. S. Treasury Note, 1%, Ser. A, 1946	3,000.00	3,000.00
U. S. Treasury Bond, $2\frac{3}{4}$%, 1947	1,000.00	1,020.00
U. S. Treasury Bonds, 2%, 1950	3,000.00	3,060.00
U. S. Treasury Bonds, $1\frac{3}{4}$%, 1948	2,000.00	2,020.00
U. S. Savings Bonds, $2\frac{1}{2}$%, Ser. G, 1953	3,000.00	2,850.00
U. S. Savings Bonds, $2\frac{1}{2}$%, Ser. G, 1954	8,200.00	7,872.00
Canadian Nat. Ry. Co. Bonds, $4\frac{1}{2}$%, 1956	2,000.00	2,300.00
Gatineau Power Co. 1st Mort. Bonds, $3\frac{3}{4}$%, Ser. A, 1969	2,000.00	2,080.00
Shawinigan W. and P. Co. 1st Mort. Bonds, $4\frac{1}{2}$%, 1970	2,000.00	2,060.00
C. and O. Ry. Co. Ref. Mort. Bonds, $3\frac{1}{2}$%, Ser. D, 1996	3,000.00	3,150.00
Penn. R. R. Co. Gen. Mort. Bonds, $3\frac{3}{4}$%, Ser. C, 1970	2,000.00	2,100.00
Union Pacific R. R. Co. Deb. Bonds	3,000.00	3,210.00
Amer. Tel. and Tel. Co. Conv. Deb. Bonds, 3%, 1956	2,000.00	2,460.00
Columbus and So. Ohio Elec. Co. 1st Mort. Bonds, $3\frac{1}{4}$%, 1970	2,000.00	2,180.00
Commonwealth Edison Co. Conv. Deb. Bonds, $3\frac{1}{2}$%, 1958	2,000.00	2,340.00
Montana Power Co. 1st Ref. Mort. Bonds, $3\frac{3}{4}$%, 1966	3,000.00	3,150.00
New York Steam Corp. 1st Mort. Bond, $3\frac{1}{2}$%, 1963	1,000.00	1,050.00
Texas Power and Light Co. 1st Mort. Bond, 5%, 1956	1,000.00	1,060.00
Phelps Dodge Corp. Conv. Deb. Bond, $3\frac{1}{2}$%, 1952	1,000.00	1,050.00
		$49,622.00

VI. Carus Fund

Balance, Jan. 1, 1944	$7,016.73	Decrease in value of securities	$ .26
Sale of monographs	1,136.99		
Interest	199.91	Balance, Dec. 31, 1944	8,353.37
	$8,353.63		$8,353.63

VII. Chace Fund

Balance, Jan. 1, 1944	$8,974.07	Decrease in value of securities	$ .33
Sale of Papyrus	118.61		
Interest	255.58	Balance, Dec. 31, 1944	9,347.93
	$9,348.26		$9,348.26

VIII. Houck Fund

Balance, Jan. 1, 1944	$8,436.24	Decrease in value of securities	$ .31
Interest	240.21	Balance, Dec. 31, 1944	8,676.14
	$8,676.45		$8,676.45

IX. Chauvenet Fund

Balance, Jan. 1, 1944	$ 645.85	Decrease in value of securities	$ .03
Interest	18.22	Chauvenet prize	50.00
		Balance, Dec. 31, 1944	614.04
	$ 664.07		$ 664.07

X. Life Membership Fund

Balance, Jan. 1, 1944	$ 777.52	Decrease in value of securities	$ .03
Interest	22.26	Transferred to General Fund	44.72
		Balance, liability as of Dec. 31, 1944	755.03
	$ 799.78		$ 799.78

XI. General Fund (Invested)

Balance, Jan. 1, 1944	$21,479.59	Decrease in value of securities	$ .79
Transferred from Current Fund	600.00	Transferred to Current Fund	1,941.78
Transferred from Savings Acct., Cleveland Trust Company	1,734.52		
Transferred from Life Memb. Fund	44.72	Balance, Dec. 31, 1944	21,916.26
	$23,858.83		$23,858.83

XII. Total Funds of the Association, December 31, 1944

Current Fund (checking account)		$ 5,474.49
Savings Account (Ithaca Savings Bank)		1,000.00
Savings Account (Oberlin Savings Bank)		648.96
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 8,353.37	
Chace Fund	9,347.93	
Houck Fund	8,676.14	
Chauvenet Fund	614.04	
Life Membership Fund	755.03	
General Fund	21,916.26	49,662.77
		$56,786.22

DECEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, December 2, 1944. Professor Anna Pell Wheeler, Chairman of the Section, presided at the morning and afternoon sessions.

There were twenty-seven present, including the following nineteen members of the Association: H. W. Brinkmann, P. A. Caris, S. F. Cavalli, J. W. Clawson, F. L. Dennis, Arnold Dresden, W. H. Gottschalk, V. H. Haag, Marguerite Lehr, F. L. Manning, A. E. Meder, Lillian Moore, F. D. Murnaghan, W. R. Murray, C. A. Nelson, C. O. Oakley, J. C. Oxtoby, A. D. Wallace, Anna Pell Wheeler.

At the business meeting the following officers were elected for the coming year: Chairman, C. A. Nelson, New Jersey College for Women, Rutgers University; Secretary, W. H. Gottschalk, University of Pennsylvania. The program committee for the next meeting will be F. L. Dennis, Ursinus College, J. C. Oxtoby, Bryn Mawr College (Chairman), and A. D. Wallace, University of Pennsylvania. The next meeting will be held at Philadelphia on December 1, 1945. At the instance of Professor Dresden, appropriate action was taken in memory of the late Professor James A. Shohat of the University of Pennsylvania.

The program consisted of the following papers:

1. *Mapping problems in aerial photography*, by Professor Marguerite Lehr, Bryn Mawr College.

This paper dealt with the grid method developed by the Canadian Topographical Survey for mapping unexplored land by means of oblique aerial photographs. Relation to the use of vertical photographs for surveying was outlined.

2. *Spherical triangles on a slide rule*, by Professor F. L. Dennis, Ursinus College.

The speaker derived an auxiliary law of sines which made possible the solution of all oblique spherical triangles on a specially designed slide rule. Several examples in navigation and astronomy were used to illustrate the reasonably high degree of accuracy of the method.

3. *Continuous flows and AP functions*, by Dr. W. H. Gottschalk, University of Pennsylvania.

A point x of a metric space in which there is defined a continuous flow is said to be almost periodic provided that to each $\epsilon > 0$ there corresponds a relatively dense sequence $\{t_n\}$ of dates such that the distance between each x_{t_n} and x is less than ϵ. Let X denote the set of all bounded uniformly continuous complex-valued functions defined for all reals. Let X be metrized in the usual fashion, and let a continuous flow be defined in X by translation of the functions. Then a point x of X is almost periodic if and only if x is an almost peri-

odic function in the sense of Bohr. Bochner's characterization of almost periodic functions is then given an abstract topological setting. A point of a complete metric space in which there is defined a distance-preserving continuous flow is almost periodic if and only if its orbit is conditionally compact.

4. *The uniform tension of an elastic cylinder*, by Professor F. D. Murnaghan, Johns Hopkins University.

This paper will be printed in a subsequent issue of this MONTHLY.

P. M. WHITMAN, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Johns Hopkins University on Saturday, December 9, 1944. Professor C. H. Wheeler III, Chairman of the Section, presided at the morning and afternoon sessions.

There were fifty-two persons present, including the following thirty-three members of the Association: R. A. Baumgartner, T. E. Berry, S. G. Bourne, G. R. Clements, A. C. Cohen, Jr., A. T. Craig, A. E. Currier, J. A. Duerksen, B. H. Gere, Michael Goldberg, D. W. Hall, L. M. Kells, J. A. Larrivee, Florence P. Lewis, M. H. Martin, T. W. Moore, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, Irwin Roman, R. E. Root, E. D. Schell, A. D. Sollins, M. F. Smiley, C. V. L. Smith, J. H. Taylor, Marian M. Torrey, V. J. Varineau, R. W. Wagner, J. A. Ward, C. H. Wheeler, III, R. H. Wilson, Jr., Oscar Zariski.

It was decided to hold the spring meeting at George Washington University on either the first or second Saturday in May, 1945.

The morning program consisted of the following papers:

1. *Integral solutions of* $x^3+py^3=1$, *p a prime*, by Professor A. E. Currier, United States Naval Academy.

It was shown that this equation has a solution other than (1, 0) if and only if the fundamental unit of the ring 1, $\sqrt[3]{p}$, $\sqrt[3]{p^2}$ is of the form $x_1+y_1\sqrt[3]{p}$. Then (x_1, y_1) is the only such solution. A necessary condition is that $p=2$, or p be one of a finite set of primes of the form $17+18k$, or p be of one of the forms $1+18k$, $7+18k$.

2. *On the commutativity of certain rings*, by Professor N. Jacobson, Johns Hopkins University, introduced by Dr. W. K. Morrill.

3. *Configuration theorems*, by Dr. J. C. Abbott, United States Naval Academy, introduced by the Chairman.

This paper consisted of a partial classification of various configuration theorems according to their relative strength and rank, with particular emphasis on special cases of the Desargueian and Pappus theorems.

4. *Coördinates*, by T. J. Benac, United States Naval Academy, introduced by the Secretary.

Using the methods of introducing coördinates in the classical Desargueian plane, coördinates were introduced from alternative fields and Veblen-Wedderburn number systems in the non-Desargueian D_9 plane and the affine Little-Desargueian plane respectively. The concept of geomorphism was defined and illustrated, two algebraic systems being geomorphic if they define the same projective plane.

In the afternoon the Session heard an address by Professor Oscar Zariski on *The Riemann Manifolds of an Algebraic Function Field.*

W. K. MORRILL, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, BROOKLYN, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF

COLLEGIATE MATHEMATICS

VOLUME 52 NUMBER 5

CONTENTS

MAY 1945

The Summer Meeting of the Association to have been held at Montreal, June 23-25, 1945, has been cancelled at the request of the Office of Defense Transportation.

The AMERICAN MATHEMATICAL MONTHLY

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THIS MONTHLY WAS FOUNDED IN 1894 BY BENJAMIN F. FINKEL

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BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the SECRETARY-TREASURER, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 10th of the month in which the change becomes effective.

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DAVID EUGENE SMITH

W. BENJAMIN FITE, Columbia University

David Eugene Smith died at his home in New York City July 29, 1944, at the age of eighty-four after a long illness. He was born in Cortland, New York, and was fortunate in that his mother took a deep and intelligent interest in his early education. She did not leave this important work entirely to the schools, but early taught him Latin and Greek and other subjects, to such good purpose that he could converse in both of these languages.

He entered the State Normal School at Cortland the year it was opened and later enrolled in Syracuse University from which he was graduated at the age of twenty-one. As an undergraduate he acquired a good knowledge of Hebrew and developed a fondness for the Hebrew Bible. Thus he left college with a mastery of the three most important ancient languages, and a strong inclination towards the arts and humanities.

His father was a lawyer and wanted his son to take up the same profession. Accordingly David Eugene studied law in his father's office after his graduation and was admitted to the bar in 1884. But he had no enthusiasm for the Law, and the same year he accepted an offer of an instructorship in mathematics in the Cortland Normal School which he had attended before he entered college. It does not appear that up to this time mathematics had appealed to him strongly. But he entered upon his new duties with enthusiasm and thoroughness, and after seven years he was appointed Professor of Mathematics in the Michigan State Normal School at Ypsilanti. He later served as Principal of the New York State Normal School at Brockport, and in 1901 he become Professor of Mathematics in Teachers College, Columbia University, a position he held until his retirement in 1926, at which time he was made Professor Emeritus.

In 1887 he married Fannie Taylor. She died in 1928 and in 1940 he married Eva May Luce in collaboration with whom he had written some textbooks and who had been Director of Teachers Training in the Iowa State Teachers College.

From the time he gave up the practice of law and began teaching he was unusually active as a writer of text-books and works on the history of mathematics. Among the latter should be mentioned *History of Modern Mathematics*, *Rara Arithmetica*, *History of Japanese Mathematics*, *Our Debt to Greece and Rome in Mathematics*, *Historical Mathematical Paris*, and, in collaboration with Jekuthiel Ginsburg, *A History of Mathematics in America before 1900*. On the pedagogical side, besides the numerous text-books already referred to, there should be mentioned his *Teaching of Arithmetic*. In 1936 and 1937 he brought out two series of Portraits of Eminent Mathematicians.

He translated René Descartes' *La géométrie* from the Latin and French in collaboration with Marcia Lutham and Klein's *Vorträge über ausgewählte Fragen Elementar-geometrie* under the title *Famous Problems of Elementary Geometry* in collaboration with Professor W. W. Beman. He was active in editorial work for

many years. Thus he was Mathematical Editor of the *New International Encyclopedia* from 1902 to 1916, of Monroe's *Cyclopedia of Education* from 1911 to 1913, the *New Practical Reference Library* in 1912, the *Encyclopedia Britannica* in 1927, and the *National Encyclopedia* in 1933. He also served on the editorial board of this MONTHLY and on that of *Scripta Mathematica*. From 1908 to 1920 he was Vice President of the International Committee on the Teaching of Mathematics, President from 1928 to 1932, and Honorary President thereafter. He was Librarian of the American Mathematical Society and Associate Editor of the *Bulletin* of the Society for eighteen years, and Vice President of the Society in 1932.

In addition to all this activity he gave his time freely and generously to the many problems in connection with the organization of the Mathematical Association of America. He was a charter member of the Association, a member of its board of trustees for a number of years, and one of its early presidents. Shortly after the First World War one of the editors of *Bibliotheca Mathematica* suggested to him the desirability of transferring that journal to this country. After conferences with Professor H. E. Slaught of the University of Chicago and Mrs. Mary Hegeler Carus it was decided it would be better to establish here a series of mathematical monographs for which Mrs. Carus would undertake to furnish the financial support. This is the origin of the Carus Monographs.

His love of art which showed itself in his undergraduate days remained with him to the end and prompted him to make a metrical version of the *Rubáiyát* of Omar Khayyám based on a verbatim translation of Haslinn Hussein. For this work he received a decoration from Rega Khan Pahleri who was Shah of Iran at the time. He was an indefatigable collector of material connected with the early history of mathematics, of fine oriental rugs, and, in general, of things of artistic merit. Some years ago he presented his library and collection of mathematical material to Columbia University, where they are suitably housed along with the collections presented by G. A. Plimpton and S. S. Dale.

We have here an impressive list of activities continued over a long period of years, but it gives only a faint and inadequate picture of David Eugene Smith the man.

"Not on the vulgar mass
Called 'work' must sentence pass."

He was not merely an active worker in more or less narrow fields of scholarship. He was learned in a broad field and his learning was tempered by a delightful sense of humor and a mellow and charming personality. In 1921 he delivered the presidential address before the Mathematical Association of America. It was entitled *Religio Mathematici* and was published in volume 28 of this MONTHLY, as well as separately. It reveals with great clearness the depth and richness of his religious feeling. He was a humanist in the finest sense of the word. Those who saw him often in his class room or in his home and who knew him well came to have a great admiration and a deep affection for him.

CERTAIN GENERATING FUNCTIONS AND ASSOCIATED POLYNOMIALS

E. D. RAINVILLE, University of Michigan

1. Introduction. Let $G(x)$ be analytic at $x=0$ and let

$$G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

We shall study the set of polynomials $g_n(x)$ defined by

$$e^t G(xt) = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!}. \tag{1}$$

The polynomials $g_n(x)$ are said to be generated by the function $e^t G(xt)$. Since it involves no loss in generality of the sets of polynomials so defined, let us take $a_0 \neq 0$. Then $g_0(x) \neq 0$.

Milne-Thomson[1] used a generating function resembling that in (1). A forthcoming paper, *On generating functions of polynomial systems*, by W. C. Brenke[2] has equations (3) and (12) below in common with this paper. Our aims and results are, except in an instance or two, not at all similar to those of the two papers mentioned.

We shall obtain a few simple properties of the $g_n(x)$ in general and then proceed to study a specific set of such polynomials.

Application of the Cauchy product of two series gives us at once

$$\begin{aligned} e^t G(xt) &= \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} a_n \frac{x^n t^n}{n!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a_k x^k}{k!(n-k)!} t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{n,k} a_k x^k \frac{t^n}{n!}. \end{aligned}$$

Hence, using (1),

$$g_n(x) = \sum_{k=0}^{n} C_{n,k} a_k x^k. \tag{2}$$

2. A recurrence relation. Differentiation of both sides of (1) with respect to x yields

$$te^t G'(xt) = \sum_{n=1}^{\infty} g_n'(x) \frac{t^n}{n!},$$

while differentiation with respect to t yields

$$e^t G(xt) + xe^t G'(xt) = \sum_{n=1}^{\infty} g_n(x) \frac{t^{n-1}}{(n-1)!}.$$

From these two equations it follows at once that

$$\sum_{n=0}^{\infty} g_n(x) \frac{t^{n+1}}{n!} + \sum_{n=1}^{\infty} x g_n'(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} g_n(x) \frac{t^n}{(n-1)!},$$

or

$$\sum_{n=1}^{\infty} g_{n-1}(x) \frac{t^n}{(n-1)!} + \sum_{n=1}^{\infty} x g_n'(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} g_n(x) \frac{t^n}{(n-1)!}.$$

Therefore, for $n \geqq 1$,

$$n g_{n-1}(x) + x g_n'(x) = n g_n(x).$$

Throughout we shall find it convenient to use the operator $\theta = x(d/dx)$. With it, the above equation appears as

$$n g_{n-1}(x) = (n - \theta) g_n(x). \tag{3}$$

The relation (3) is equivalent to the well-known result

$$\alpha_n'(x) = \alpha_{n-1}(x)$$

for Appell polynomials $\alpha_n(x)$ defined by

$$A(t)e^{xt} = \sum_{n=0}^{\infty} \alpha_n(x) t^n.$$

Everything in this paper can be reworded in terms of Appell polynomials by putting $A(t) \equiv G(t)$ and passing from the $g_n(x)$ to the $\alpha_n(x)$ by means of

$$n! \alpha_n(x) = x^n g_n\left(\frac{1}{x}\right).$$

3. **Connection with differential equations.** Let $G(x)$ satisfy a linear differential equation with polynomial coefficients. Such an equation can always be written in a certain form which is very well suited to many theoretical discussions. That form is obtained by writing the linear differential operator as a sum of terms each containing the product of a polynomial in θ and a single power of x. That is, any linear differential equation with polynomial coefficients may be written as

$$\sum_{k=0}^{m} x^k F_k(\theta) G(x) = 0, \tag{4}$$

where the $F_k(\theta)$; $k = 0, 1, \cdots, m$, are polynomials in the operator θ. It is wise to keep in mind that the operators x and θ are not commutative. Indeed, $\theta x G(x) = x(\theta + 1) G(x)$ and $x F_k(\theta) G(x) = F_k(\theta - 1) x G(x)$.

We shall concern ourselves primarily with the case in which $G(x)$ satisfies the simple equation

$$[F_0(\theta) + xF_1(\theta)]G(x) = 0. \tag{5}$$

This is an equation (not the only one) which, in the parlance of the subject of power series solutions, may be solved with a two-term recurrence relation.

Let us now determine from (3) and (5) a linear differential equation satisfied by the polynomial $g_n(x)$. Since the generating function involves $G(xt)$, we rewrite (5) with $z=xt$, $\theta_1=z(d/dz)$, as

$$[F_0(\theta_1) + zF_1(\theta_1)]G(z) = 0.$$

Differentiation of the fundamental relation (1) k times with respect to x yields

$$t^k e^t G^{(k)}(xt) = \sum_{n=0}^{\infty} g_n^{(k)}(x) \frac{t^n}{n!}.$$

Hence

$$e^t x^k t^k G^{(k)}(xt) = \sum_{n=0}^{\infty} x^k g_n^{(k)}(x) \frac{t^n}{n!},$$

or

$$e^t\theta_1(\theta_1 - 1) \cdots (\theta_1 - k + 1)G(z) = \sum_{n=0}^{\infty} \theta(\theta - 1) \cdots (\theta - k + 1)g_n(x) \frac{t^n}{n!}.$$

Now successive eliminations yield

$$e^t\theta_1^k G(z) = \sum_{n=0}^{\infty} \theta^k g_n(x) \frac{t^n}{n!}; \qquad k = 0, 1, 2, \cdots. \tag{6}$$

But, from (5),

$$e^t[F_0(\theta_1) + zF_1(\theta_1)]G(z) = 0.$$

Therefore

$$\sum_{n=0}^{\infty} F_0(\theta)g_n(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} xF_1(\theta)g_n(x) \frac{t^{n+1}}{n!} = 0,$$

or

$$\sum_{n=0}^{\infty} F_0(\theta)g_n(x) \frac{t^n}{n!} + \sum_{n=1}^{\infty} xF_1(\theta)g_{n-1}(x) \frac{t^n}{(n-1)!} = 0.$$

We may conclude that $F_0(\theta)g_0(x)=0$, or $F_0(0)=0$, which is merely a reflection of the fact that $G(x)$ is analytic and not zero at $x=0$.

Next, for $n \geqq 1$,

$$F_0(\theta)g_n(x) + nxF_1(\theta)g_{n-1}(x) = 0. \tag{7}$$

We already knew one relation between $g_n(x)$ and $g_{n-1}(x)$, namely,

$$(3) \qquad ng_{n-1}(x) = (n - \theta)g_n(x).$$

The elimination of $g_{n-1}(x)$ from (7) and (3) shows that $g_n(x)$ satisfies the differential equation

$$(8) \qquad [F_0(\theta) - xF_1(\theta)(\theta - n)]g_n(x) = 0.$$

4. A less restricted equation for $G(x)$. Suppose that $G(x)$ satisfies the less special equation

$$(4) \qquad \sum_{k=0}^{m} x^k F_k(\theta) G(x) = 0.$$

From (6) it follows at once that

$$\sum_{k=0}^{m} \sum_{n=0}^{\infty} x^k F_k(\theta) g_n(x) \frac{t^{n+k}}{n!} = 0,$$

or

$$(9) \qquad \sum_{k=0}^{m} \sum_{n=k}^{\infty} x^k F_k(\theta) g_{n-k}(x) \frac{t^n}{(n-k)!} = 0,$$

which is the counterpart of (7).

Now, from (3) above it may be seen that

$$(n-1)g_{n-2}(x) = (n - 1 - \theta)g_{n-1}(x),$$
$$n(n-1)g_{n-2}(x) = (n - \theta)(n - 1 - \theta)g_n(x),$$

and, in general,

$$n(n-1)\cdots(n-k+1)g_{n-k}(x) = (n-\theta)(n-1-\theta)\cdots(n-k+1-\theta)g_n(x).$$

Here it is convenient to introduce a common notation which will simplify the writing of several formulas in the remainder of the work. Let us define $(\alpha)_k$ by

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1); \qquad (\alpha)_0 = 1.$$

Then the relation preceding the definition of $(\alpha)_k$ may be written

$$(10) \qquad \frac{g_{n-k}(x)}{(n-k)!} = (-1)^k(\theta - n)_k \frac{g_n(x)}{n!}.$$

Next, with the aid of (9),

$$\sum_{k=0}^{m} \sum_{n=k}^{\infty} (-1)^k x^k F_k(\theta)(\theta - n)_k g_n(x) \frac{t^n}{n!} = 0,$$

from which the differential equation,

$$\sum_{k=0}^{m} (-1)^k x^k F_k(\theta)(\theta - n)_k g_n(x) = 0 \tag{11}$$

for the $g_n(x)$ follows for $n \geqq m$.

5. Pure recurrence relation. If (3) is put in the form

$$\theta g_n(x) = n[g_n(x) - g_{n-1}(x)]$$

and considered together with

$$F_0(\theta)g_n(x) + nxF_1(\theta)g_{n-1}(x) = 0, \tag{7}$$

in which F_0 and F_1 are polynomials in θ, it is evident that the operator θ may be eliminated. The result will be a pure recurrence relation expressing $g_n(x)$ in terms of a certain number of $g_k(x)$ with subscripts less than n. The coefficients in this relation will be polynomials at most linear in x. As an example, see equation (23) below. When $G(x)$ satisfies the less simple equation (4), a similar relation may be obtained, but with coefficients of higher degree in x.

6. Two identities. Let us return to the original definitions of $g_n(x)$ and $G(x)$ in section 1. Of course,

$$G(xt) = e^{-t}\sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!}\right).$$

The Cauchy product of the two series on the right leads us at once to

$$G(xt) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^{n-k} \frac{g_k(x)}{(n-k)!k!} t^n,$$

or

$$\sum_{n=0}^{\infty} a_n \frac{x^n t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^{n-k} C_{n,k} g_k(x) \frac{t^n}{n!}.$$

Therefore

$$\sum_{k=0}^{n} (-1)^k C_{n,k} g_k(x) = (-1)^n a_n x^n. \tag{12}$$

We shall now obtain another identity involving the $g_k(x)$. From the definition (1) it follows that

$$e^t G(xyt) = \sum_{n=0}^{\infty} g_n(xy) \frac{t^n}{n!},$$

and also

$$e^{yt} G(xyt) = \sum_{n=0}^{\infty} g_n(x) \frac{y^n t^n}{n!}.$$

Now,

$$e^tG(xyt) = e^{t(1-y)}e^{yt}G(xyt),$$

hence

$$\sum_{n=0}^{\infty} g_n(xy)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}\frac{(1-y)^n t^n}{n!}\right)\left(\sum_{n=0}^{\infty} g_n(x)\frac{y^n t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{n,k}y^k(1-y)^{n-k}g_k(x)\frac{t^n}{n!}.$$

Therefore

$$g_n(xy) = \sum_{k=0}^{n} C_{n,k}y^k(1-y)^{n-k}g_k(x). \tag{13}$$

Three special cases of (13) appear to be of particular interest.[3] They are

$$g_n(x) = \sum_{k=0}^{n} C_{n,k}g_k(1)x^k(1-x)^{n-k}, \tag{13a}$$

$$g_n(-x) = \sum_{k=0}^{n}(-1)^kC_{n,k}2^{n-k}g_k(x), \tag{13b}$$

$$g_n\left(\frac{x}{2}\right) = 2^{-n}\sum_{k=0}^{n} C_{n,k}g_k(x). \tag{13c}$$

There is a symbolic notation which is convenient here. In this notation it is agreed that exponents shall be lowered to subscripts on any symbol, such as $g(x)$ in our work, which has been undefined except with subscripts. That is, such a symbol as $\{1-g(x)\}^3$ is to be taken to mean

$$g_0(x) - 3g_1(x) + 3g_2(x) - g_3(x).$$

In order to emphasize the presence of the symbolic notation we shall use $\doteqdot$ to replace $=$ whenever a symbolic equation is given.

The identities of this section may now be written

$$\{1 - g(x)\}^n \doteqdot \{-ax\}^n, \tag{12}$$

and

$$g_n(xy) \doteqdot \{1 - y + yg(x)\}^n. \tag{13}$$

The symbolic notation could have been used to arrive at the results of this section in the following way. First, from

$$e^tG(xt) \doteqdot e^{tg(x)} \quad\text{and}\quad G(xt) \doteqdot e^{axt},$$

we conclude that

$$e^{t\{g(x)-1\}} \doteqdot e^{axt},$$

or
$$\{g(x) - 1\}^n \doteqdot \{ax\}^n,$$
which is essentially equation (12). Next, from
$$G(xt) \doteqdot e^{t\{g(x)-1\}}$$
we see that
$$G(xyt) \doteqdot e^{yt\{g(x)-1\}}.$$
Hence
$$e^t G(xyt) \doteqdot e^{tg(xy)} \doteqdot e^{t\{1-y+yg(x)\}}$$
so that
$$\{g(xy)\}^n \doteqdot \{1 - y + yg(x)\}^n \tag{13}$$
follows.

7. The Laguerre polynomials. For the particular choice
$$G(x) = J_0(2\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n!)^2},$$
the polynomials $g_n(x)$ are called Laguerre polynomials[4] and are usually written $L_n(x)$. That is, the $L_n(x)$ may be defined by the well-known generating function
$$e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!} \cdot$$
The function $G(x) = J_0(2\sqrt{x})$ satisfies the differential equation
$$xG''(x) + G'(x) + G(x) = 0,$$
or
$$(\theta^2 + x)G(x) = 0.$$
The material in the preceding sections leads at once to the familiar results
$$L_n(x) = \sum_{k=0}^{n} (-1)^k C_{n,k} \frac{x^k}{k!},$$
$$\theta L_n(x) = xL_n'(x) = n[L_n(x) - L_{n-1}(x)],$$
and
$$[\theta^2 - x(\theta - n)]L_n(x) = 0,$$
or
$$xL_n''(x) + (1 - x)L_n'(x) + nL_n(x) = 0.$$
With the procedure outlined in Section 5 it is easy to establish the recurrence relation

$$nL_n(x) = (2n - 1 - x)L_{n-1}(x) - (n - 1)L_{n-2}(x).$$

But the preceding sections also yield the less commonly known results of Feldheim[5]

$$\sum_{k=0}^{n} (-1)^k C_{n,k} L_k(x) = \frac{x^n}{n!}, \tag{14}$$

and

$$L_n(xy) = \sum_{k=0}^{n} C_{n,k} y^k (1 - y)^{n-k} L_k(x). \tag{15}$$

Formula (14) is attractive symbolically, being

$$\{1 - L(x)\}^n \doteq \frac{x^n}{n!}.$$

It may be used to define the $L_n(x)$, with $L_0(x) = 1$.

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, for which

$$G(x) = x^{-\alpha/2} J_\alpha(2\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!\Gamma(n + \alpha + 1)},$$

may be treated in the same way.

It must be emphasized that our work has no direct bearing on the extremely important orthogonality property of the Laguerre polynomials. Naturally, all properties of the polynomials are, directly or otherwise, results of the generating function employed in defining the polynomials. But, the orthogonality does not follow from the form of the generating function used in (1), as simple counter examples show. Take $G(x) = e^x$, for instance.

8. Some classical formulas. In order to discuss the polynomials which we wish to introduce in the next section, we need certain classical formulas relating to two well-known functions.

Gauss' hypergeometric series or function is

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma + n)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!}.$$

It satisfies the differential equation

$$[\theta(\theta + \gamma - 1) - x(\theta + \alpha)(\theta + \beta)]F = 0.$$

We shall use the relation

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x).$$

A function such as $F(\alpha, \beta+1; \gamma; x)$ in which one and only one of the parame-

ters α, β, and γ, is increased or decreased by unity is said to be contiguous to $F(\alpha, \beta; \gamma; x)$. There are evidently six functions contiguous to $F(\alpha, \beta; \gamma; x)$. For them we shall use a common notation illustrated by

$$F_{\gamma-} = F(\alpha, \beta; \gamma - 1; x),$$
$$F_{\alpha+} = F(\alpha + 1, \beta; \gamma; x).$$

Gauss showed that F and any two of its contiguous functions are related linearly with coefficients at most linear polynomials in x. Of the fifteen relations which Gauss thus obtained the other ten follow from the five below by elimination of $F_{\alpha+}$ between the various pairs. The five relations are:

$$\begin{aligned}
(\alpha - \beta)F &= \alpha F_{\alpha+} - \beta F_{\beta+}, \\
(\alpha - \gamma + 1)F &= \alpha F_{\alpha+} - (\gamma - 1)F_{\gamma-}, \\
[2\alpha - \gamma + (\beta - \alpha)x]F &= \alpha(1 - x)F_{\alpha+} + (\alpha - \gamma)F_{\alpha-}, \\
(\alpha + \beta - \gamma)F &= \alpha(1 - x)F_{\alpha+} + (\beta - \gamma)F_{\beta-}, \\
\gamma[\alpha + (\beta - \gamma)x]F &= \alpha\gamma(1 - x)F_{\alpha+} - (\alpha - \gamma)(\beta - \gamma)xF_{\gamma+}.
\end{aligned}$$

The Pochhammer-Barnes confluent hypergeometric function may be defined by

$$M(\alpha, \gamma, x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{x^n}{n!} \cdot$$

This function satisfies the differential equation

$$[\theta(\theta + \gamma - 1) - x(\theta + \alpha)]\, M = 0.$$

The only additional property of $M(\alpha, \gamma, x)$ which we shall use is Kummer's formula,

$$M(\alpha, \gamma, x) = e^x M(\gamma - \alpha, \gamma, - x).$$

9. A set of hypergeometric polynomials. Let us consider the polynomials $\phi_n(\alpha, \gamma, x)$ defined by

$$e^t M(\alpha, \gamma, xt) = \sum_{n=0}^{\infty} \phi_n(\alpha, \gamma, x) \frac{t^n}{n!} \cdot \tag{16}$$

Whenever the parameters α and γ remain unchanged throughout a relation or discussion we shall omit them and write simply $\phi_n(x)$.

From the earlier sections we have at once the results

$$\phi_n(x) = \phi_n(\alpha, \gamma, x) = \sum_{k=0}^{n} C_{n,k} \frac{(\alpha)_k}{(\gamma)_k} x^k, \tag{17}$$

$$n\phi_{n-1}(x) = (n - \theta)\phi_n(x), \tag{18}$$

$$\theta(\theta + \gamma - 1)\phi_n(x) - nx(\theta + \alpha)\phi_{n-1}(x) = 0, \tag{19}$$

$$[\theta(\theta+\gamma-1)+x(\theta-n)(\theta+\alpha)]\phi_n(x)=0, \tag{20}$$

$$\{1-\phi(x)\}^n \doteqdot (-1)^n\frac{(\alpha)_n}{(\gamma)_n}x^n, \tag{21}$$

$$\phi_n(xw)\doteqdot\{1-w+w\phi(x)\}^n. \tag{22}$$

The relations (18) and (19) together yield a pure recurrence relation for $\phi_n(x)$. From (18) in the form

$$\theta\phi_n(x)=n[\phi_n(x)-\phi_{n-1}(x)]$$

it follows that

$$\theta^2\phi_n(x)=n[n\phi_n(x)-(2n-1)\phi_{n-1}(x)+(n-1)\phi_{n-2}(x)].$$

Then direct substitution of these two results into (19) leads to the desired relation,

$$\begin{aligned}(n+\gamma-1)\phi_n(x) &= [2n+\gamma-2+(n+\alpha-1)x]\phi_{n-1}(x)\\ &\quad -(n-1)(1+x)\phi_{n-2}(x).\end{aligned} \tag{23}$$

An inspection of equation (20) indicates that $\phi_n(x)$ is a hypergeometric function. In fact, since

$$\begin{aligned}C_{n,k}=\frac{n(n-1)\cdots(n-k+1)}{k!}&=(-1)^k\frac{(-n)(-n+1)\cdots(-n+k-1)}{k!}\\ &=(-1)^k\frac{(-n)_k}{k!},\end{aligned}$$

it is possible to write

$$\phi_n(\alpha,\gamma,x)=\sum_{k=0}^{n}(-1)^k\frac{(-n)_k(\alpha)_k}{(\gamma)_k}\frac{x^k}{k!}=F(\alpha,-n;\gamma;-x). \tag{24}$$

Now it is evident that $\phi_n(x)$ is a terminating hypergeometric series. Indeed, every terminating hypergeometric series is a $\phi_n(x)$, but we must not demand too much of this fact. It is inherent in our definition of $\phi_n(\alpha,\gamma,x)$ that the parameters α and γ be independent of n. Otherwise we would not dare use results obtained on the basis of the generating function and would be forced to reexamine each separate formula. A pertinent case is the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ which can be expressed in terms of $\phi_n(\sigma,\epsilon,x)$ but with σ dependent upon n. Each individual Jacobi polynomial may be written as a ϕ_n but the theory of the ϕ_n's as developed here may not be applied to Jacobi polynomials.

10. Further properties of the polynomials. The following six formulas are immediate results of those properties of $F(\alpha,\beta;\gamma;x)$ given in Section 8. They are obtained by replacing β by $-n$, x by $-x$:

$$\gamma\phi_n'(\alpha, \gamma, x) = n\alpha\phi_{n-1}(\alpha + 1, \gamma + 1, x), \tag{25}$$

$$(\alpha + n)\phi_n(\alpha, \gamma, x) = \alpha\phi_n(\alpha + 1, \gamma, x) + n\phi_{n-1}(\alpha, \gamma, x), \tag{26}$$

$$(\alpha - \gamma + 1)\phi_n(\alpha, \gamma, x) = \alpha\phi_n(\alpha + 1, \gamma, x) - (\gamma - 1)\phi_n(\alpha, \gamma - 1, x), \tag{27}$$

$$\begin{aligned}[2\alpha - \gamma + (\alpha + n)x]\phi_n(\alpha, \gamma, x) &= \alpha(1 + x)\phi_n(\alpha + 1, \gamma, x) \\ &\quad + (\alpha - \gamma)\phi_n(\alpha - 1, \gamma, x),\end{aligned} \tag{28}$$

$$\begin{aligned}(\alpha - n - \gamma)\phi_n(\alpha, \gamma, x) &= \alpha(1 + x)\phi_n(\alpha + 1, \gamma, x) \\ &\quad - (n + \gamma)\phi_{n+1}(\alpha, \gamma, x),\end{aligned} \tag{29}$$

$$\begin{aligned}\gamma[\alpha + (n + \gamma)x]\phi_n(\alpha, \gamma, x) &= \alpha\gamma(1 + x)\phi_n(\alpha + 1, \gamma, x) \\ &\quad - (\alpha - \gamma)(n + \gamma)x\phi_n(\alpha, \gamma + 1, x).\end{aligned} \tag{30}$$

The pure recurrence relation (23) may be obtained by eliminating $\phi_n(\alpha+1, \gamma, x)$ from (26) and (29) and then replacing n by $(n-1)$.

From Kummer's formula for $M(\alpha, \gamma, x)$ we see that

$$e^tM(\alpha, \gamma, xt) = e^te^{xt}M(\gamma - \alpha, \gamma, -xt),$$

and thus that

$$\sum_{n=0}^{\infty} \phi_n(\alpha, \gamma, x)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^nt^n}{n!}\right)\left(\sum_{n=0}^{\infty} \phi_n(\gamma - \alpha, \gamma, -x)\frac{t^n}{n!}\right).$$

Hence

$$\phi_n(\alpha, \gamma, x) = \sum_{k=0}^{n} C_{n,k}x^{n-k}\phi_k(\gamma - \alpha, \gamma, -x), \tag{31}$$

or

$$\phi_n(\alpha, \gamma, x) \doteqdot \{x + \phi(\gamma - \alpha, \gamma, -x)\}^n.$$

11. Relation of the $\phi_n(x)$ to some well-known polynomials. The following results are not difficult to verify:

For Laguerre polynomials,

$$L_n(x) = \lim_{\sigma\to\infty} \phi_n(\sigma, 1, -x/\sigma),$$

$$L_n^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \lim_{\sigma\to\infty} \phi_n(\sigma, \alpha + 1, -x/\sigma).$$

For Legendre polynomials,

$$P_n(x) = (x - \sqrt{x^2 - 1})^n\phi_n\left(\frac{1}{2}, 1, \frac{2\sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}}\right).$$

For Hermite polynomials, see Szegö, *op. cit.*, p. 102, and apply the above result for $L_n^{(\alpha)}(x)$.

References

1. L. M. Milne-Thomson, Two classes of generalized polynomials, Proc. Lond. Math. Soc., (2) vol. 35, 1933, 514–522.
2. I am indebted to Professor Brenke for an opportunity to examine his manuscript.
3. Note also that equation (12) may be obtained by equating coefficients of y^n in (13).
4. See G. Szegö, Orthogonal Polynomials, Colloquium Lectures, 1939, 96–98. Our notation agrees with that of Szegö whenever we use classical polynomials.
5. E. Feldheim, Développements en série de polynômes d'Hermite et de Laguerre à l'aide des transformations de Gauss et Hankel, I et II, Nederl. Akad. Wetensch., Proc. 43, 1940, 224–248.

MEAN LENGTHS OF LINE SEGMENTS

HERMANCE MULLEMEISTER, University of Washington

1. Introduction. If we assume a line segment of unit length divided at random into two parts, it is obvious that the mean length of the longer of the two parts will be three-fourths, that of the shorter one-fourth.

If the random division is into three parts, it has been shown* that the mean lengths of the parts, taken in the descending order of magnitude are 11/18, 5/18 and 2/18 respectively.

The general case is more difficult. A line segment is broken at random into $(n+1)$ parts. What are the mean lengths of the separate parts taken in descending order of magnitude?

2. Four segments. We consider first the case when $n=3$. Let us denote the lengths of the four parts taken in descending order of magnitude by x, y, z, w and their mean values by $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{w}$, respectively.

By definition:

$$\bar{x} = \frac{\iiint x dxdydz}{\iiint dxdydz}, \qquad \bar{y} = \frac{\iiint y dxdydz}{\iiint dxdydz}, \qquad \bar{z} = \frac{\iiint z dxdydz}{\iiint dxdydz},$$

$$\bar{w} = 1 - \bar{x} - \bar{y} - \bar{z}, \tag{1}$$

where the intervals of integration are subject to the restrictions:

$$x > y, \qquad y > z, \qquad z > w, \qquad w = 1 - x - y - z > 0. \tag{2}$$

Consider now the region of space where coordinates are the values of x, y, z subject to the restrictions (2). The plane $x=y$ separates the region in which $x>y$ from the region in which $x<y$. The plane $y=z$ separates the region in which $y>z$ from the region in which $y<z$, and similar remarks apply to the planes $z=1-x-y-z$ and $1-x-y-z=0$.

The region of integration of the integrals (1) is therefore limited to some one

* See Czuber: Geometrische Wahrscheinlichkeiten und Mittelwerte, p. 207.

of the fifteen regions into which ordinary three-space is divided by the four planes:

$$(3)\qquad x = y, \qquad y = z, \qquad z = 1 - x - z, \qquad 1 - x - y - z = 0.$$

Moreover, since x, y, z and $w = 1 - x - y - z$, are each limited, the region of integration must be the interior of the tetrahedron bounded by the four planes (3).

Let us denote the interior region of the tetrahedron by T. We may now write:

$$(4)\qquad \bar{x} = \frac{\int_T x dV}{\int_T dV}, \qquad \bar{y} = \frac{\int_T y dV}{\int_T dV}, \qquad \bar{z} = \frac{\int_T z dV}{\int_T dV},$$

where $dV = dxdydz$, and T is the region of integration. The direct integration of the integrals in the numerators offer difficulties which may be avoided by observing that the quantities $\bar{x}$, $\bar{y}$, $\bar{z}$ define the centroid of the tetrahedron. They may therefore be computed indirectly from the well-known formulas:

$$(5)\qquad \bar{x} = \tfrac{1}{4}\sum x_i, \qquad \bar{y} = \tfrac{1}{4}\sum y_i, \qquad \bar{z} = \tfrac{1}{4}\sum z_i, \qquad i = 1, 2, 3, 4,$$

where x_i, y_i, z_i are the coordinates of the vertices of the tetrahedron. On solving the equations (3) taken three at a time, we obtain the coordinates of the vertices:

$$P_1 = (1, 0, 0), \qquad P_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0), \qquad P_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), \qquad P_4 = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}),$$

and thence, by (5)

$$(6)\qquad \begin{aligned} \bar{x} &= \tfrac{1}{4}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) = \tfrac{25}{48}, & \bar{y} &= \tfrac{1}{4}(0 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) = \tfrac{13}{48}, \\ \bar{z} &= \tfrac{1}{4}(0 + 0 + \tfrac{1}{3} + \tfrac{1}{4}) = \tfrac{7}{48}, & \bar{w} &= 1 - \bar{x} - \bar{y} - \bar{z} = \tfrac{3}{48}. \end{aligned}$$

3. The general case. Let the lengths of

$$x_1, x_2, \cdots, x_n, w,$$

represent the $n+1$ parts, taken in descending order of magnitude, in which a line segment of unit length is divided at random. The mean value of the ith of these parts is:

$$(7)\qquad \bar{x}_i = \frac{\int\int \cdots \int x_i dx_1 dx_2 \cdots dx_n}{\int\int \cdots \int dx_1 dx_2 \cdots dx_n}, \qquad i = 1, 2, \cdots, n$$

where the x's are subject to the restrictions

$$x_1 > x_2 > \cdots > x_n > w > 0.$$

Let us interpret the x's as the coordinates of a point in a euclidean space (R_n) of n dimensions. The equation $x_i = x_{i+1}$ may then be interpreted as the equation of a hyper-plane which separates the region of R_n in which $x_i > x_{i+1}$ from the region in which $x_i < x_{i+1}$. The $n+1$ equations

$$
\begin{aligned}
&x_1 = x_2,\ x_2 = x_3, \cdots, x_n = w = 1 - x_1 - x_2 \cdots - x_n, \\
&w = 1 - x_1 - x_2 \cdots - x_n = 0
\end{aligned} \tag{8}
$$

being linearly independent, represent the boundary planes of a simplex in R_n, and since each of the x's is limited, x_i being less than unity, the region of integration of (7) must be the interior region of this simplex. The evaluation of the integrals (7) is thus reduced to that of finding the centroid of a simplex in R_n, in terms of the coordinates of the vertices of the simplex. These may be readily computed by solving the equations (8) in sets of n.* Let P_i denote the vertex of the simplex determined by those equations, omitting the ith. An easy computation gives the coordinates of P_i as tabulated below.

$$
\begin{array}{ccccccccc}
 & x_1 & x_2 & x_3 & & x_{n-2} & x_{n-1} & x_n \\
P_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
P_2 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\
P_3 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
P_{n-1} & \dfrac{1}{n-1} & \dfrac{1}{n-1} & \dfrac{1}{n-1} & \cdots & \dfrac{1}{n-1} & \dfrac{1}{n-1} & 0 \\
P_n & \dfrac{1}{n} & \dfrac{1}{n} & \dfrac{1}{n} & \cdots & \dfrac{1}{n} & \dfrac{1}{n} & \dfrac{1}{n} \\
P_{n+1} & \dfrac{1}{n+1} & \dfrac{1}{n+1} & \dfrac{1}{n+1} & \cdots & \dfrac{1}{n+1} & \dfrac{1}{n+1} & \dfrac{1}{n+1} .
\end{array}
$$

Applying now the theorem referred to we obtain the mean values sought.

$$
\begin{aligned}
\bar{x}_1 &= \frac{1}{n+1}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n+1}\right) \\
\bar{x}_2 &= \frac{1}{n+1}\left(0 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n+1}\right) \\
\bar{x}_3 &= \frac{1}{n+1}\left(0 + 0 + \tfrac{1}{3} + \cdots + \frac{1}{n+1}\right) \\
&\cdots\cdots\cdots\cdots\cdots \\
\bar{x}_n &= \frac{1}{n+1}\left(0 + 0 + 0 + \cdots + \frac{1}{n} + \frac{1}{n+1}\right) \\
\bar{w} &= 1 - \bar{x}_1 - \bar{x}_2 - \cdots - \bar{x}_n = \frac{1}{(n+1)^2} .
\end{aligned}
$$

* The assumption is made that the centroid formulas are valid for n-dimensional space.

AN ANALYTIC GEOMETRY FOR *N* VARIABLES

R. S. UNDERWOOD, Texas Technological College

While it is known* that we may set up a one-to-one correspondence between the points in n-space and those on a line or plane, no device for doing this of which the writer is aware would yield anything practical in the way of a picture. Yet any system which provides a framework for a plane and solid analytic geometry involving n variables should broaden the base for the impact of euclidean geometry upon algebra, and vice versa. With this general goal in mind the writer has experimented with various coordinate systems, seeking one which meets satisfactorily the tests of simplicity, consistence with the developed subject for the cases of two and three variables, and practical usefulness in the interpretation of such things as partial derivatives. Herewith is presented his best result in the light of these criteria.

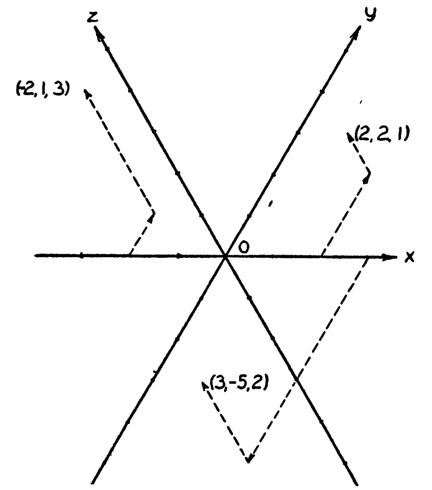

FIG. 1

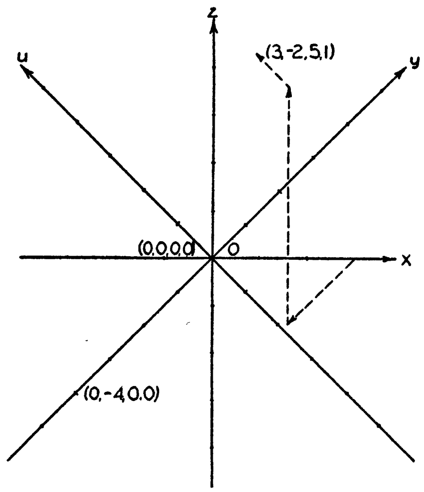

FIG. 2

A. THE TWO-DIMENSIONAL CASE

1. The n-axes plane. The basic feature of the plan is a set of n equally-spaced axes passing through the origin on *the n-axes plane*, with one axis horizontal, and with the positive direction always upward along a slant or vertical axis and to the right on the horizontal one. Figures 1 and 2 show the planes with three and four axes respectively. Some sample points are shown to indicate the method of locating them by means of the assigned coordinates, which specify the positive or negative distances to be measured parallel to the respective axes. Except in the case of two variables, here always denoted by X and Y, the coordinates x, y, z, u, v and w are used in order as needed to correspond with successive axes read counterclockwise, beginning with the horizontal one. In the

* Ch.-J. de la Vallée Poussin, Cours d'Analyse Infinitésimale, Paris, 1914, p. 49.

general case, they will be designated as x_i ($i=1, 2, \cdots, n$). The angle between successive axes is always π/n.

A given set of coordinates x_i, as applied in vector sequence from the origin, will of course, as in the case of vectors, lead always to the same point, regardless of the order of application. Hence each set of coordinates designates one and only one point, though the converse is not true except when $n=2$. On the n-axes plane the lack of this converse feature is responsible for a "piling of points on top of each other" in the graphs of equations in three or more variables. We shall see, however, that this apparent flaw may be practically nullified by use of a simple device.

The usual definition of a locus must now be modified slightly, thus:

The locus of an equation in n variables is the totality of points on the n-axis plane which have coordinates satisfying the equation.

Such coordinates of a locus point, as distinguished from the others it will have when $n>2$, will be called its *proper* coordinates.

To anticipate, we shall show that the normal locus of an equation in n variables, when $n>2$, is a continuous area such as a *filled circle* (a circle plus the points inside it), a *filled parabola, etc.* Degenerate loci include all the points, straight lines and curves of the two-variables case. But the appearance and derivation of these loci (Section **3**) is in practice irrelevant to the interpretation of partial derivatives, as discussed in the next section.

2. Complanes and derivatives. The n-axes plane may be considered as made up of an infinity of coincident planes, on each of which all variables except two are held fixed. These we shall call *complanes* (from "component planes"). For example, the 3-axes plane contains the complane $(xy)_{z=2}$ on which are the points $(x, y, 2)$. The special origin for this complane is the point $(0, 0, 2)$. Only two axes are used for it, designated as x' and y' and drawn parallel to the x-axis and y-axis respectively.

Whatever may be the locus of the equation

$$f(x, y, z) = 0, \tag{1}$$

on the 3-axes plane, only the curve

$$f(x, y, 2) = 0 \tag{2}$$

appears on the complane $(xy)_{z=2}$. And when all the complanes $(xy)_{z=k}$ $(-\infty<k<\infty)$ are placed on each other like so many sheets of glazed paper of zero thickness, the various curves taken together black out the total locus. It is natural to speak of these curves as *traces* of the locus on the respective complanes. In addition to those mentioned, two more groups of traces would appear in the complanes $(xz)_{y=k}$ and $(yz)_{x=k}$; but neither of these would add any points to the total locus as described above, since every real set (x, y, z) satisfying (1) fixes a locus point on a curve in one or more of the complanes $(xy)_{z=k}$.

The geometric interpretation of a partial derivative follows from an extension of the 2-axes practice. To state it precisely some definitions are needed.

Let x and y stand for any two of the variables x_i, and let (xy) represent any com-plane using these variables. Designate by m_{xy} the "slope" of a straight line L in (xy), with the first letter in the subscript standing for the independent variable. Then

$$m_{xy} = (y_2 - y_1)/(x_2 - x_1)$$

where (x_1, y_1) and (x_2, y_2) are any two distinct points on L.

It will now be evident that $\partial y/\partial x$ at a point P having specified coördinates is simply the slope m_{xy} of the line tangent at P to the curve in the complane (xy) containing P. Similarly, $\partial x/\partial y = m_{yx}$ for the same line.

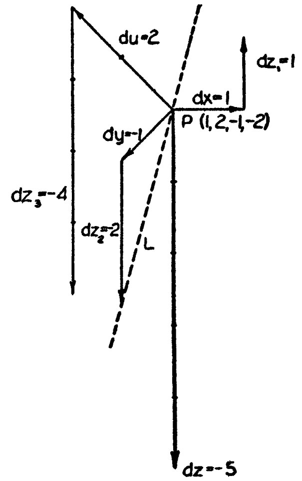

FIG. 3

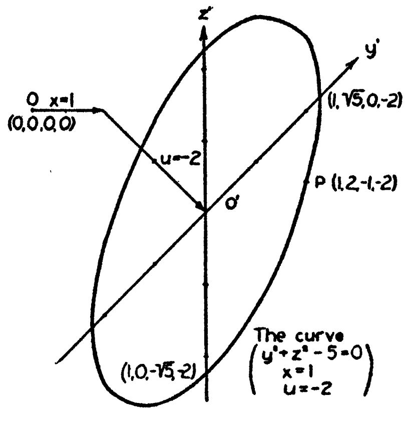

FIG. 4

To illustrate, Figure 3 shows visual interpretations of some of the quantities: $\partial z/\partial x$, $\partial z/\partial y$, $dx=1$, $dy=-1$, $du=2$, and the total differential dz, at the point $P(1, 2, -1, -2)$ on the locus of

$$(3) \qquad f(x, y, z, u) \equiv x^2 + y^2 + z^2 + u^2 - 10 = 0.$$

The value of $\partial z/\partial y$, for example, is the slope m_{yz} at P of the trace of (3) which appears in the complane $(yz)_{x=1,\, u=-2}$, and whose equation is

$$(4) \qquad f(1, y, z, -2) \equiv y^2 + z^2 - 5 = 0.$$

Since $\partial z/\partial y = 2$ at P, this is the slope m_{yz} of the dotted line L. Similarly, at P, $\partial z/\partial x = 1$ and $\partial z/\partial u = -2$. Now let $dz_1 = (\partial z/\partial x)dx$, $dz_2 = (\partial z/\partial y)dy$, and $dz_3 = (dz/\partial u)du$, so that

$$(5) \qquad dz = dz_1 + dz_2 + dz_3.$$

For the assigned value of dx, $dz_1 = 1$, and hence the directed segment representing

dz_1 must be drawn in the positive z-direction. The segments for $dz_2=-2$, $dz_3=-4$, and $dz=-5$ are of course oppositely directed.

The locus of (4) is shown in Figure 4. This curve is an ellipse whose equation in X and Y can be found by use of (4) and (7). But this method involves cumbersome algebra and is unnecessary for a rough sketch, since it is simpler to plot directly on the complane some of the points $(1, y, z, -2)$ whose coordinates satisfy (4).

The traces of (3) in the complanes $(xz)_{y=2,u=-2}$, $(yz)_{x=1,u=-2}$, and $(uz)_{x=1,y=2}$ are ellipses (one of them a circle) intersecting at P. Each of them, being a part of the total locus of (3), must lie within that locus. The latter, incidentally, is shown in Section **3** to be a filled circle, of radius $\sqrt{20}$, with center at the origin.

Note: By a given set of coordinates (an algebraic concept) there is indicated one and only one point-location (a geometric concept). In speaking of "a point on the n-axes plane," as in our definition of "locus," the geometric aspect is emphasized. However, when a point is said to be "on" or "not on" a given complane the algebraic aspect is dominant. Thus $(0, 0, 0)$, $(1, -1, 1)$ and $(-1, 1, -1)$ are different coordinates of the same point on the 3-axes plane, but no two of them are on the same complane.

3. Loci.

a. *The general case.* Over every n-axes system we may superimpose a 2-axes, or XY, system, with the positive and negative halves of the X and x_1 axes coinciding respectively. The coordinates X and Y of a point on the n-axes plane refer always, in this paper, to the superimposed system.

It may readily be seen, upon inspection of Figure 1, that for a point $P(x, y, z)$ on the 3-axes plane the relation between the two sets of coordinates is as follows:

$$X = x\cos 0^\circ + y\cos 60^\circ + z\cos 120^\circ;$$
$$Y = x\sin 0^\circ + y\sin 60^\circ + z\sin 120^\circ. \tag{6}$$

Similarly, for the n-axes case,

$$X = \sum_{i=1}^{n} x_i\cos\theta_i; \qquad Y = \sum_{i=1}^{n} x_i\sin\theta_i; \qquad \theta_i = (i-1)\pi/n, \tag{7}$$

where the θ_i are in order the least positive angles made with the positive half of the x_1 axis by the successive axes.

We may call (7) *the fundamental equations of the n-axes plane.* Of course, for the 2-axes system they yield the trivial equations: $X=x_1$; $Y=x_2$.

It may be noted that (7) would still be valid with other choices for the θ_i. This might indeed be kept in mind as a factor of flexibility yielding systems adapted to special problems. However, some advantages of the equal spacing of axes will appear in what follows, as in Theorem 1.

Here, as well as in the solid geometry variant introduced in Part B, something new is added to the technical tools of orthodox analytic geometry. For we need but substitute for X and Y, in any two-variable equation, the values given in (7), and we get an n-variable equation *with the same locus*, which will now, for

$n>2$, be a degenerate form of a typical locus. This interesting result could conceivably have direct physical or mechanical applications.

b. *Quadric loci on the 3-axes plane.* From (6) we get, for the 3-axes case,

$$2x+y-z=2X; \qquad y+z=2Y/\sqrt{3}. \tag{8}$$

We seek now the total locus of

$$f(x, y, z)=0. \tag{9}$$

When f is a second degree function of the independent variables, whatever their number, the locus will be called a *quadric locus*. Since such loci often fail to cover the whole plane, the problem posed is to find the equation of the bounding curve in terms of X and Y. This may be done for (9) by eliminating two of the variables x, y and z from (8) and (9) and equating to zero the discriminant of the quadratic in the remaining variable. The device is illustrated as applied to

$$f(x, y, z) \equiv x^2+y^2+z^2-a^2=0. \tag{10}$$

From (8) we get

$$y=Y/\sqrt{3}+X-x; \qquad z=Y/\sqrt{3}-X+x. \tag{11}$$

These values, substituted in (10), yield

$$Ax^2+Bx+C=0, \tag{12}$$

where

$$A=3; \qquad B=-4X; \qquad C=2X^2+2Y^2/3-a^2. \tag{13}$$

Evidently x, as well as, through (11), y and z, will be real for any real pair X and Y so chosen that $D \equiv B^2-4AC \geqq 0$. Here $D=12a^2-8X^2-8Y^2$. For $D=0$,

$$X^2+Y^2=3a^2/2. \tag{14}$$

Thus any point (X, Y) on the circle (14) will yield one and only one set of coordinates (x, y, z) satisfying (10). Any point inside the circle will have two such sets of proper coordinates. For example, the proper coordinates of the point $(0, 0)$ as solutions of (10) are $(\pm a/\sqrt{3}, \mp a/\sqrt{3}, \pm a/\sqrt{3})$, and not $(0, 0, 0)$.

As a second example of a typical result, we find that the locus of the equation $x=y^2+z^2$ is the filled parabola

$$Y^2=\tfrac{3}{2}(X+\tfrac{1}{8}). \tag{15}$$

c. *A quadric locus on the 4-axis plane, and its generalization.*

The algebraic solution for the total locus of an equation in n variables is in general more difficult when $n>3$. Some available methods may be suggested by the following treatment of the equation

$$x^2+y^2+z^2+u^2=a^2. \tag{16}$$

For $n=4$, equations (7) yield

$$\sqrt{2}\,x + y - u = \sqrt{2}\,X; \qquad y + \sqrt{2}\,z + u = \sqrt{2}\,Y. \tag{17}$$

Eliminating y and z from (16) by means of (17), and equating to zero the discriminant of the quadratic in u, we have

$$8(-2X^2 - Y^2 + 4Xx - 4x^2 + 2a^2) = 0. \tag{18}$$

This is equivalent to the family of ellipses

$$2(X - x)^2 + Y^2 = 2(a^2 - x^2) \tag{19}$$

representing the bounding curves in terms of the parameter x. Evidently $-a \leqq x \leqq a$, as is also apparent by inspection of (16). For any permissible x the locus area is the corresponding filled ellipse, and the total locus is the filled envelope of (19).

Seeking the maximum (and minimum) X for a given Y, we consider X in (19) as a function of x, and set $dX/dx = 0$, whence $x = X/2$. Replacing x by $X/2$ in (19), we have

$$X^2 + Y^2 = 2a^2. \tag{20}$$

Thus the locus of (16) is the filled circle (20).

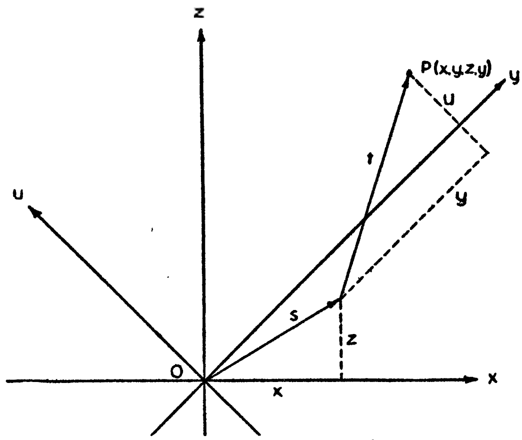

FIG. 5

The important role which geometry may play in this field is well illustrated by a geometric proof of result (20), which leads to the following generalization:

THEOREM 1. *When n is even and exceeds 2, the locus of*

$$\sum_{i=1}^{n} x_i^2 = a^2 \tag{21}$$

is the filled circle

$$X^2 + Y^2 = na^2/2. \tag{22}$$

The proof for the case $n=4$ follows: Consider a point $P(x, y, z, u)$ (Figure 5) on the locus of (16). Since the pairs x, z and y, u are mutually perpendicular

$$a^2 = (x^2 + z^2) + (y^2 + u^2) = s^2 + t^2. \tag{23}$$

For any given pair s, t satisfying (23), various choices of the pairs x, z and y, u are available. The ones which place P as far as possible from 0 are those which make one straight segment from the given segments s and t, so that $OP=s+t$, or, by (23),

$$OP = s + \sqrt{a^2 - s^2}. \tag{24}$$

By calculus, OP reaches its maximum when $s=t=a/\sqrt{2}$, whence $OP=\sqrt{2}a$.

In the general case, with n even, the pairs x_i, $x_{n/2+i}$ $(i=1, 2, \cdots, n/2)$ are mutually perpendicular, and hence, by grouping the variables properly in pairs,

$$a^2 = (x_1^2 + x_{n/2+1}^2) + \cdots + (x_{n/2}^2 + x_n^2) = s_1^2 + s_2^2 + \cdots + s_{n/2}^2. \tag{25}$$

For a given choice of the s_i satisfying (25), OP will reach its maximum when

$$OP = s_1 + s_2 + \cdots + s_{n/2}. \tag{26}$$

The rest of the proof is by induction. Given $k_m^2=s_1^2+s_2^2+ \cdots +s_m^2$, where the k_i $(i=1, 2, \cdots, m)$ are constants, assume that, for each value of m,

$$OP = s_1 + s_2 + \cdots + s_m \tag{27}$$

attains its maximum value, which is $\sqrt{m}\, k_m$, when $s_1=s_2= \cdots =s_m$. The assumption is verifiable by calculus when $m=2$. Let $k_m=t$ and $k_{m+1}=a$. Then

$$a^2 = (s_1^2+ \cdots + s_m^2) + s_{m+1}^2 = t^2 + s_{m+1}^2 = t^2 + (a^2 - t^2), \tag{28}$$

and

$$OP = (s_1 + \cdots + s_m) + s_{m+1} = \sqrt{m}\, t + \sqrt{a^2 - t^2}. \tag{29}$$

Treating OP as a function of t, we find that it reaches its maximum when $t=a\sqrt{m/(m+1)}$, so that $OP=a\sqrt{m+1}$ and $s_{m+1}=\sqrt{a^2-x^2}=a/\sqrt{m+1}$ $=OP/(m+1)$, completing the induction.

Note: It seems likely that Theorem 1 holds also when n is odd. By way of partial verification, the writer has proved that the locus of (21) contains the circle (22) when $n\geqq 2$.

d. *Loci of first degree equations.*

THEOREM 2. *The locus of the equation*

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = A_0 \tag{30}$$

is normally the whole n-axes plane. It is a straight line if and only if the rank of the matrix

$$\left\| \begin{matrix} A_1 & \cdots & A_n \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \sin\theta_1 & \cdots & \sin\theta_n \end{matrix} \right\| \tag{31}$$

is two.

Proof. The elements in the rows of (31) are the coefficients of the homogeneous terms in x_i $(i=1, 2, \cdots, n)$ in equations (30) and the two parts of (7). If the rank is 3, all of the x_i except three which are properly selected, say x, y and z, may be set equal to zero. The equations may then be solved, giving x, y and z as linear functions of X and Y. Thus all the points of the plane are on the locus, since values for X and Y may be assigned at will. If, on the other hand, the rank is 2 (it cannot be 1 in view of the linear independence of the two bottom rows), there exist constants A, B and C, not all zero, such that $AA_i+B\cos\theta_i+C\sin\theta_i=0$, and hence

$$A\sum_{i=1}^{n} A_i x_i + B\sum_{i=1}^{n} x_i\cos\theta_i + C\sum_{i=1}^{n} x_i\sin\theta_i = 0, \tag{32}$$

since the total coefficient of each x_i is zero. Therefore unless, with reference to the terms A_0, X and Y in (30) and (7),

$$AA_0 + BX + CY = 0, \tag{33}$$

the three equations in (30) and (7) are inconsistent. In other words, only points of (33) are on the locus of (30).

An example of such a degenerate case is the line

$$2x + (2\sqrt{3}+1)y + (2\sqrt{3}-1)z = 5, \tag{34}$$

whose equation is two variables is

$$2X + 4Y = 5. \tag{35}$$

Here (35) was written first and then changed to (34) by use of (8).

4. Simultaneous equations. A point is on the ***intersection*** of two simultaneous equations if it has a set of coordinates which satisfy both equations.

As in solid analytic geometry, the intersection of the loci of two equations in three variables will normally be a curve; but in this case the curve lies directly on the 3-axes plane instead of twisting inconveniently into space.

Consider, for example, the planes

$$5x + 2y - z = 2; \qquad x + 2y + z = 2. \tag{36}$$

Eliminating z and then y, we get successively: $y=1-3x/2$ and $z=2x$. The three general coordinates of an intersection point are therefore x, $1-3x/2$, and $2x$; so that by assigning two different values to x we find at once two points on the line of intersection. The proof that this intersection is the straight line

$$X + \sqrt{3}\,Y = 2 \tag{37}$$

may be carried through as indicated in the next paragraph.

Evidently a straight line or curve will be the normal type of intersection of the loci of $n-1$ equations in n variables, where $n \geqq 3$. For one letter may be chosen as a parameter in terms of which alone each of the others may be expressed after the proper eliminations. Then by means of the form of (7) which applies, X and Y may be expressed in terms of the chosen letter, thus yielding the equation of the intersection in parametric form. When the parameter is eliminated, if this is convenient or desirable, the equation of the intersection appears directly in terms of X and Y.

If the number of variables exceeds the number of simultaneous equations by more than one, the intersection will normally be an area. Numerous special cases of course occur, especially when the loci of some of the simultaneous equations are degenerate.

B. THE THREE-DIMENSIONAL VARIANT

In this system we shall use, in addition to the equally-spaced axes on a plane, one extra axis which is perpendicular to that plane at the origin. The use of the capital Z to designate this special axis serves not only to show its unique status when the number of variables exceeds three, but also to give a consistent notation (with variables X, Y and Z) for the 3-axes case, which is that of orthodox solid analytic geometry.

It is true that analogy suggests for the solid case an equal spacial distribution of the axes. But the scheme here suggested, besides being more practical algebraically, permits the use of all the theory developed for the plane case. In addition, it exhibits in a useful way the special status of one variable, to be represented by Z, which is expressed explicitly as a function of the others.

The locus of an equation in four or more variables is normally the totality of points in a solid which may or may not be bounded. The revealing cross-sections of such loci made by the planes $Z=k$ are often sufficient to indicate their general contours, and may even suggest the equations of the bounding surfaces. For example, consider the equation

$$Z = x^2 + y^2 + z^2 + u^2 - a^2. \tag{38}$$

By Theorem 1, we know that the trace of this locus on the plane $Z=k$ is the filled circle

$$X^2 + Y^2 = 2(a^2 + k). \tag{39}$$

It follows that the locus of (38) is the filled paraboloid

$$Z = (X^2 + Y^2)/2 - a^2, \tag{40}$$

since the trace of (40) in the plane $Z=k$ is identical with (39).

It is evident that in this system the interpretation of a partial derivative not involving Z is exactly that of Section A 2, as applied in the proper plane $Z=k$. To interpret a partial derivative of Z we use a Z-plane, or a plane parallel to the Z-axis. For example, in the solid 5-axes system, the Z-plane $Zx)_{y=4,\ z=0,\ u=4}$ has a

2-axes system whose conveniently perpendicular x' and z' axes are parallel respectively to the x and Z axes, and whose origin is the point (0, 4, 0, 4, 0), the Z-coordinate being always last. The trace of the locus of

$$f(x, y, z, u, Z) = 0 \tag{41}$$

on this Z-plane is the curve

$$f(x, 4, 0, 4, Z) = 0 \tag{42}$$

whose slope at $x = x_1$ and $Z = Z_1$ is the value of $\partial Z/\partial x$ at the point $(x_1, 4, 0, 4, Z_1)$.

It should be noted that many different Z-planes may coincide, with each one contributing its special curve to the total locus. Thus the Z-planes are somewhat analogous to the complanes on the planes $Z = k$. For example, the Z-planes $Zx)_{y=4,z=0,u=4}$ and $Zx)_{y=0,z=4\sqrt{2},u=0}$ coincide; but the former contains the curve (42) and the latter the curve

$$f(x, 0, 4\sqrt{2}, 0, Z) = 0. \tag{43}$$

Again, to illustrate graphically the relation between any three variables which include Z, we may use a 3-axes *subsystem*, in which all variables except Z and two others are held fixed. For example, the subsystem $yzZ_{x=4,u=3}$ has its origin at the point (4, 0, 0, 3, 0), with its axes y', z' and Z' parallel, respectively, to the y, z and Z axes. The portion of the locus of (41) which appears as a trace in this subsystem is the surface

$$f(4, y, z, 3, Z) = 0. \tag{44}$$

As on the n-axes plane, equations in n variables having degenerate loci may be obtained readily. In this case, substitutions (7) are made upon equations in X, Y and Z, with Z remaining unchanged.

To summarize, in this paper we have sketched the essential features of a plane and solid analytic geometry for n variables. Sample loci have been obtained, sample methods have been suggested, and some general results have been stated as theorems.

It is clear that the investigation of the two-dimensional phase should come first, since knowledge of the loci on the n-axes plane greatly facilitates the study of the loci in space.

Question and answer. How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?—Einstein.

Nature has not embarrassed itself with mathematical difficulties.—Fresnel.

Man and Beast. So celebrated was this proposition [that the square root of 2 is irrational] among the ancient philosophers that Plato declared anyone ignorant of it was not a man but a beast.—Isaac Barrow.

Irrational numbers do not exist.—Leopold Kronecker. *Contributed.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE "TIME-UNIT" SYSTEM IN NAVIGATION

R. A. ROSENBAUM, U.S.N.R.

1. Introduction. In celestial navigation it is rare that the observations for a fix are made simultaneously, so that the lines of position resulting from the earlier observations must be "advanced" to the time of the last. This is often done as follows (Fig. 1): Any point G is chosen on l, the line of position. In the direction of the ship's track, GG' is measured equal to the distance covered during the time-interval for which l is to be advanced. Then l', the line parallel to l through G', is the required advanced line, for, if P is the actual position of the ship on l, its advanced position, P', will lie on l'.

This method has the disadvantage, in aerial navigation, that frequently neither the track nor the speed which the aircraft is making good is accurately known. The error resulting from use of an incorrect track and speed will ordinarily have a negligible effect on the usefulness of the fix, but some perfectionists like to use the following method, when it is applicable.

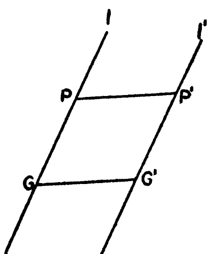

FIG. 1

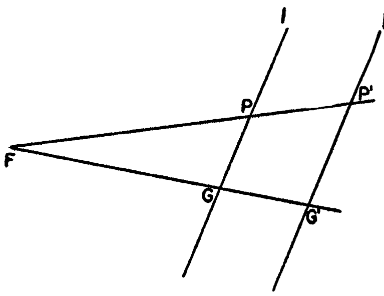

FIG. 2

2. The "Time-Unit" System. This method may only be used when it can be assumed that track and speed have been constant since the last fix. Suppose, in Fig. 2, that F is the position of a fix at time T, and that l is a line of position obtained t minutes later. This means that the aircraft is at F at time T and is somewhere on l at $T+t$. Suppose further that it is desired to advance l for τ minutes, *i.e.*, to find the possible positions for the aircraft at time $T+t+\tau$. The method consists in choosing any convenient unit of length on a marked straight-edge to represent one minute of time, and to move the straight-edge about on the chart until a point G is obtained on l such that $FG=t$, according to

the chosen unit. Then FG is prolonged to G' such that $GG'=\tau$, and l' is drawn parallel to l through G'. The advanced line of position is l', for:

Suppose that the actual track is FP, that the observer t minutes after the fix is at P, and that $t+\tau$ minutes after the fix he is at P'. Then, under the assumption of constant track and speed, the distance covered is proportional to the elapsed time; hence,

$$FG/GG' = t/\tau = FP/PP'.$$

Therefore, P' lies on l'.

Note that this method requires that the track and speed of the aircraft be constant, but it does not require that the navigator know their values.

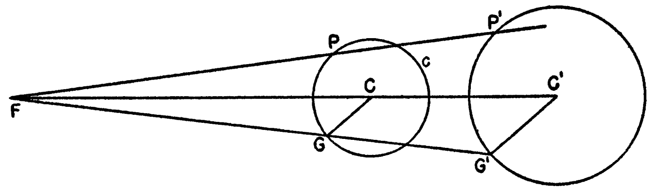

FIG. 3

3. Extension. The line of position resulting from the usual celestial observation is an approximation to part of a large circle of position. In case the altitude of the observed body is close to 90°, the circle is small, and a straight-line approximation is not sufficiently accurate—the circle itself must be plotted. It is the purpose of this paragraph to show how the "time-unit" method may be extended to the case of advancing circles of position. The method illustrates the perhaps unfamiliar fact that the advanced line or curve of position need not be simply the original one displaced; in this case it will appear that the advanced circle of position is not congruent to the original.

Suppose that, in Fig. 3, F is again the position of a fix at time T; c, with center C, is a circle of position t minutes later; and it is desired to advance c for τ minutes. As before, choose a unit of length to represent one minute; with this scale find G on c such that $FG=t$; extend FG to G' such that $GG'=\tau$; and through G' draw $G'C'$ parallel to GC, meeting FC produced at C'. Then C' is the center and $C'G'$ the radius of the advanced circle. The reasoning is based on proportions as before.

Note by the Editor. While his note *Checking the SAS Case in Trigonometry* in the April issue of this MONTHLY was in press, Professor Eves called my attention to the fact that Professor E. J. Moulton had pointed out the same discrepancy in the customary check of the *SAS* case in trigonometry in this MONTHLY, vol. 31, 1924, p. 292. M.J.W.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Mathematical Society, Hofstra College

During the year the following talks were given:

Mathematics in art, an illustrated lecture by Professor Constant Van de Wall of the Fine Arts department

Greek mathematics, by Harry Durham

Mathematical logic, by Robert Ackerson

Signifying nothing, by Professor L. F. Ollmann.

One meeting was devoted to a discussion of mathematical brain-teasers. A joint picnic was held with *Kappa Mu Epsilon*.

Officers for the year were: President, Edith Hufman; Vice-President, Joseph Consoli; Secretary, Dorothy McFarland; Treasurer, Vera Pabo; Historian, Rheba Handler; Faculty Adviser, Professor E. R. Stabler.

Kappa Mu Epsilon, Hofstra College

The following papers were presented during the year to the New York Alpha Chapter of *Kappa Mu Epsilon*:

Mechanics of structural design, by Professor A. D. Capuro

Postulational methods, by Professor E. R. Stabler

Mathematics of finance, by Professor L. F. Ollmann

Photography, by Mr. Stanley Rodgers of the Physics department

The motion of projectiles, by Professor W. L. Ayres of Purdue University.

In addition, a banquet was held, and there was a joint picnic with the Mathematical Society.

Officers were as follows: President, Edith Hufman; Vice-President, Harry Durham; Secretary, Wanda Scala; Treasurer and Historian, Jean Ruppel; Corresponding Secretary, Professor E. R. Stabler; Faculty Sponsor, Professor A. D. Capuro.

Pi Mu Epsilon, Washington Square College, New York University

The Executive Council agreed that the war-time curtailment of Chapter activities should be continued during this year, omitting the former annual banquet and inter-scholastic contest. Chapter meetings were restricted to two lectures, two business sessions, and two induction ceremonies at which fifteen members were inducted. The two lectures, with refreshments following, were:

Demonstrations of soap film in connection with minimal surfaces and problems of stability, by Professor Richard Courant.

The foundations of probability, by Professor Hans Reichenbach of the University of California, at a meeting held jointly with the New York University Philosophical Society.

Officers elected for 1944–45 were: Chapter Director, Professor W. M. Maiden: Vice-Director, Naomi Rosenstein; Treasurer, Peter Lax; Secretary, Eleanor Karasak; Permanent Secretary, Professor F. W. John.

Pi Mu Epsilon, University of Arkansas

Because of the fewness of older members in our University of Arkansas chapter and the heavy schedules imposed by the Air Corps on our faculty members there has been no presentation of papers this past year. The annual banquet and initation of new members was revived this year and took place on April 30. Eight candidates were initiated and each presented a short humorous theme on some phase of mathematics which served as the program for the banquet.

Actually there were fifteen candidates initiated this year, but the other group was initiated at a special meeting. The necessity of the other initiation arose from the quarter system here at the University. It is interesting to note that a greater percentage of girls were initiated this year than ever before.

Officers for the year were: Director, J. W. Keller; Vice-Director, Dan Welchel; Secretary, Roger Harris; Treasurer, Carl Gamel.

AN APPROACH TO THE NORMAL CURVE AND THE CYCLOID

J. S. FRAME, Michigan State College

1. Attractive forces. At first sight the parabola, the cosine curve, the cycloid, and the areas and ordinates of the normal curve of error do not seem to have much in common, except that they all seem to be popular subjects for exercises in a first course in the calculus. But the inner harmony of mathematics is illustrated once again in the fact that these curves may all be considered as different special cases of the parametric relationship between the distance y and the time t for a particle moving in a straight line under an attractive force proportional to a power of the distance.

At a combined meeting of the Mathematics Club and the Physics Club, four speakers might each present one of these four special curves from the common viewpoint, and a fifth speaker might discuss the general case which is treated below.

The common ancestor of these curves is the differential equation

$$\frac{dv}{dt} = -ky^{n-1}, \quad \text{where} \quad v = \frac{dy}{dt}. \tag{1}$$

It will be convenient to take $t=0$ when $v=0$, and let $y=a$ at that time. Since the velocity v will be negative for small positive t, we shall introduce a positive parameter u defined by the equations

$$u = -v/c, \qquad c^2 = ka^n. \tag{2}$$

Multiplying both sides of equation (1) by dy/c^2, dt may be eliminated and we obtain

$$\frac{vdv}{c^2} = -\frac{y^{n-1}dy}{a^n}. \tag{3}$$

Integrating, and noting that $y=a$ when $v=0$, we have

$$\frac{v^2}{2c^2} = \frac{u^2}{2} = \frac{1}{n}\left\{1 - \left(\frac{y}{a}\right)^n\right\}, \qquad n \neq 0, \tag{4}$$

or

$$\frac{v^2}{2c^2} = \frac{u^2}{2} = \ln\frac{a}{y}, \qquad n = 0. \tag{4'}$$

2. **The parabola and the cosine curve.** For $n=1$, equation (1) is simply the law for the motion of a body falling vertically from a height a under a constant gravitational force. The path of the motion is a straight line, but the graph expressing y as a function of t is a *parabola.*

For $n=2$, equation (1) represents the simple harmonic motion of a weight hanging on a stretched spring and oscillating up and down with amplitude a about its position of equilibrium. The graph expressing y in terms of t is a *cosine curve.* These facts are well known to teachers of the calculus, and should also be known by students who have had a year of calculus.

3. **The cycloid and the normal curve.** It is not common knowledge, however, nor is it mentioned in most calculus texts, that the cases $n=-1$ and $n=0$ of equation (1) lead respectively to the *cycloid* and the *normal curve of error.* The case $n=-1$ is the *inverse square law* which governs the motion of a body falling to the earth from a distance, or the motion of a charged particle attracted to a particle of opposite charge. The case $n=0$ is the *inverse first power law* of force, and can be realized physically in the motion of a charged particle attracted to a long wire of opposite charge.

Perhaps the natural way to integrate equations (4) and (4′) would be to express $dt=dy/v$ in terms of y and integrate. But this procedure leads to square roots under the integral sign. Thus

$$t = \frac{-1}{c}\int_a^y \frac{dy}{\sqrt{2(a-y)/y}} \quad \text{for} \quad n = -1, \tag{5}$$

and

$$t = \frac{-1}{c}\int_a^y \frac{dy}{\sqrt{2\ln(a/y)}} \quad \text{for} \quad n = 0. \tag{5'}$$

Simpler results are obtained by expressing y and t both parametrically in terms of u. Thus for $n=-1$, we have

$$(6)\qquad \frac{u^2}{2}=\frac{a}{y}-1,\quad y=\frac{a}{1+u^2/2},\quad t=\int_0^v\frac{dy}{-cu}=\frac{a}{c}\int_0^u\frac{du}{(1+u^2/2)^2}.$$

The substitution $u=\sqrt{2}\tan\phi$ gives

$$y=a\cos^2\phi,\qquad t=\frac{a}{c}\sqrt{2}\int_0^\phi\cos^2\phi d\phi=\frac{a\sqrt{2}}{4c}\int_0^\phi(1+\cos 2\phi)d2\phi,$$

or

$$(7)\qquad \sqrt{2}\,ct=\tfrac{1}{2}a(2\phi+\sin 2\phi),\qquad y=\tfrac{1}{2}a(1+\cos 2\phi).$$

With y as ordinate and $\sqrt{2}ct$ as abscissa, these are parametric equations of a cycloid which has the top point of its arch on the y-axis, and for which 2ϕ is the angle of rotation of a rolling wheel which generates the curve.

Similarly for $n=0$, we have from (4′) and (5′) the parametric equations

$$(6')\qquad \frac{u^2}{2}=\ln\frac{a}{y},\qquad y=ae^{-u^2/2},\qquad t=\int_0^v\frac{dy}{-cu}=\frac{a}{c}\int_0^u e^{-u^2/2}du,$$

or

$$(7')\qquad \frac{ct}{a}=\int_0^u e^{-u^2/2}du,\qquad \frac{y}{a}=e^{-u^2/2}.$$

We see that y/a and ct/a are respectively the ordinate and area for the unit normal curve of error, in which $u=-v/c$ is the abscissa. Since these functions are tabulated in parallel columns in many mathematical tables, the numerical values for plotting y in terms of t are readily accessible.

4. The solution of the general case. In general we may express both y and t in terms of the parameter $u=-v/c$, and we obtain from equation (4)

$$(8)\qquad y=a(1-nu^2/2)^{1/n},\qquad ct=a\int_0^u(1-nu^2/2)^{(1-n)/n}du.$$

In particular, for $n=1$, equation (8) defines the parabola

$$(8')\qquad y=a(1-u^2/2),\qquad ct=au;\quad\text{or,}\quad y=a-c^2t^2/2a.$$

Furthermore, for $n=2$, equation (8) defines the cosine curve

$$(8'')\qquad y=a\sqrt{1-u^2},\qquad ct=a\sin^{-1}u;\quad\text{or,}\quad y=a\cos(ct/a).$$

In general, for positive n, if we set $n=2/p$, $u=\sqrt{p}\sin\phi$, we obtain

$$(9)\quad y=a\cos^p\phi,\quad ct=a\sqrt{p}\int_0^\phi\cos^{p-1}\phi d\phi,\quad v=-c\sqrt{p}\sin\phi,\quad p=2/n.$$

In general, for negative n, if we set $n=-2/m$, $u=\sqrt{m}\tan\phi$, we obtain

$$(9') \quad y = a\cos^m \phi, \quad ct = a\sqrt{m}\int_0^{\phi} \cos^m \phi d\phi, \quad v = -c\sqrt{m}\tan\phi, \quad m = -2/n.$$

For $n=0$, the expressions in (8) must be looked upon as indeterminate forms whose limits for $n \to 0$ are given by (7').

5. The total time. The total time T required for the trip from $y=a$ to $y=0$ is given by definite integrals which may be evaluated in terms of the Beta and Gamma functions. For $n=2/p>0$ we have

$$(10) \quad \begin{aligned} T &= \frac{a\sqrt{p}}{c}\int_0^{\pi/2} \cos^{p-1}\phi d\phi = \frac{a\sqrt{p}}{c2} B\left(\frac{p}{2}, \frac{1}{2}\right) \\ &= \frac{a}{c}\sqrt{\frac{\pi}{2}}\sqrt{\frac{p}{2}}\,\Gamma\left(\frac{p}{2}\right)\Big/\Gamma\left(\frac{p+1}{2}\right), \quad \frac{p}{2} = \frac{1}{n}. \end{aligned}$$

Similarly for $n=-2/m<0$, we have

$$(10') \quad \begin{aligned} T &= \frac{a\sqrt{m}}{c}\int_0^{\pi/2} \cos^{m}\phi d\phi = \frac{a\sqrt{m}}{c2} B\left(\frac{m+1}{2}, \frac{1}{2}\right) \\ &= \frac{a}{c}\sqrt{\frac{\pi}{2}}\,\Gamma\left(\frac{m+1}{2}\right)\Big/\sqrt{\frac{m}{2}}\,\Gamma\left(\frac{m}{2}\right), \quad \frac{m}{2} = -\frac{1}{n}. \end{aligned}$$

These two results may be combined in a single formula

$$(11) \qquad T = \frac{a}{c}\sqrt{\frac{\pi}{2}}\, e^{f(n)},$$

by means of the odd function $f(n)$, which we define by the equations

$$(12) \quad \begin{aligned} &f(n) = \ln\sqrt{\frac{1}{n}}\,\Gamma\left(\frac{1}{n}\right) - \ln\Gamma\left(\frac{1}{n}+\frac{1}{2}\right), \qquad n>0; \\ &f(0) = 0; \qquad f(-n) = -f(n). \end{aligned}$$

It can be shown that $f(n)$ has the power series expansion

$$(13) \quad f(n) = \frac{n}{8} - \frac{n^3}{192} + \frac{n^5}{640} - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^k(1-4^k)B_k}{(2k-1)(2k)}\left(\frac{n}{2}\right)^{2k-1},$$

where B_k are the Bernoulli numbers ($B_1=1/6$, $B_2=1/30$, $B_3=1/42$, *etc*). It may be an interesting exercise to evaluate three terms of the series for $n=\pm 1, \pm 2$, and compare (11) with the values for T in (10) and (10').

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Education of T. C. Mits. By H. G. Lieber and L. R. Lieber. New York, W. W. Norton and Company, Inc., 1944. 230 pages. $2.50.

This book, as the title indicates, is concerned with the education of "The Celebrated Man In the Street." It is "an attempt to get a bird's eye view of T. C.'s predicament, and to look for a possible egress." To do this vividly, the author uses pictures whenever possible, and to do it clearly, he uses the clearest language man has invented, mathematics.

The contents of the book are divided into two parts. Part I is what the author calls *The Old.* Among other things he points out by examples from mathematics, that the average person should not depend on intuition to draw conclusions; that, as in algebra, the use of generalization is a distinct advantage; that, as in geometry, abstraction is a powerful tool; that the mathematician by means of using "the marriage of algebra and geometry," analytic geometry, and its "offspring," calculus, can easily study our ever changing world; and most important, that science and mathematics can not only protect us from actual physical dangers, but also from the errors of our loose thinking "and thus be a veritable defense against all evil—a Totem Pole." The Totem Pole contains five divisions, from the bottom floor where all the scientific gadgets are kept, to the top floor where the pure mathematicians dwell. The author draws the conclusion in his summary of Part I "that mathematics is not only for someone who needs its formulas. It is a way of thinking, a way of life, very important for everyone."

Part II is what the author calls *The New.* In this section the author points out how the study of any science or system of thought can be developed starting with a few basic ideas from which all the other ideas or "propositions" are derived by logic. By giving several examples from mathematics, he presents the pitfalls that may lie ahead, those of using guess work, limited knowledge, and not keeping an open-minded attitude. Man must use his own reason to the best of his ability and he will obtain respectable results, but he must not brag that he "knows" the truth. As in mathematics, future observations may change the present. T. C. Mits must adapt himself to a continually changing world.

The reviewer would like to picture the book as an excellent attempt to popularize mathematics for ordinary people, an amusing work, and yet full of a deep philosophy of life. It is presented in a unique and interesting fashion, greatly aided by the illustrations. The book deserves greater popularity with the non-mathematician as well as the mathematician.

E. P. VANCE

Algebra of Analysis. By Karl Menger. Notre Dame, Indiana (Notre Dame Mathematical Lectures, Number 3), University of Notre Dame, 1944. 3+50 pages. $1.00.

The author takes a new approach to the theory of functions of a single variable by starting with a system of elements (to be called functions) which form a tri-operational algebra. The operations of this algebra of functions are addition, multiplication and substitution. The commutative, associative and distributive laws for these operations which are satisfied by ordinary functions are postulated. The author defines neutral or identity elements in regard to each operation, the neutral element for the substitution operation corresponding to the function $y=x$ in the ordinary function theory.

Topics covered include the theory of constant functions, the theory of inverses (where these exist) for the three operations of the algebra of functions, definitions and properties of exponential, logarithmic, power and trigonometric functions, the algebra of calculus including anti-derivatives. The last chapter explores the concept of algebra of functions of several variables.

The booklet is written in a fluent style and it can be readily appreciated by a reader who has an elementary knowledge of the concepts of ring and field. The limited distributivity available in regard to the substitution operation impedes the development of any very deep algebraic theory for the algebra of functions. In some topics, the author's notation and method achieves considerable elegance; this is, however, balanced on occasion by formalism and expedient postulation.

A number of minor errors were observed by the reviewer. In the footnote on p. 11, third line from the bottom, "and $=0$" should read "and $=c$." On p. 16 the formulas for j.rec and f.rec f should have 1 not j as their right members. On p. 19 the discussion of exponential functions where the constants form a finite field requires slight correction. On page 31, line 7, $(Dg)f$ should be $(Df)g$, and on page 32, line 2, the left member should be D neg j.

C. J. NESBITT

Mathematics. Second Edition. By J. W. Breneman. New York and London, McGraw-Hill Book Company, Inc., 1944. 12+224 pages. $1.75.

This text covers arithmetic, algebra to the solution of quadratics, geometrical constructions without proofs, trigonometry, and logarithms in an elementary and practical manner. It contains nearly five hundred simple and practical problems taken from engineering applications whenever possible.

The emphasis on theory varies from no proof in geometry to careful proofs of the sine and cosine laws. The laws of logarithms are presented by simple numerical examples but no formal proofs are given.

Besides the usual tables the appendix contains some tables ordinarily placed in engineering texts and a splendid table of specific gravities and weights of common substances.

R. D. WAGNER

Air Navigation Made Easy. By J. F. Naidich. New York and London, McGraw-Hill Book Company, Inc., 1944. 9+124 pages. $1.75.

This book is intended for private civilian flyers who plan flights under three hundred or four hundred miles and who have had very little technical training. It covers the subjects of piloting, *i.e.*, flying by use of land marks; dead reckoning, *i.e.*, flying by calculated data; and, briefly, flying by radio aids. It contains numerous questions and examples to fix in mind the principles that the flyer should know thoroughly. Answers to the odd-numbered questions and examples follow the main body of the text.

The book is divided into four parts. Part 1, *Air Piloting*, consists of one chapter on maps and charts, and their uses in air piloting. Mercator, Lambert, and polar projections are explained. Part 2, *Dead Reckoning*, is divided into two chapters. Chapter II deals with the aircraft compass, its errors, and use in flying. Chapter III is on the wind triangle, the solutions of which are obtained graphically. Part 3, *Locating Position and Other Problems*, has two chapters. Locating position while in flight is given in Chapter IV. Flying time, range, radius of action, and a brief discussion of Federal Aids to Navigation are given in Chapter V. Part 4, *Review Tests*, has five tests covering the main points in the other three parts.

The reviewer found no errors. The mechanical make-up of the book is excellent. All printing is clear and distinct. The author's style of presentation is good and one is able to read without tiring. The only adverse criticism the reviewer is inclined to make is concerning the expression on page 1 about the "top" of the earth, when the polar region is meant.

Anyone learning to fly, whether he be preparing to fly for pleasure or preparing to operate a plane for other reasons, will find this a very helpful book.

H. H. DOWNING

Vital Mathematics. By E. B. Allen, Dis Maly, and S. H. Starkey, Jr. New York, The Macmillan Company, 1944. 7+456+22 pages. $1.80.

This text covers arithmetic, algebra through quadratic equations, plane geometry without formal proofs, some solid geometry, plane trigonometry, and an introduction to spherical trigonometry. The simple operations of arithmetic are carefully presented. The theory and the practice of rounding off numbers are thoroughly discussed.

There is a large collection of examples and interesting problems. Many of them emphasize practical applications of mathematics.

Each chapter is headed by a set of problems typical of those occurring in the chapter. This makes the text especially adaptable for self study.

Some unusual materials included are elementary statistics and an introduction to cartography.

R. D. WAGNER

Plane and Spherical Trigonometry. By H. P. Doole. New York, Thomas Crowell Co., 1944. 8+183 pages. $1.75.

This rearrangement of the standard materials of trigonometry is intended for the instructor who wishes to complete the theory before turning to the applications. For example, radian measure and inverse functions appear in the first lessons. Three chapters present theory (17 exercises, 569 problems); two chapters are devoted to logarithms and the solution of triangles (14, 232); there are chapters on graphs, mil measure, slide rule, and vectors (5, 88); and a chapter on spherical trigonometry (8, 74). A companion handbook of tables is required, only two brief tables being given in the text. An index and a list of answers to odd-numbered problems are included.

The volume is a handy size, all figures are clear, and there are stimulating drawings as chapter headings. Brief statements are given of the use of each new concept, and explanations are brief and simple. It may be valuable in teaching identities to use the author's symbols A and T to distinguish those steps of a proof which are algebraic and those which are trigonometric.

The text assumes that the student knows the rectangular coördinate system. There is little review of the material from solid geometry used in spherical trigonometry. Proofs for some identities are not extended to all cases. The term "distance" is introduced for "radius vector." The term "knots per hour" is used.

Doole's text will be found suitable, subject to the comments above, for either a brief or an extended course.

B. M. STEWART

New List of Mathematical Tables. Washington, D. C., National Bureau of Standards, 1945. 14 pages. No charge.

The following paragraphs are quoted from a communication recently received from the National Bureau of Standards:

"A list of the 48 mathematical tables which have thus far been prepared by the Mathematical Tables Project and made available to the public is given in Letter Circular LC-777, just issued by the National Bureau of Standards.

"31 of these tables may be purchased from the Bureau, 2 from the Government Printing Office, 4 from the Columbia University Press, and nearly all of the remainder can be consulted in the several mathematical journals referred to.

"The Project was conducted by the Work Projects Administration for the City of New York until March 1943, at which time the Bureau (the sponsoring agency) took over its operation with the support of the Office of Scientific Research and Development. As time permits, various tables under way when the WPA was discontinued, will be completed.

"Copies of Letter Circular LC-777, 'Mathematical Tables,' may be had by writing to the Information Section, National Bureau of Standards, Washington 25, D. C."

H. P. EVANS

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solution to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 671. *Proposed by Alice Malice, Owen Sound, Ontario*

A hobo has crossed three-eighths of a high railway bridge when he hears behind him an express train coming at sixty miles per hour. He can just save his life by running to either end of the bridge. How fast can he run? (There is no question of acceleration or deceleration; the engineer is merciless and the man gets going at once.)

E 672. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the cyclic numbers in the duodenary scale. (A number is said to be *cyclic* if every cyclic permutation of its digits produces a multiple of the number.)

E 673. *Proposed by L. S. Shively, Ball State Teachers College*

Does there exist a regular polygon having both these properties: (a) a diagonal is equal to the sum of two other diagonals; (b) a diagonal is equal to the sum of a side and another diagonal?

E 674. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that a pandiagonal magic square of order 4, when regarded as a determinant, has the value zero.

E 675. *Proposed by Howard Eves, Syracuse University*

Show that the ratio of the curvatures of two curves in a plane at a point of contact is invariant under projection.

SOLUTIONS

An Infinite Matrix

E 634 [1944, 405]. *Proposed by H. S. Wall, Northwestern University*

Let a matrix be constructed according to the following rules:

$$a_{1,1} = 1, \qquad a_{p,q} = 0 \quad \text{if} \quad p < q,$$

$$a_{p,1} = a_{p-1,2} \qquad\qquad (p = 2, 3, \cdots),$$

$$a_{p,q} = a_{p-1,q-1} + a_{p-1,q+1} \qquad\qquad (p, q = 2, 3, \cdots).$$

Show that the sum of the products of corresponding elements in any two rows is equal to an element in the first column, *viz.*,

$$\sum_{r=1}^{\infty} a_{p,r}\, a_{q,r} = a_{p+q-1,1} \qquad (p,\, q = 1,\, 2,\, \cdots).$$

I. *Solution by N. J. Fine, Naval Ordnance Plant, Indianapolis.* Define $\|b_{p,q}\|$ by the equation

$$a_{p,1} = \sum_{q=1}^{\infty} b_{k,q}\, a_{p-k,q} \qquad (k = 0,\, 1,\, 2,\, \cdots).$$

Clearly $b_{0,1}=1$, and $b_{k,q}=0$ if $q>k+1$. Using the recursion formulas for the $a_{p,q}$,

$$\begin{aligned} a_{p,1} &= b_{k,1}\, a_{p-k-1,2} + \sum_{q=2}^{\infty} b_{k,q}(a_{p-k-1,q-1} + a_{p-k-1,q+1}) \\ &= b_{k,2}\, a_{p-k-1,1} + \sum_{q=2}^{\infty} (b_{k,q-1} + b_{k,q+1})\, a_{p-k-1,q}; \end{aligned}$$

but also

$$a_{p,1} = \sum_{q=1}^{\infty} b_{k+1,q}\, a_{p-k-1,q}.$$

Hence

$$\begin{aligned} b_{k+1,1} &= b_{k,2} && (k = 0,\, 1,\, \cdots), \\ b_{k+1,q} &= b_{k,q-1} + b_{k,q+1} && (k = 0,\, 1,\, \cdots;\ q = 2,\, 3,\, \cdots). \end{aligned}$$

Since the matrix $\|b_{p-1,q}\|$ has the same initial conditions and the same recursion formulas as $\|a_{p,q}\|$, the two are identical, that is,

$$b_{p-1,q} = a_{p,q} \qquad (p,\, q = 1,\, 2,\, \cdots).$$

Hence

$$a_{p,1} = \sum_{r=1}^{\infty} a_{k+1,r}\, a_{p-k,r} \qquad (k = 0,\, 1,\, \cdots).$$

Setting $P=k+1$, $Q=p-k$, so that $p=P+Q-1$, we have, finally,

$$a_{P+Q-1,1} = \sum_{r=1}^{\infty} a_{P,r}\, a_{Q,r} \qquad (P,\, Q = 1,\, 2,\, \cdots).$$

II. *Solution by the Proposer.* Define $a_{p,0}=0$ and

$$C_{p,q} = \sum_{r=1}^{\infty} a_{p,r}\, a_{q,r} \qquad (p,\, q = 1,\, 2,\, \cdots).$$

Then

$$C_{p,q+1} = \sum_{r=1}^{\infty} a_{p,r}\, a_{q+1,r} = \sum_{r=1}^{\infty} a_{p,r}(a_{q,r-1} + a_{q,r+1})$$
$$= \sum_{r=1}^{\infty} a_{p,r+1}\, a_{q,r} + \sum_{r=1}^{\infty} a_{p,r}\, a_{q,r+1},$$

which involves p and q symmetrically, so that

$$C_{p,q+1} = C_{p+1,q}$$
$$= C_{p+2,q-1} = \cdots = C_{p+q,1}.$$

Hence

$$C_{p,q} = C_{p+q-1,1} = \sum_{r=1}^{\infty} a_{p+q-1,r}\, a_{1,r} = a_{p+q-1,1}.$$

Also solved by E. P. Starke, who points out that

$$a_{p,q} = \binom{p-1}{\left[\frac{p-q}{2}\right]} - \binom{p-1}{\left[\frac{p-q-1}{2}\right]}.$$

The Linking Rings

E 636 [1944, 472]. *Proposed by P. R. Halmos, Syracuse University*

What is the least number of links that have to be cut, in a chain of n links, so that every integer between 1 and n can be represented as a sum of numbers of links in the disconnected chains obtained? (Cutting one link which is not at either end of a connected chain produces three pieces: the severed link and the two ends which it held together.)

Solution by Monte Dernham, San Francisco. Assuming an indefinitely extended chain, let k links be cut, so that the maximum number of successive integers, say m, can be represented in the manner described. The severed links permit representation of every integer from 1 to k. To represent the next integer, $k+1$, most economically, one of the disconnected pieces must contain $k+1$ links. This piece, in combination with one or more of the severed links, further permits representation of each of the succeeding k integers. Continuing in like manner, we find that additional segments, if any, must contain respectively

$$2(k+1),\ 4(k+1),\ \cdots,\ 2^r(k+1),\ \cdots$$

links; and, since the number of disconnected segments which the k severed links held together cannot exceed $k+1$,

$$m = k + \sum_{r=0}^{k} 2^r(k+1) = 2^{k+1}(k+1) - 1.$$

It follows that, in a chain of n links, the least number of links that have to be cut is the integer k for which

$$2^k k \leqq n < 2^{k+1}(k+1).$$

Thus $k=1$ for $2 \leqq n \leqq 7$,
$k=2$ for $8 \leqq n \leqq 23$,
$k=3$ for $24 \leqq n \leqq 63$,
$k=4$ for $64 \leqq n \leqq 159$, and so on.

Finally, it may be observed that there are ways of producing disconnected chains, and reconnecting them, without cutting any links whatsoever—exemplified by such finished artists as Dante, the Great Leon, Jean Hugard, Dr. Harlan Tarbell, Chung Ling Soo. Monographs on this effect appear occasionally in *The Linking Ring*, official journal of the International Brotherhood of Magicians. But extreme care must be taken to preclude interference by disembodied spirits; otherwise the chains are apt to become unmanageable.

Also solved by Shepard Bartnoff, D. H. Browne, W. E. Buker, E. D. Schell, E. P. Starke, and the proposer.

Seven Sevens at the End of a Cube

E 638 [1944, 472]. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Without the use of tables, find the smallest integer whose cube terminates in seven sevens.

Solution by L. R. Chase, Rogers High School, Newport, R. I. Let a denote an n-digit number, such that a^3 ends in n sevens preceded by a digit y. Let a new digit x be prefixed to a to form an $(n+1)$-digit number $x \cdot 10^n + a$, whose cube is

$$x^3 \cdot 10^{3n} + 3ax^2 \cdot 10^{2n} + 3a^2x \cdot 10^n + a^3.$$

Here x may be selected so that the number formed by the last two terms shall end in $n+1$ sevens; the other terms cannot affect the last $n+1$ digits. Since $3a^2 \equiv 147 \equiv 7 \pmod{10}$, these last two terms are

$$7x \cdot 10^n + a^3 \pmod{10^{n+1}}.$$

Hence we require $7x+y \equiv 7 \pmod{10}$, *i.e.*,

$$x \equiv 7y + 1 \pmod{10}.$$

By trial we find, when $n=1$, $a=3$, so that $a^3=27$, $y=2$, $x=5$.

When $n=2$, we have	$a=53$,	$a^3=\cdots 877$,	$y=8$, $x=7$.
When $n=3$,	$a=753$,	$a^3=\cdots 7777$,	$y=7$, $x=0$.
When $n=4$,	$a=0753$,	$a^3=\cdots 57777$,	$y=5$, $x=6$.
When $n=5$,	$a=60753$,	$a^3=\cdots 577777$,	$y=5$, $x=6$.
When $n=6$,	$a=660753$,	$a^3=\cdots 4777777$,	$y=4$, $x=9$.
When $n=7$,	$a=9660753$.		

Also solved by Shepard Bartnoff, Colin Blyth, D. H. Browne, W. E. Buker F. M. Carpenter, Monte Dernham, William Douglas, Margaret Olmsted, E. D. Schell, E. P. Starke, J. A. Tierney, Jeanette Van Os, and the proposer.

The Pole of a Clothoid

E 639 [1944, 472]. *Proposed by Howard Eves, Syracuse University*

A *clothoid* (or *transition spiral,* used in highway engineering) is defined as a curve whose curvature varies directly with the arc length. Locate the geometrical pole of this spiral.

Solution by the Proposer. Let the start O of the spiral be taken as origin of rectangular coordinates, and let the initial tangent OX be taken as the x-axis. Then, if s is the arc length of the spiral from O to any point P on the curve, and ϕ the angle which the tangent at P makes with OX, we have, by the definition of the curve,

$$d\phi/ds = ks,$$

where k is a constant of proportionality. Since $s=0$ when $\phi=0$, it follows that

$$s = 2K\phi^{1/2},$$

where $K=(2k)^{-1/2}$. Hence

$$dx = ds \cos \phi = K\phi^{-1/2} \cos \phi d\phi, \qquad dy = ds \sin \phi = K\phi^{-1/2} \sin \phi d\phi,$$

and

$$x = K\int_0^\phi \phi^{-1/2} \cos \phi d\phi, \qquad y = K\int_0^\phi \phi^{-1/2} \sin \phi d\phi.$$

But the pole of the spiral corresponds to $\phi=\infty$, and

$$\int_0^\infty \phi^{-1/2} \cos \phi d\phi = \int_0^\infty \phi^{-1/2} \sin \phi d\phi = \sqrt{\pi/2}.$$

Therefore the pole is located at the point $(K\sqrt{\pi/2}, K\sqrt{\pi/2})$, or, in terms of the original constant, $(\frac{1}{2}\sqrt{\pi/k}, \frac{1}{2}\sqrt{\pi/k})$.

An Exponential Equation

E 640 [1944, 472]. *Proposed by E. D. Schell, Arlington, Virginia*

Solve in integers the equation

$$x^y = y^x \qquad (x > y).$$

Solution by M. D. D. Burrow, McGill University. Writing the equation in the form $x^{1/x}=y^{1/y}$, we see that the problem reduces to finding integral values of x which give coincident values of

$$f(x) = x^{1/x}.$$

Now, $f(x)$ increases for $0<x<e$, and then decreases but remains greater than 1. Thus the smaller value of x, if positive, must lie between 1 and e, where the only integer is 2. Hence the only positive integral solution is $y=2$, $x=4$.

Also solved by Shepard Bartnoff, Colin Blyth, D. H. Browne, W. E. Buker, L. H. Bunyan, H. N. Carleton, L. R. Chase, Monte Dernham, Howard Eves, Daniel Finkel, L. R. Ford, Irving Kaplansky, J. B. Kelly, Samuel Kramer, C. D. Olds, E. S. Pondiczery, Alexander Russ, E. P. Starke, Martha Sved, Jeanette Van Os, and the proposer.

Many (like Burrow and the proposer) overlooked the second solution

$$x = -2, \qquad y = -4.$$

Buker and Olds pointed out that Euler solved the given equation in the form

$$x = \left(1 + \frac{1}{n}\right)^{n+1}, \qquad y = \left(1 + \frac{1}{n}\right)^{n},$$

and by D. Bernoulli in the equivalent form $x = c^{c/(c-1)}$, $y = c^{1/(c-1)}$. Here $n = 1/(c-1)$ and $c = \log_y x$. For a simple proof that Euler's expressions, for integers n, are the only positive rational solutions, see R. L. Goodstein, *Mathematical Gazette*, vol. 28 (May 1944), p. 76. See also this MONTHLY, vol. 28 (1921), pp. 141–143, and vol. 38 (1931), pp. 444–447.

Swimming Around a Lighter

E 641 [1944, 530]. *Proposed by F. M. Garnett, Savannah, Ga.*

A lighter, twenty by thirty feet, travels upstream at a uniform speed of 2 m.p.h., against a current of $1\frac{1}{2}$ m.p.h. A swimmer, whose rate of swimming is 4 m.p.h. in still water, swims around the lighter. How long does he take to complete the circuit?

I. *Solution by Frank Hawthorne, Allegheny College.* It is assumed that the swimmer starts and finishes his circuit at the same position relative to the lighter, and that the given speed of 2 m.p.h. is relative to the land.

The man's net speed, relative to the lighter, while swimming up one side and down the other, will be $\frac{1}{2}$ and $7\frac{1}{2}$ m.p.h., or 11/15 and 11 f.p.s. His time for the two trips together is thus

$$30\left(\frac{15}{11} + \frac{1}{11}\right) = \frac{480}{11}$$

seconds. In crossing the bow and stern he must maintain a forward component of $3\frac{1}{2}$ m.p.h. relative to the water; so his transverse component is

$$\sqrt{4^2 - (3\tfrac{1}{2})^2} = \tfrac{1}{2}\sqrt{15}$$

m.p.h., or $11/\sqrt{15}$ f.p.s. Since he travels a total distance of 40 feet transversely his time for these parts of the circuit is $40 \cdot \sqrt{15}/11$ seconds. His total time is thus

$$\frac{480}{11} + \frac{40\sqrt{15}}{11} = \frac{40(12 + \sqrt{15})}{11} = 57.72$$

seconds.

Also solved thus by Murray Barbour, W. E. Buker, Leon Bunyan, Monte Dernham, Howard Eves, E. P. Starke and W. Unterberg. *Cf.* problem 3218 [1927, 388].

II. *Solution by D. H. Browne, Buffalo, N. Y.* Because of the pressure wave at the bow and the drag at the stern, the swimmer is virtually in still water at either end. (I incline to this simpler view, having experienced it!) He passes the length of the boat at $\frac{1}{2}$ m.p.h. upstream and $7\frac{1}{2}$ m.p.h. downstream, covering the transverses at 4 m.p.h. and making the total encirclement in

$$\frac{2}{44}+\frac{30}{44}+\frac{5}{44}=\frac{37}{44}$$

minutes, or $50\frac{5}{11}$ seconds.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4158. *Proposed by P. D. Thomas, Lumberton, Miss.*

Integrate the partial differential equation

$$(1+q^2)r-2pqs+(1+p^2)t+(rt-s^2)/(1+p^2+q^2)^{1/2}+(1+p^2+q^2)^{3/2}=0;$$

and give a geometrical interpretation of the general integral.

4159. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given the 18 numbers $a_i, b_i, c_i, i=1, 2, 3, 4, 5, 6$ such that

$$\begin{vmatrix} (bc)_{13} & (ca)_{13} & (ab)_{13} \\ (bc)_{25} & (ca)_{25} & (ab)_{25} \\ (bc)_{46} & (ca)_{46} & (ab)_{46} \end{vmatrix} = \begin{vmatrix} (bc)_{24} & (ca)_{24} & (ab)_{24} \\ (bc)_{16} & (ca)_{16} & (ab)_{16} \\ (bc)_{35} & (ca)_{35} & (ab)_{35} \end{vmatrix} = 0$$

where

$$(mn)_{ij}=\begin{vmatrix} m_i & m_j \\ n_i & n_j \end{vmatrix},$$

show that

$$\begin{vmatrix} (bc)_{12} & (ca)_{12} & (ab)_{12} \\ (bc)_{34} & (ca)_{34} & (ab)_{34} \\ (bc)_{56} & (ca)_{56} & (ab)_{56} \end{vmatrix} = 0.$$

4160. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show how to construct four spheres passing through a given point and tangent respectively to the planes of three faces of a given tetrahedron so that the points of contact are twelve points of the same sphere.

SOLUTIONS

Factorial Coefficients

4108 [1944, 96]. *Proposed by G. Pólya, Stanford University*

Let ${}_nP_r$ be the number of those permutations of n elements which are the products of exactly r cycles without common elements. For instance, ${}_4P_2=11$. Let ${}_nQ_r$ be the number of different classifications of n distinct elements into exactly r classes. For instance, ${}_4Q_2=7$. Prove that

$$(1) \qquad {}_nP_1x + {}_nP_2x^2 + \cdots + {}_nP_nx^n = x(1 + x)(2 + x) \cdots (n - 1 + x),$$

$$(2) \qquad {}_nQ_1x + {}_nQ_2x(x - 1) + \cdots + {}_nQ_nx(x - 1) \cdots (x - n + 1) = x^n.$$

Note. In the Calculus of Finite Differences, ${}_nP_r$ and ${}_nQ_r$ are considered in various notations and are often called factorial coefficients; see, for instance, Steffensen, *Interpolation*, Baltimore, 1927, pp. 53–58.

Solution by P. R. Halmos, Syracuse University. A permutation of $n+1$ elements which is the product of exactly r cycles may be obtained in one of two ways from permutations of n elements. The first way is to adjoin a cycle of length one involving a new element to a permutation which is the product of $r-1$ cycles; the second way is to adjoin a new element to any cycle of a permutation which is the product of r cycles. Since in the second operation the new element may be inserted after any one of the r old elements, we see that

$${}_{n+1}P_r = {}_nP_{r-1} + n{}_nP_r.$$

(If ${}_nP_0$ is interpreted to be 0, this equation is valid for $n=1, 2, \cdots, r=1, \cdots, n$.) If we write $p_n(x)=\sum_{r=1}^{n} {}_nP_rx^r$ then the relation between the P's implies that $(x+n)p_n(x)=p_{n+1}(x)$, from which (1) follows immediately by mathematical induction.

Similarly, a classification of $n+1$ elements into r classes may be obtained in one of two ways from classifications of n elements. The first way is to adjoin a class consisting of exactly one new element to a classification into $r-1$ classes; the second way is to adjoin a new element to any class of a classification into r classes. Since in the second operation the new element may be inserted into any one of the r existing classes, we see that

$$_{n+1}Q_r = {}_nQ_{r-1} + r\,{}_nQ_r.$$

(If $_nQ_r$ is interpreted to be 0 unless $1 \leqq r \leqq n$, this equation is valid for n, $r=1, 2, \cdots$). Write now $f_r(x)=x(x-1)\cdots(x-r+1)$ and $q_n(x)=\sum_{r=1}^{n} {}_nQ_r f_r(x)$. Since, as is easily verified, $f_{r+1}(x)+rf_r(x)=xf_r(x)$, (2) follows inductively from the equations

$$\begin{aligned} q_{n+1}(x) &= \sum_{r=1}^{n+1} {}_{n+1}Q_r f_r(x) = \sum_{r=1}^{n+1} ({}_nQ_{r-1} + r\,{}_nQ_r) f_r(x) \\ &= \sum_{r=1}^{n} {}_nQ_r (f_{r+1}(x) + rf_r(x)) = x\sum_{r=1}^{n} {}_nQ_r f_r(x) = xq_n(x). \end{aligned}$$

Solved also by H. W. Becker, G. F. Morecroft, C. D. Olds, and Louis Weisner.

Editorial Note. The solution by Morecroft is similar to the above. Olds stated that $P=m!{}_nQ_m/m^n$ is the probability that, in n drawings of a single number from an urn containing the numbers $1, 2, \cdots, m$, replacing the number in the urn before the next drawing, each number will be drawn at least once. He referred to Ch. Jordan's *Calculus of Finite Differences*, Budapest, 1939, p. 604. Weisner used in his proofs the formulas

$$_nP_r = \sum_\alpha \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_n!2^{\alpha_2}3^{\alpha_3}\cdots n^{\alpha_n}},$$

$$_nQ_r = \sum_\alpha \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_n!(2!)^{\alpha_2}(3!)^{\alpha_3}\cdots(n!)^{\alpha_n}},$$

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n, \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = r$$

where the α_i's are non-negative integers. Becker gave interesting remarks with references to E 461 [1941, 701], E 565 [1944, 47], Riordan, *Moment Recurrence Relations*, Ann. Math. Stat. VIII, 103; Steffensen, *Statistics and Actuarial Science*, Cambridge, 1930, 24; E. T. Bell, *Generalized Stirling Transforms*, Am. Jour. Math. 61, 1939, 89, and *Iterated Exponential Integers*, Annals of Math. 39, 1938, 539.

The equation (2) of the problem may be written

$$x^n = \sum_{r=1}^{n} {}_nQ_r x^{(r)}, \qquad x^{(r)} = x(x-1)\cdots(x-r+1), \qquad x^{(0)} = 1,$$

and the coefficients $_nQ_r$ can be computed by a process similar to the synthetic division method for reducing the roots of a polynomial by a constant a. Here in the first division we use $a=1$, in the second $a=2$, and so on. For $n=5$ the divisions are

	1		0		0		0		0
			1		1		1		1
${}_1Q_1$	1	${}_2Q_1$	1	${}_3Q_1$	1	${}_4Q_1$	1	${}_5Q_1$	1
			2		6		14		
${}_2Q_2$	1	${}_3Q_2$	3	${}_4Q_2$	7	${}_5Q_2$	15		
			3		18				
${}_3Q_3$	1	${}_4Q_3$	6	${}_5Q_3$	25				
			4						
${}_4Q_4$	1	${}_5Q_4$	10						
${}_5Q_5$	1								

$$x^5 = x^{(5)} + 10x^{(4)} + 25x^{(3)} + 15x^{(2)} + x^{(1)}.$$

For a larger table we extend the rows and columns and for each value of n we read the results along the secondary diagonal from ${}_nQ_n n^{(n)}$ to ${}_nQ_1 x^{(1)}$. From two consecutive rows we have

$$\begin{array}{ll} \cdots {}_{n-1}Q_{r-1} & {}_nQ_{r-1} \\ \cdots r_{n-1}Q_r & r_nQ_r \\ \hline \cdots {}_nQ_r & {}_{n+1}Q_r \end{array} \qquad \begin{array}{r} (r-1) \\ \\ (r) \end{array}$$

and thus ${}_{n+1}Q_r = {}_nQ_{r-1} + r_nQ_r$. Since $\Delta x^{(r)} = rx^{(r-1)}$, we have

$$\Delta^r x^n = \sum_{j=r}^{n} {}_nQ_j j^{(r)} x^{(j-r)},$$

and it follows that $\Delta^r 0^n = r!_nQ_r$, ${}_nQ_r = \Delta^r 0^n / r!$ The above expression for x^5 may be written

$$x^5 = \frac{\Delta x^{(6)}}{6} + 10\frac{\Delta x^{(5)}}{5} + 25\frac{\Delta x^{(4)}}{4} + 15\frac{\Delta x^{(3)}}{3} + \frac{\Delta x^{(2)}}{2}\cdot$$

Hence

$$\sum_{x=1}^{n-1} x^5 = \frac{n^{(6)}}{6} + 10\frac{n^{(5)}}{5} + 25\frac{n^{(4)}}{4} + 15\frac{n^{(3)}}{3} + \frac{n^{(2)}}{2}\cdot$$

From (1) in the problem it is seen that ${}_nP_{n-r} = \sigma_r(n)$, the rth elementary symmetric function of $1, 2, \cdots, n-1$ which is considered in 3940 [1941, 641]. It was shown that

$$\sigma_r(n) = \sum_{t=1}^{r} \overset{r}{b}_t n^{(r+t)},$$

$$\sigma_r(n+1) = \sigma_r(n) + n\sigma_{r-1}(n), \qquad \sigma_0(n) = 1,$$

$$\overset{r}{b}_1 = \frac{1}{r+1}, \qquad \overset{r}{b}_r = \frac{1}{r!2^r}, \qquad \overset{r}{b}_t = \frac{\overset{r-1}{b}_{t-1} + (r+t-1)\overset{r-1}{b}_t}{r+t}.$$

The values of $\overset{r}{b}_t$ are there given for $0 \leqq r \leqq 5$. A variation of the method used gives also

$$\sigma_r(n) = \sum_{t=1}^{r} (-1)^{r+t} \overset{r}{a}_t (n+t-1)^{(r+t)};$$

$$\overset{r}{a}_1 = \frac{1}{(r+1)!}, \qquad \overset{r}{a}_r = \frac{1}{r!2^r}, \qquad \overset{r}{a}_t = \frac{\overset{r-1}{a}_{t-1} + t\overset{r-1}{a}_t}{r+t};$$

$$\overset{3}{a}_2 = \frac{1}{12}, \qquad \overset{4}{a}_2 = \frac{5}{144}, \qquad \overset{4}{a}_3 = \frac{1}{48}, \qquad \overset{5}{a}_2 = \frac{1}{90},$$

$$\overset{5}{a}_3 = \frac{7}{(4!)^2}, \qquad \overset{5}{a}_4 = \frac{1}{2!3!4!}.$$

We shall now show that

$$\sigma_r(-n) = {}_{n+r}Q_n = \frac{\Delta^n 0^{n+r}}{n!}.$$

We may write the difference equations in the forms

$$\sigma_r(-n) = \sigma_r[-(n-1)] + n\sigma_{r-1}(-n)$$

$${}_{n+r}Q_n = {}_{(n-1)+r}Q_{n-1} + n_{n+(r-1)}Q_n$$

which are of the form $f_r(n) = f_r(n-1) + nf_{r-1}(n)$, for $r=0$ in the two cases we have $\sigma_0(-n) = {}_nQ_n = 1$, and this suffices to prove the desired result. Thus

$${}_nQ_{n-r} = \sum_{t=1}^{r} \overset{r}{a}_t n^{(r+t)} = \sum_{t=1}^{r} (-1)^{r+t} \overset{r}{b}_t (n+t-1)^{(r+t)}.$$

Sum of Two Squares Equal to a Square

4110 [1944, 96]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the smallest integer whose square is the sum of two squares in seven different ways.

I. *Solution by Colin Blyth, Queen's University Student.* The following theorems are used:

1. If $p=4n+1$, a prime, then p^2 is the sum of two squares in exactly one way. If $p=4n+3$, a prime, then it is impossible to express p^2 as the sum of two squares.

2. If $N = m\prod_{i=1}^{n} p_i^{\alpha_i}$ where every prime factor of m is of the form $4k+3$ or 2,

and the p_i are distinct primes of the form $4k+1$, then N^2 is the sum of two squares in the same number of ways as is $\prod_{i=1}^{r} p_i^{\alpha_i}$, namely

$$2^{n-1}+\sum \text{ Number of ways for every possible factor.}$$

Since we are looking for the smallest integer we can find, we must obviously take $m=1$.

The first few primes of type $4k+1$ are:

$$5,\ 13,\ 17,\ 29,\ 37,\ 41,\ \cdots .$$

By making a few trials we quickly arrive at the required integer:

5^2 is expressible in 1 way

$(5.5)^2$ is expressible in $1 + 1 = 2$ ways

$(5.13)^2$ is expressible in $2 + 1 + 1 = 4$ ways

. . . .

$(5.5.13)^2$ is expressible in $2 + 1 + 2 + 1 + 1 = 7$ ways.

This solution does not provide a means of finding these actual sums. The seven ways are readily found to be:

$$\begin{aligned}
(5.5.13)^2 &= 204^2 + 253^2\\
&= 36^2 + 323^2\\
&= 13^2.7^2 + 13^2.24^2 = 91^2 + 312^2\\
&= 5^2.33^2 + 5^2.56^2 = 165^2 + 280^2\\
&= 5^2.63^2 + 5^2.16^2 = 315^2 + 80^2\\
&= (5.5)^2.5^2 + (5.5)^2.12^2 = 125^2 + 300^2\\
&= (5.13)^2.3^2 + (5.13)^2.4 = 195^2 + 260^2.
\end{aligned}$$

II. *Note by E. P. Starke, Rutgers University.* The following is an obvious adaptation of a solution of problem no. 350 in the *National Mathematics Magazine*, Dec. 1940, p. 146, by G. W. Wishard, whose problem was to show that $5525=5^2\cdot 13\cdot 17$ has twenty-two representations as a sum of two squares.

If $c^2=a^2+b^2$, $ab\neq 0$, it is well known that we must have

$$a = 2kxy, \qquad b = k(x^2 - y^2), \qquad c = k(x^2 + y^2),$$

where x and y have no common factor, one of them is even, and $x>y$. Further, a product $P=L\cdot M$ can be represented as a sum of two squares if each factor can, *viz.*

$$(1) \qquad (r^2 + s^2)(u^2 + v^2) = (ru + sv)^2 + (rv - su)^2.$$

Conversely, every representation of P as a sum of two squares can be obtained from representations for L and M by use of (1). (See, *e.g.*, Carmichael, *Diophantine Analysis*, pp. 10, 24, ff.)

Now $325=5^2\cdot 13$, and $13=2^2+3^2$, $5=1^2+2^2$, $25=3^2+4^2$, whence various factorizations of 325 and repeated applications of (1) give the required seven representations as follows:

k	x^2+y^2	x	y	a	b
65	5	2	1	260	195
25	13	3	2	300	125
13	25	4	3	312	91
5	65	8	1	80	315
		7	4	280	165
1	325	17	6	204	253
		18	1	36	323

It is obvious from the theorems and methods employed that no smaller value of c than 325 can have the required property.

Solved also by E. D. Schell and the proposer: the latter did not give his method.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Arvid Jacobson has been appointed to an instructorship at Wayne University.

Professor R. D. James of the University of British Columbia has been granted a leave of absence to serve as operations analyst with the Third Air Force at Tampa, Florida.

Professor Mabel M. Heren of Knox College has retired as chairman of the department but will continue her teaching there. Professor R. C. Stephens succeeds her in the chairmanship.

Associate Professor W. H. Meyers of San Jose State College, California, has been appointed head of the department.

Professor Emeritus G. E. Robinson of the University of British Columbia died January 24, 1945.

SUMMER COURSES

The University of North Carolina: From July 2 to August 29 the following advanced courses in mathematics will be offered: By Professor Brauer: complex variable. By Professor Browne: history of mathematics. By Professor Henderson: differential equations. By Professor Lasley: analytic projective geometry. By Professor Winsor: college geometry. By Dr. Wong: mathematical theory of statistics.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

USAFI COURSES IN COLLEGE MATHEMATICS

Returning veterans who have studied courses under the auspices of the U. S. Armed Forces Institute are now entering colleges and universities in larger numbers. To assist departments of mathematics in planning a program for these men, the following information is given in regard to U.S.A.F.I. courses in mathematics. The descriptions here given are restricted essentially to courses which are regarded by the Institute as upon the college level. The statements of subject-matter are those provided by the U.S.A.F.I.

There are eleven correspondence courses in mathematics on the college level now offered directly to service personnel by the U.S.A.F.I. These courses are listed below. Credit equivalents are not given, of course, since the Institute does not grant or recommend credit.

C 712. *Plane Trigonometry.* Trigonometric functions; graphs; analysis; right and oblique triangle solution by natural functions and by logarithms; applications to surveying, physics, astronomy; inverse, exponential, and hyperbolic functions; trigonometric equations; De Moivre's Theorem. H 138 and H 139 (Beginning and Intermediate High School Algebra) are necessary preparation for this course. H 140 (Advanced High School Algebra) is desirable. Sixteen assignments. Text: *Plane and Spherical Trigonometry* (With Tables) by Nelson and Folley, 1943.

C 713. *College Algebra and Trigonometry.* Trigonometric functions; identities and equations; graphs; solution of right and oblique triangles; logarithms; inequalities; progressions; mathematical induction; theory of equations; probability; permutations and combinations. H 138, H 139, and H 140 (or equivalent courses) are required preparation for the course. H 143 and H 144 (Plane Geometry, two terms) are desirable. Thirty-two assignments. Texts: *College Algebra* by Rietz and Crathorne, 1939; *Brief Trigonometry* by E. A. Cameron, 1941.

C 714. *Plane Analytic Geometry.* Rectangular, oblique, and polar coordinates in the plane; the relation between a curve and its equation; properties of straight lines, circles, conic sections, and certain other plane curves. C 713 or H 138, H 139, H 140, and H 147 (High School Trigonometry) or C 712 (or equivalent courses) are necessary preparation. Sixteen assignments. Text: *Analytic Geometry* by C. E. Love, 1938.

C 715. *Descriptive Geometry.* The making and interpretation of the perspective drawings necessary in engineering, architecture and various fields of design. It considers the projections of points, lines, planes, and curved surfaces on

appropriate reference planes. Plane geometry and one semester of college mechanical drawing are required preparation for the course. Forty-five assignments. Text: *Descriptive Geometry* by F. H. Cherry, 1933.

C 716. *Spherical Trigonometry.* Solutions of right and oblique spherical triangles, and their applications. C 712 or any course in plane trigonometry is necessary preparation. Eight assignments. Text: *Plane and Spherical Trigonometry* (With Tables) by Rietz, Woods, and Reilly, 1942.

C 717. *Differential Calculus.* A study of the elements of calculus and their applications in finding areas, volumes, velocity and acceleration, motions of celestial bodies and atoms, flow of heat and electricity, probabilities, stresses of structural steel, compound interest, gas pressures, laws of growth and decay, and countless others. C 712 and C 714, or their full equivalent, is necessary preparation for the course. Forty assignments. Text: *Elements of the Differential and Integral Calculus* by Granville, Smith, and Longley, 1941.

C 718. *Integral Calculus.* Methods of integration with applications to areas of plane curves, volumes of solids of revolution, length of curve, areas of surfaces of revolution, moments of area; centroids of solids of revolution; and other problems in physics and mechanics. A knowledge of C 717, or similar background, is necessary preparation for this course. Forty assignments. Texts: *Elements of the Differential and Integral Calculus*, 1941 Edition, by Granville, Smith, and Longley; *Short Table of Integrals* by B. O. Peirce, 1929.

C 719. *Solid Analytic Geometry.* Continues the development and application of the methods of analytical geometry in the study of curves and surfaces, and the application of calculus to partial derivatives, multiple integrals, and infinite series. C 714 or equivalent is necessary preparation for this course. Sixteen assignments. Text: *Analytic Geometry* by C. E. Love, 1938.

C 724. *Differential Equations.* Fundamental types of ordinary differential equations, with applications to problems that are geometrical in nature and particularly to problems arising in physics and mechanics. A knowledge of C 717 and C 718, or equivalent, is necessary preparation for the course. Sixteen assignments. Text: *Differential Equations* by D. A. Murray, 1924.

C 725. *Engineering Mathematics—Part I.* A course in elementary analysis, covering the essentials of algebra, trigonometry, analytic geometry, and elementary calculus. One year each of algebra and geometry on the high school level are necessary, and an additional half-unit of algebra is desirable preparation for the course. Forty-six assignments. Text: *Introductory College Mathematics* by Milne and Davis, 1941.

C 726. *Engineering Mathematics—Part II.* A continuation of C 725. C 725 is necessary preparation. Twenty-two assignments. Text: *Introductory College Mathematics* by Milne and Davis, 1941.

The U.S.A.F.I. also makes available to service personnel a series of special reprints of standard textbooks to be used for group study or in off-duty classes. By requisition, an organization commander may obtain a quantity of these

books without charge. The following books in mathematics are now available.

EM 315. *College Algebra* by W. L. Hart, 1938. Covers the fundamentals of algebra on the college level.

EM 318. *Plane and Spherical Trigonometry with Tables* by Kells, Kern, and Bland, 1940. Covers the principles of both plane and spherical trigonometry with a unit on the slide rule, its construction and use.

EM 321. *Elements of Analytic Geometry* by C. E. Love, 1940. Covers the field of plane analytic geometry.

EM 324. *Elements of Differential and Integral Calculus* by Granville, Smith, and Longley, 1941. Covers fundamentals of differential and integral calculus.

EM 333. *Mathematics of Investment* by W. L. Hart. Provides an elementary study of the theory and application of annuities certain and of the mathematical aspects of life insurance.

EM 906. *A Course in the Slide Rule and Logarithms* by E. J. Hills, 1943. Explains the principles of the slide rule and logarithms, and their many practical uses. A polyphase slide rule is provided with each book. This is regarded as a high school course.

All other studies in mathematics now available to service men are definitely upon the elementary or secondary level, with the exception of two courses in trigonometry. These are listed below.

H 147. *Trigonometry*. The relations of the sides and angles of plane and spherical triangles; trigonometric functions; definitions; measurement and functions of angles; areas of plane figures; trigonometry and its usefulness in solving problems in navigation, mapping, artillery fire, surveying, engineering, and related fields. A knowledge of H 138 (Beginning Algebra) and H 139 (Intermediate Algebra), or similar background, is required. A knowledge of high school geometry is essential. Twenty assignments. Text: *Plane Trigonometry With Tables*, by Rosenbach, Whitman, and Moskovitz, 1941. This subject appears in the list of correspondence courses on the high school level.

Plane Trigonometry, Self-Teaching Manual EM 311. This manual is based on *Essentials of Trigonometry With Applications*, by Curtiss and Moulton, 1943. Covers trigonometric functions, reduction formulas, graphs, identities and equations, solution of triangles, and logarithms. A quantity of these self-teaching manuals may be obtained by organization commanders for use in regularly organized off-duty classes. An individual desiring to study the manual must enroll with the Institute, and make application to study the course.

RESEARCH BOARD FOR NATIONAL SECURITY

The recently announced Research Board for National Security is composed of ten Army officers of general rank and ten Naval officers of flag rank, and twenty civilians chosen from the fields of science, engineering, and industry. The executive committee of the Board is composed of Dr. Roger Adams, Head of the Department of Chemistry, University of Illinois; Dr. A. R. Dochez,

Professor of Experimental Medicine and Surgery, College of Physicians and Surgeons, Columbia University; Brigadier General W. A. Borden, Director of the New Developments Division, War Department Special Staff; Rear Admiral J. A. Furer, Coordinator of Research and Development, Navy Department; and Dr. K. T. Compton, President of the Massachusetts Institute of Technology, Chairman. The only mathematician on the Board is Dr. Oswald Veblen, Institute for Advanced Study, Princeton, New Jersey.

The new Board will be concerned with the advancement of science and technology in those directions which may have application in the conduct of future warfare. The ultimate establishment of the Research Board for National Security as an independent agency of government is now being considered by the Select Committee of the House on Post-War Military Policy. Temporarily the Board will continue as an agency of the National Academy of Sciences, thereby making available to it the resources and contacts of the National Research Council. For the duration of the war, it has been agreed that the new Board shall not engage in activities which are properly a function of the Office of Scientific Research and Development; this latter organization, of course, is an emergency wartime agency which will be disbanded shortly after the war.

EDUCATIONAL SURVEY OF THE ARMED FORCES

Recently the Information and Education Division of the Army Service Forces released Report No. B 121, pertaining to the educational demands of the members of the Armed Forces upon demobilization. The information which the report contains is based upon a survey of 20,000 white enlisted men. The following statements are quoted from the bulletin.

"Approximately 7 per cent of the white enlisted men in the Army are now definitely planning to return to full-time school after they leave the Army. On the basis of the current white enlisted strength of about $6\frac{3}{4}$ million, this represents close to a half million men, definitely planning to return to full-time school at the present time. Another 4 per cent of the men may possibly return to full-time school. Eighteen per cent of the men are planning to attend part-time school although one-third of these men would prefer to return to full-time school.

"Young men are more likely to be planning further education. Over ninety per cent of the men definitely planning to return to full-time school are less than twenty-five years old. Among men who want part-time education, less than two-thirds are under twenty-five.

"Over ninety per cent of the men who plan full-time school attendance have the formal educational requirements for college entrance. About two-thirds of them actually want college education. The remainder are primarily interested in trade and business schools. Over half the men who want part-time education are formally qualified for college courses, but three-fourths of them are planning trade or business courses.

"Probably not all the men now planning to go to full-time school will return, but others without definite plans at the present may decide to return. Seven

per cent is the best estimate at present of the number of men actually going back to full-time school. Factors such as the length of the war and favorable economic conditions may tend to decrease this proportion when demobilization actually occurs. Increasing knowledge of the educational provisions of the 'G.I. Bill' and less favorable conditions would tend to raise it."

A representative of the American Council on Education recently questioned fourteen education officers in the several overseas branches of the Armed Forces. These men indicated their belief that the educational interests of veterans would be primarily in technical and professional fields rather than in general education. Some officers made the qualification that younger veterans who had not completed their college education would be more interested in general education, while older veterans who had not been in college would be more interested in studies of a technical or professional nature. Two of the officers stated that service men had expressed special interest in courses in business administration and in engineering.

COMMITTEE ON THE TEACHING OF THE BASIC SCIENCES

On May 1, 1943, U. S. Commissioner of Education J. W. Studebaker asked for the formation of a committee to "canvass carefully the question of whether the basic sciences should not be included along with vocational education in a federally subsidized program. Such a subsidization would include, of course, the appropriate training of teachers for such courses." This resulted in the creation of a committee of four members representing the Commission on Teacher Education (Karl W. Bigelow), research in science teaching (R. J. Havighurst), the division of higher education of the United States Office of Education (F. J. Kelly), and the sciences (K. Lark-Horovitz, chairman of the committee).

After thorough consideration of the problems involved, it seemed apparent to the members of the committee that federal aid of some kind will be necessary to strengthen the teaching of the basic sciences. Accordingly, it has been proposed by the committee that existing legislation relating to vocational and technical education be amended, and future legislation be formulated to include provision, in addition to that now provided for such fields as vocational agriculture, home economics, trades and industries, and distributive occupations, for "the sciences basic thereto and (or including) mathematics." Essentially it is recommended that the Smith-Hughes and the George-Deen Acts be extended; these two acts provide for the cooperation between the federal government and the states in the promotion of certain types of vocational education.

According to a recent report of the Committee, the desired amendment, specifically as relating to the sections of the law having to do with the preparation and employment of teachers, would have the following effects:

"(1) As in the case of vocational agricultural and other vocational teachers, the standards which are set up by the state department of education in cooperation with the U. S. Office of Education will have to be met. This guarantees that certain minimum requirements would have to be fulfilled if federal aid

is to be obtained for the teaching of the sciences and mathematics as basic to agriculture, trades and industries, distributive occupations, and the proposed vocational-technical training.

"(2) Since the effectiveness of engineering, technical and vocational education and training is largely dependent on the soundness of preparatory education in mathematics, physics and chemistry, programs of teacher training under existing and future acts supplying federal support for teacher training should then include provision for teacher training in these fields."

The proposed amendment might well be part of a further modification of existing legislation that would also provide for:

"(1) Expansion of the present vocational and technical education program (now limited to work 'of less than college grade') to encompass activities 'of scholastic standard commonly associated with work done in high school and in the first two years beyond high school, but excluding that done as part of a regular four-year curriculum';

"(2) Authorization to the states to make use of and allocate support to the facilities of tax-exempt, though not publicly supported, universities, junior colleges, technical institutes, and the like, as well as to those publicly controlled;

"(3) Requirement that states—if they desire to participate—designate or create state boards to exercise control over the program within their boundaries, such boards to consist of not less than seven members so selected as to be representative of the various fields of interest for which the vocational and technical programs prepare."

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following fifty-one persons have been elected to membership on applications duly certified:

SILVIO AURORA. Student, Columbia Univ., New York, N. Y.

D. R. BEY, A.M.(Illinois) Asst. Prof., Illinois State Normal Univ., Normal, Ill.

W. H. BROTHERS, Jr., Ph.D.(Michigan) Prof., Talladega Coll., Talladega, Ala.

REV. W. F. BURNS, M.S.(Boston Coll.) Asso. Prof., Coll. of the Holy Cross, Worcester, Mass.

JOSEPHINE H. CHANLER, Ph.D.(Illinois) Asst. Prof., Univ. of Illinois, Urbana, Ill.

WAY MING CHEN, B.S.(Nat'l. Southwest Asso. Universities, Kunming, China) Teaching Asst., Univ. of California, Berkeley, Calif.

SISTER MARIA LOYOLA CONLAN, A.M.(Fordham) Asso. Prof., Coll. of Mount St. Vincent, New York, N. Y.

W. R. DOLL, A.B.(Columbia Coll.) Engr., Radio Station WKNE, Keene, N. H.

MARY A. ENGLISH. Student, Spelman Coll., Atlanta, Ga.

E. J. EVENSEN. Actuarial Section, Metropolitan Life Ins. Co., Pacific Coast Head Office. On leave. T/4, U. S. Army,

HANS FRIED, Ph.D.(Vienna) Lecturer, Swarthmore Coll., Swarthmore, Pa.

A. H. S. GILLSON, M.A.(Cambridge) Prof., McGill Univ., Montreal, P.Q., Canada

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L. B. GUELPA, JR., A.M.(New York Univ.) Lt., U.S.N.R. Asso. Educational Officer in Math., U. S. Merchant Marine Acad., Kings Point, N. Y.

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F. F. HELTON, A.M.(Missouri) Asso. Prof., Dir. of Morrison Observatory, Central Coll., Fayette, Mo.

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JEAN B. WALTON, A.M.(Brown) Instr., Swarthmore Coll., Swarthmore, Pa.

MRS. JEAN A. WILBURN, A.B.(California) Teaching Asst., Univ. of California, Berkeley, Calif.

THE COORDINATING COMMITTEE

Some Sections of the Association have active committees devoted to the furthering of the interests of sound education in general and of mathematics in particular. Some of these committees have real accomplishment to their credit as a result of advising local educational groups, state boards of education, and legislative committees. They generally find that their advice is treated with respect in such matters as curriculum changes and teacher training.

As post-war educational problems press upon us, it is probable that most if not all of the Sections will find it desirable to establish such committees, possibly in conjunction with other organizations whose aims and ideals are similar to our own. Since education is still a local matter in this country, it is only through local committees, preferably at least one committee for each State, that progress can be made. Not only do the problems vary from place to place, but persons in authority are amenable only to committees of local origin.

With these thoughts in mind, the Board of Governors of the Association has authorized the appointment of a Coordinating Committee whose function it shall be to keep in close touch with all educational movements in the United States and Canada, and to make the information thus acquired available to all committees of the Association and its Sections who can profit by this information. It is hoped that in return all Sectional committees will make contact with the Coordinating Committee and report the problems which are facing them, their plans to meet these problems, and their successes or failures. The Coordinating Committee may be able to supply the local committees with reprints of pertinent articles, or at least with references to them. In any case the local committees should be able to work with more enthusiasm and confidence if they are fortified with exact information about what is going on in other places.

The Association is fortunate in having secured the services of an able committee. The chairman is Professor C. V. Newsom of Oberlin College, Oberlin, Ohio, and the other two members are Professor M. S. Knebelman of the State College of Washington at Pullman, and Professor W. V. Parker of the Louisiana State University at Baton Rouge.

C. C. MacDuffee, *President*

THE FALL MEETING OF THE INDIANA SECTION

The twenty-second annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Friday, November 10, 1944, in conjunction with the meeting of the Indiana Academy of Science. Professor Emil Artin, Chairman of the Section, presided at the morning session, and Professor P. M. Pepper, Chairman of the Mathematics Section of the Academy, presided at the afternoon session.

Thirty-two persons registered at the meeting, including the following eighteen members of the Association: Emil Artin, Juna Lutz Beal, G. E. Carscallen, J. E. Dotterer, W. E. Edington, P. D. Edwards, G. H. Graves, Cora B. Hennel,

M. W. Keller, Mark Lotkin, H. A. Meyer, P. M. Pepper, J. C. Polley, D. H. Porter, J. W. Wiley, K. P. Williams, H. E. Wolfe, Sister Gertrude Marie Zieroff.

At the business meeting the following officers were elected for the next year: Chairman, Juna Lutz Beal, Butler University; Secretary, M. W. Keller, Purdue University.

The following papers were presented:

1. *The great mathematics books in the college curriculum*, by Sister Gertrude Marie Zieroff, O. S. F., Marian College.

In this paper the speaker evaluated the method of learning college mathematics directly from the great mathematics classics. The technique employed for this type of instruction at St. John's College was described. Representative classics were compared with college text-books. The feasibility of using mathematics classics for collateral reading as part of the regular course, for honors courses, and for seminars, was discussed.

2. *On certain recursion inequalities with applications*, preliminary report, by Professor P. M. Pepper, University of Notre Dame.

Professor Pepper dealt with certain problems relating to a switchboard with n terminals, and with wires connecting the terminals in pairs. He considered the determination of the greatest number of cross-connections which can be made without there being somewhere three terminals each two of which are joined by a wire. Knowing the answer to this question, one may ask for a distribution of the maximum number of wires in such a way as to form no triangles (*i.e.*, no three terminals each two of which are connected). In the study of such questions he was led to the consideration of the following auxiliary problem: Let a, b, c, and u_0 be given integers with $a \geqq 0$; find a simple formula for u_n in terms of a, b, c, u_0 and n if u_n is the least integer satisfying the inequality

$$u_n \geqq [(n+a+c)u_{n-1} - (n+b)]/(n-a), \quad n = 1, 2, 3, \cdots.$$

The present paper contains a solution of the first two problems and the solution of a two-parameter family of the recursion inequalities with restricted u_0.

3. *What are we teaching mathematics for?* by Professor G. H. Graves, Purdue University.

In this paper the author pointed out that an essential feature of mathematics is the development of the implications of a set of assumptions. He stated that one of the prime purposes of mathematics is to convey an appreciation of this viewpoint, and that we should examine whether this objective is not in danger of being submerged in the many applications of mathematical processes. It was affirmed that there is a higher practicality to the grasp of a technique for drawing conclusions from a set of data, to the assembling and criticizing of data, and to the investigation of the assumptions on which the argument proceeds, than to any particular results of this process, however important these may be.

4. *Some illustrations of the Hamilton-Jacobi theory*, by Professor K. P. Williams, Indiana University.

Professor Williams explained the importance of the Hamilton-Jacobi differential equation theorems as they apply to planetary theories. An example, in which all integrations could be carried through, was given to show how the solution of one Hamilton system could be made to furnish the solution of a modified system.

5. *Some remarks on final grades in freshman mathematics*, by Professors M. W. Keller and H. S. F. Jonah, Purdue University.

Some data was presented which indicates from a preliminary study that certain tendencies exist in giving final grades when ordinary final examinations are given, when no examinations are given, and when uniform objective final examinations are given.

6. *Determinants*, by Professor Emil Artin, Indiana University.

Professor Artin presented a new set of axioms for determinants. The axioms were: (1) linearity and homogeneity as a function of the columns of a matrix; (2) the vanishing, if two adjacent cloumns are equal; (3) the value one in case of the unit matrix. From these axioms the speaker led very quickly to all important properties of determinants without introducing more than the elementary notions of permutations.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

The Office of Defense Transportation has refused permission for our previously announced meeting at Montreal, June 23–25, 1945, and this meeting has therefore been cancelled.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in November 1944 to Professor R. H. Cameron for his paper, "Some introductory exercises in the manipulation of Fourier transforms," published in the *National Mathematics Magazine*, vol. 15 (1941), pp. 331-356.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

THE CARUS MONOGRAPHS

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935; Fourth Impression, 1944.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930; Third Impression, 1943.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929; Third Impression, 1936; Fourth Impression, 1943.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSORS DAVID EUGENE SMITH and JEKUTHIEL GINSBURG. (First Impression, 1934.)

No. 6. *Fourier Series and Orthogonal Polynomials,* by PROFESSOR DUNHAM JACKSON. (First Impression, 1941.)

No. 7. *Vectors and Matrices,* by PROFESSOR C. C. MACDUFFEE. (First Impression 1943.)

Price $1.25 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary, McGRAW HALL, Cornell University, ITHACA, N.Y.

Additional copies for members, and copies for non-members, may be purchased from the Open Court Publishing Co., La Salle, Illinois, at the regular price of $2.00 per copy.

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ON GENERATING FUNCTIONS OF POLYNOMIAL SYSTEMS

W. C. BRENKE, University of Nebraska

1. Generating functions and generating sequences. If the function $g(x, t)$ admits a formal expansion in powers of t in which the coefficient of $t^n/n!$ is a polynomial in x, $y_n(x)$, of degree n for $n=0, 1, 2, \cdots$, it readily follows that $g(x, t)$ may be written in the form

$$(1) \qquad g(x, t) = \sum y_n(x)t^n/n! = \sum \phi_n(t)(xt)^n/n!,$$

where $\phi_n(t)$ admits a formal expansion in powers of t; to insure the existence of a complete sequence of polynomials $y_n(x)$ we must have $\phi_n(0) \neq 0$ for all n. All indicated summations have the range 0 to ∞.

We shall call $g(x, t)$ the *generating function* (g.f.) and $[\phi_n(t)]$ the *generating sequence* (g.s.) of the sequence of polynomials $[y_n(x)]$.

By successive differentiation of (1) with respect to x we obtain

$$(2) \qquad t^{-k} D_x^{(k)} g(x, t) = \sum \phi_{n+k}(t)(xt)^n/n! = \sum \frac{y_{n+k}^{(k)}(x)}{(n+1)\cdots(n+k)} \cdot \frac{t^n}{n!},$$

where the superscripts (k) indicate orders of derivatives. We have here the g.f. and the g.s. of the sequence of polynomials $[y_{n+k}^{(k)}/(n+1)\cdots(n+k)]$, $k=1, 2, \cdots$.

Differentiation of (1) with respect to t and reduction by use of (2) for $k=1$ gives

$$(3) \qquad \sum \phi_n'(t)(xt)^n/n! = \sum [y_{n+1} - xy_{n+1}'/(n+1)]t^n/n!,$$

and by iteration, with

$$y_{n,1} \equiv y_{n+1} - xy_{n+1}'/(n+1) \quad \text{and} \quad y_{n,r+1} \equiv y_{n+1,r} - xy_{n+1,r}'/(n+1),$$

$$(4) \qquad \sum \phi_n^{(r)}(t)(xt)^n/n! = \sum y_{n,r}t^n/n!.$$

That is, the sequence of rth derivatives of the terms of $[\phi_n]$ is the g.s. of $[y_{n,r}]$. To make the last sequence complete we assume $\phi_n^{(r)}(0) \neq 0$ for all n and r.

2. Appell polynomials. If, for all n,

$$\phi_n(t) = \phi(t) = \sum a_n t^n/n!, \qquad a_0 \neq 0,$$

we have

$$(5) \qquad g(x, t) = \phi(t)e^{xt} = \sum A_n(x)t^n/n!,$$

where $A_n(x)$ is the nth Appell polynomial [1]. These polynomials were introduced by Appell as solutions of the defining equation

$$(6) \qquad D_x A_n(x) = nA_{n-1}(x),$$

which is a generalization of $D_x x^n = nx^{n-1}$, and have received considerable atten-

tion in the literature. They may be written conveniently in the form of the symbolic binomial expansion

$$A_n(x) = (a + x)^{(n)}, \tag{7}$$

where, after expanding, we replace a^r by a_r from the expansion of $\phi(t)$.

Equation (6) follows readily from (5) by use of (2) with $k=1$.

It is obvious that, when $\phi_n(t)$ is independent of n, the $y_{n,r}$ in (4) are Appell polynomials for all values of n and of r.

3. The polynomials $X_n(x)$ and $Y_n(x)$. If, in (1), we take $\phi_n(t)=b_n\phi(t)$, $b_n\neq 0$ for all n, and designate the resulting polynomials by $X_n(x)$, we obtain

$$g(x, t) = \sum b_n\phi(t)(xt)^n/n! = \sum X_n(x)t^n/n!. \tag{8}$$

As before, let

$$\phi(t) = \sum a_n t^n/n! \tag{9}$$

and also let

$$f(xt) = \sum b_n(xt)^n/n!. \tag{10}$$

Then

$$g(x, t) = \phi(t)f(xt) = \sum X_n(x)t^n/n!, \tag{11}$$

and the $X_n(x)$ may be written in the symbolic form

$$X_n(x) = (a + bx)^{(n)}, \tag{12}$$

where, after expanding, we replace a^r, b^r by a_r, b_r respectively.

The Appell polynomials are a subclass of the $X_n(x)$'s for the case $b_n=1$, all n.

Another interesting subclass of (12) results when we take $a_n=1$, all n. We shall designate this subclass by $Y_n(x)$. Then

$$g(x, t) = e^t f(xt) = \sum Y_n(x)t^n/n!; \tag{13}$$

$$Y_n(x) = (1 + bx)^{(n)} \quad \text{and} \quad f(xt) = \sum b_n(xt)^n/n!. \tag{14}$$

When necessary to indicate the dependence of $Y_n(x)$ on the b's, we use $Y_n(x, b_r)$.

Equation (3), with $\phi_n(t)=\phi_n'(t)=b_ne^t$, gives the fundamental relation

$$xY_n' = n(Y_n - Y_{n-1}) = n\Delta Y_{n-1}. \tag{15}$$

By iteration there is indicated the relation

$$x^k D_x^{(k)} Y_n = \frac{n!}{(n-k)!}\Delta^k Y_{n-k}, \tag{16}$$

which is readily proved by induction.

Every $g(x, t)$ of the form (13) satisfies the partial differential equation

$$tg_t - xg_x = tg, \tag{17}$$

where subscripts indicate partial derivatives. From this equation and (13) we again obtain (15).

By use of well known properties of the binomial coefficients we find the relation

$$b_n x^n = (Y-1)^{(n)}; \qquad Y^{(r)} \equiv Y_r = Y_r(x, b_s). \tag{18}$$

By means of this relation a polynomial in x, of degree n, can be expressed as a linear combination of $Y_0, Y_1, \cdots, Y_n$.

$$\sum_{r=0}^{n} b_r x^r = \sum_{r=0}^{n} A_{n,r} Y_r(x, b_s); \qquad A_{n,r} = \sum_{i=0}^{n-r} (-1)^i \binom{r+i}{r}.$$

To obtain examples of $Y_n(x)$ we choose for $f(xt)$ some simple power series without missing terms and indicate the values of b_n from which the polynomials may be written down explicitly by use of (14).

(A) $f(xt) = \cos (xt)^{1/2}, \qquad b_n = (-1)^n n!/(2n)!.$

(B) $f(xt) = \sin (xt)^{1/2}/(xt)^{1/2}, \qquad b_n = (-1)^n n!/(2n+1)!.$

(C) $f(xt) = \cosh (xt)^{1/2}, \qquad b_n = n!/(2n)!.$

(D) $f(xt) = J_0(2(xt)^{1/2}), \qquad b_n = (-1)^n/n!.$

These functions satisfy a simple type of linear differential equation in x of second order, namely,

$$xf_{xx} + \alpha f_x + \beta t f = 0; \qquad \alpha, \beta \text{ constants.} \tag{19}$$

Obviously the associated $g(x, t)$ also satisfies (19). By use of (1), (15) and (19) we can obtain the following differential equation and difference equation for the corresponding $Y_n(x)$.

$$xY_n'' + (\alpha - \beta x)Y_n' + n\beta Y_n = 0. \tag{20}$$

$$(n - 1 + \alpha)Y_n = (2n - 2 + \alpha - \beta x)Y_{n-1} - (n-1)Y_{n-2}. \tag{21}$$

To get (20) and (21) we substitute from (1) in (19) and compare coefficients of t^n, obtaining

(a) $$xY_n'' + \alpha Y_n' + n\beta Y_{n-1} = 0.$$

Equation (15) is

(b) $$xY_n' = nY_n - nY_{n-1}.$$

Replacing n by $(n-1)$ in (b) gives

(c) $$xY_{n-1}' = (n-1)Y_{n-1} - (n-1)Y_{n-2}.$$

Differentiation of (b) gives

(d) $$xY_n'' = (n-1)Y_n' - nY_{n-1}'.$$

Suitable combinations of these four equations give (20) and (21).

For the four examples considered above the values (α, β) are, respectively, $(1/2, 1/4)$, $(3/2, 1/4)$, $(1/2, -1/4)$, $(1, 1)$.

We define the systems of polynomials $y_n(x)$ and $c_n y_n(a+bx)$ to be *equivalent*, where c_n does not depend on x. Accordingly, (20) is the same for all values of c_n, but (21) and the function $g(x, t)$ will depend essentially on the choice of c_n.

4. Orthogonal polynomials, (*OP*). We use the standard notation

$$(u, v) = (u, v; a, b; \theta) = \int_a^b u(x)v(x)d\theta(x), \tag{22}$$

and, when $\theta(x)$ is the integral of $w(x)$,

$$(u, v) = (u, v; a, b; w) = \int_a^b u(x)v(x)w(x)dx, \tag{22'}$$

where $w(x)$ is called the weight function.

Associated with each function $\theta(x)$ or $w(x)$ is a sequence of moments [2],

$$u_n = \int_a^b x^n d\theta = (x^n, 1) = (x^n, 1; a, b; \theta),$$

and similarly with w in place of θ.

If $y_n(x)$, $n=0, 1, 2, \cdots$, is an *OP* system with respect to a given interval and a given $\theta(x)$ or $w(x)$, we have, by definition,

$$(y_n, y_m) = (y_n, y_m; a, b; \theta \text{ or } w) = 0, \qquad n \neq m.$$

We use the further notation

$$(x^r, y_n) \equiv n!k_{r,n} \quad \text{with} \quad k_{r,0} = u_r; \qquad (y_n, y_n) \equiv n!l_n.$$

If

$$x^r = \sum_{n=0}^{r} c_{r,n} y_n(x),$$

then $c_{r,n}l_n = k_{r,n}$; $r \geqq n$. For $n > r$, $k_{r,n} = 0$, and $c_{r,n}$ is taken equal to zero by definition.

By substituting the values of $g(x, t)$ given in (1) in the integral (x^r, g) we obtain formally, for all r,

$$\sum u_{n+r} t^n \phi_n(t)/n! = \sum k_{r,n} t^n = \sum c_{r,n} l_n t^n. \tag{23}$$

If $\phi_n(t)$ is given by

$$\phi_n(t) = \sum a_{n,r} t^r, \qquad a_{n,0} \neq 0 \text{ for all } n,$$

equations (23) set up necessary relations between the $a_{n,r}$ and $k_{r,n}$. For the polynomials $Y_n(x, b_s)$ of (13) these equations become

$$b_n u_{r+n} = \sum_{i=0}^{r} (-1)^{n-i} \binom{n}{i} i! k_{r,i}.$$

When $r=0$, we have, on taking $u_0 = k_{0,0} = 1$,

$$b_n u_n = (-1)^n; \qquad n = 0, 1, 2, \cdots .$$

Therefore a necessary condition that the $Y_n(x, b_s)$ form an orthogonal system is that the reciprocals of the quantities $(-1)^n b_n$ shall form a moment sequence.

5. Orthogonal polynomials among the $Y_n(x)$. It has been shown by several writers that among the Appell polynomials the only OP are those of Hermite. That there are also OP among the $Y_n(x)$ follows immediately from the formula [3]

$$g(x, t) = e^t (xt)^{-a/2} J_a[2(xt)^{1/2}] = \sum L_n^{(a)}(x) t^n / \Gamma(n + a + 1).$$

On comparing the known equations [4] for $L_n^{(a)}(x)$, namely,

$$xD_x L_n^{(a)} = nL_n^{(a)} - (n + a) L_{n-1}^{(a)},$$

$$xD_{xx} L_n^{(a)} + (a + 1 - x) D_x L_n^{(a)} + nL_n^{(a)} = 0,$$

with our corresponding equations (15) and (20), we find that the polynomials determined by (19) are the generalized Laguerre polynomials:

$$Y_n(x) \equiv Y_n^{(\alpha,\beta)}(x) = n! L_n^{(\alpha-1)}(\beta x)/\Gamma(n + \alpha). \tag{24}$$

In the four examples previously considered, the equations (20) and (21) show that the $Y_n(x)$ in (A), (B) and (D) are OP, and also they that are special cases of (24) or equivalent systems. The question whether there are among the $Y_n(x)$ other OP than the generalized Laguerre polynomials (24) remains open.

Example (C) does not lead to OP. Thus there is at least a suggestion that, if we are to obtain OP from (19), the choice of α and β should be restricted to values which will make the differential equation (20) oscillatory.

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Algebraic or Transcendental? Let $a_k = 0$ or 1, according as the kth decimal in π is less than 5, or greater than or equal to 5. Is $\sum_1 a_k 2^{-k}$ algebraic or transcendental? The same question can be asked for any other real number.

Arnold Dresden

WHAT IS A TOPOLOGICAL GROUP?

DEANE MONTGOMERY, Smith College

1. Introduction. The concept of a topological group originated with Lie, who was concerned with a very important special class of these groups now known as Lie groups. Much of the work of Lie and of many of his followers was on the theory in the small which is a way of saying that they studied only a small part of the group near the identity element. After a theory of topological spaces had been developed it was possible to formulate Lie's idea in somewhat more abstract terms and thus obtain topological groups in full generality. Progress in topology had also provided tools and a language which made it easier to discuss these groups in the large, and it was inevitable that a greater interest in questions in the large should arise. This is not a denial of the fruitfulness of the local point of view, which with the related subject of Lie algebras, is an indispensable part of the theory.

2. Groups and spaces. A collection of elements is called a group provided that there is given for each pair of elements g_1 and g_2 an operation called the product, written as g_1g_2, satisfying the following axioms:

A. g_1g_2 is in G.

B. $g_1(g_2g_3) = (g_1g_2)g_3$.

C. There is an identity element e such that for any g in G, $ge = eg = g$.

D. For each g in G there is an inverse element g^{-1} such that $gg^{-1} = g^{-1}g = e$.

In this set of axioms as in others to follow, no attempt is made to present the minimum possible conditions.

Topological groups are spaces and to understand them it is necessary to have some notion of what a space is. Almost any idea of a space the reader may have will be adequate to find out something about topological groups. One thing which might be meant by a space is that it is any subset of a euclidean space of some number of dimensions. This would be slightly restrictive but would include many of the cases of interest. For the sake of completeness the axioms for a topological space, as now usually given, are listed below, but the chief thing to have in mind is an intuitive concept of continuous transformations and functions.

A collection G of elements is called a topological space provided a certain family of subsets called open sets is given so as to satisfy the following conditions:

A. The null set and the entire space are open sets.

B. The union of any number of open sets is an open set.

C. The intersection of a finite number of open sets is an open set.

In addition, for topological groups it is usually convenient to assume the following separation property and it is assumed here.

D. If g_1 and g_2 are two points of G, then there is an open set which includes g_1 and does not include g_2.

3. Topological groups. A collection G of elements is called a topological group provided the collection satisfies the following axioms:

A. G is a group.

B. G is a topological space.

C. The group operations g_1g_2 and g^{-1} are continuous functions in the topology of the space.

The last condition is the link between the space and group properties and without it the group and space properties would be entirely independent of each other.

Example 1. The real numbers form both a space and a group where the group "product" is defined to be the sum. The group operations are continuous in the usual topology of the line, and consequently the real numbers form a topological group.

Example 2. The complex numbers can be represented by the points in the plane. They form a group under addition, and the group operations are continuous in the topology of the plane. Thus we obtain another topological group.

Incidentally the plane can be made into a topological group in another and essentially different way as follows: If (x, y) and (x', y') are two points of the plane the product of these two points is defined to be the point with coordinates $(x+x'e^{-y},\ y+y')$. That this rule of combination really yields a topological group is left as an easy exercise for the reader. The group multiplication defined in this way is noncommutative so that this group is essentially different from the group of the complex numbers under addition although the underlying spaces in the two cases are the same.

Example 3. Let G be the collection of all complex numbers of absolute value one. This collection G forms a topological group if the rule of combination is ordinary multiplication of complex numbers. The underlying space is a circumference of unit radius. The group is isomorphic to the group of real numbers modulo 1. This group is an important one not only for the study of other topological groups but also because of applications in combinatorial topology.

Example 4. Let G be the collection of all n by n real matrices with nonvanishing determinant. These form a group under multiplication. Any matrix of this kind can be regarded as a point in n^2 dimensional space simply by regarding the elements of the first row as the first n coordinates, the elements of the second row as the next n coordinates, and so on. Hence any collection of n by n matrices forms a topological space because it may be regarded as a subset of euclidean space. Since each element of a product matrix is a polynomial in the elements of the two matrices being multiplied it is quite clear that the group product, is continuous. The group inverse can be expressed quite simply and is also continuous, and therefore the collection G is a topological group. Of course entirely analogous remarks could be made for n by n complex matrices with nonvanishing determinant.

Any subgroup of a topological group is also a topological group, and this shows that any of the well known groups of matrices are topological groups. In

particular this remark applies to the group of orthogonal matrices. From the algebraic condition for such matrices it can be seen that the associated point set in n^2 dimensional euclidean space is closed and bounded. Consequently the underlying space of the orthogonal group is compact, for a compact topological space is merely one which is analogous to a closed bounded set. This is also true for the group of unitary matrices.

Example 5. Let $C_1, C_2, C_3, \cdots$ be an infinite number of copies of the circle group, that is the group given in Example 3. The elements of any C_i may be given by means of an angle. Let $f_1, f_2, \cdots$ be a sequence of homomorphisms where f_i carries the group C_{i+1} onto the group C_i by doubling the angle of each element. This is a two to one homomorphism. Let G be the collection of elements each of which is an infinite sequence

$$(c_1, c_2, c_3, \cdots) \tag{1}$$

where $c_i = f_i(c_{i+1})$. If

$$(c_1', c_2', c_3', \cdots)$$

is any other element of G, the product of these two elements is defined to be

$$(c_1c_1', c_2c_2', c_3c_3', \cdots).$$

That this is an element of G and that this rule of multiplication satisfies the other axioms for a group can be verified quite easily. Hence the collection G forms a group.

Let i be any integer and let O_i be any open subset of C_i. Then let O^* be the collection of all elements (1) having the property that c_i is in O_i. By definition O^* will be called an open set in G. The topology of G is to be specified by choosing for the open sets all sets which may be defined in the above described manner. It can be shown that under this definition G becomes a topological space and that the group operations are continuous in this topology. The details are omitted.

The space obtained in this way is compact and one dimensional but it has a rather complicated structure; it has been called a solenoid by van Dantzig.

4. Properties of group spaces. If a is a fixed element of a topological group G and if x is a variable element then the mapping

$$T_a: \quad x \to ax$$

is a homeomorphism of G onto itself. If b and c are any two elements of G and if we choose $a = cb^{-1}$ then T_a maps the point b on the point c. We see therefore that for any two elements b and c of G there is a homeomorphism of G onto itself taking b to c. This means that neighborhoods of the points b and c are alike topologically, and this implies a drastic restriction on spaces which can be carriers of topological groups. A closed interval can not be made into a topological group, for example, because an end point and an interior point of the interval

do not have topologically equivalent neighborhoods. In the same way many other spaces can be eliminated as carriers of a topological group.

If a is not the identity then the mapping T_a does not have a fixed point. If G is arcwise connected then T_a can be shown to be a deformation. There are many spaces which do not admit a deformation without a fixed point. We can see in this way that no even dimensional sphere can be the carrier of a topological group. It can be shown in other ways that there are only two spheres which can carry a topological group, namely the one dimensional sphere (example 3) and the three dimensional sphere. The reader familiar with quaternions may see that the three dimensional sphere is a topological group by observing that the quaternions of absolute value one form a group under multiplication and that those same quaternions are the points on the surface of the unit sphere in four space.

Many other properties of group spaces are known and from the ones mentioned here and others it can be verified that the only connected and well behaved (manifolds; see next section) one and two dimensional spaces which can carry a group are the circle, line, plane, cylinder, and torus.

5. Lie groups. The groups in the first four examples above are Lie groups and the one in the fifth example is not a Lie group. In order to define Lie groups it will be necessary to define manifolds. A manifold in the sense used here is a topological space such that every point is in an open set homeomorphic to the set $x_1^2+x_2^2+ \cdots +x_n^2<1$ in euclidean n-space. For example, the circle and line are one dimensional manifolds, the torus, plane and cylinder are two dimensional manifolds, and so on. In a manifold we may clearly choose a coordinate system valid for some (possibly small) open set including each point. If there is given a set of coordinate systems which taken together cover the entire manifold and if whenever two systems overlap the transformation from one to the other is analytic then the manifold is called an analytic manifold.

A Lie group is a topological group whose underlying space is an analytic manifold and in which the group operations are specified by means of analytic functions of the coordinates.

If e is the identity element then e will be in a subset O for which there is given a definite homeomorphism with $x_1^2+ \cdots +x_m^2<1$. This gives a definite set of coordinates for each point of O. If x, y, and xy (if x and y are near enough to e and xy must be in O because of the continuity of the product) are points of O, then the ith coordinate of xy is given by a function of the coordinates of x and y

$$(xy)_i = f_i(x_1, \cdots , x_n; y_1, \cdots , y_n).$$

According to the definition of a Lie group this function must be analytic in the $2n$ variables, that is it can be expanded in a power series in these variables. This small open set O was the entity mainly studied by Lie and his followers. Such a small open set O is often called a group germ. The group germ yields much information about the group but it clearly can not give complete information be-

cause there are groups which are distinct in the large and yet have identical group germs, for example the line and circle as given in Examples 1 and 3.

For the study of the group operations in O, we may use a great many results from analysis because of the analyticity of the functions f_i. It is this fact which simplifies the theory of Lie groups and makes it possible to obtain far more information in this case than in any other.

As we have already remarked, all the groups mentioned in the first four examples are Lie groups. This is fairly clear for the groups of the first three examples. For groups of matrices it is not quite so obvious although it is not very difficult. The underlying space of the group of Example 5 is not a manifold and therefore this group can not be a Lie group.

6. Measure in groups. If the group G is locally compact then Haar has shown how a measure can be defined in G which is invariant under left translations. The measure, like any measure, is a generalization of volume and is a real non-negative function defined on certain sets, in particular on all open and closed sets. It has various useful properties which make it entirely comparable to ordinary Lebesgue measure in euclidean spaces. By means of this measure, a theory of integration can be developed which enjoys the properties of the ordinary Lebesgue integral. These integrals have been of great use in many questions related to groups, particularly in studying the representations of topological groups by means of orthogonal and unitary matrices.

It is a consequence of the results thus developed that every compact topological group can be approximated by means of compact Lie groups somewhat as the group of Example 5 is approximated by means of the circle groups used in defining it.

7. Hilbert's fifth problem. Some topological groups fail to be Lie groups because their underlying spaces are not manifolds. But if a space is a manifold can it fail to be a Lie group? In other words, if a topological group G is a manifold, can a system of coordinates always be introduced into G in such a way that G becomes a Lie group? This question has been answered affirmatively for the compact case by von Neumann, for the Abelian case by Pontrjagin, and for the solvable case by Chevalley. The general case, however, remains open and the methods so far used do not extend to it. One method of attack has been to try to show that sufficiently small neighborhoods of the identity can contain no subgroups except the trivial one containing only the identity. This might be a partial step toward the solution of the problem, but so far attempts to prove this have also failed. This problem is a part of the famous fifth problem of Hilbert which is a very important unsolved problem.

8. Transformation groups. If each element g of a topological group G is a homeomorphism $g(x)$ of a space M onto itself and if these homeomorphisms combine in the usual way and depend continuously on G and M simultaneously, then G is called a transformation group of M. Furthermore if all these homeo-

morphisms are distinct the group is called effective. If G and M are both manifolds then Hilbert asked in his fifth problem whether coordinates in G and M may be so chosen that all operations are analytic in the parameters of G and M simultaneously. J. von Neumann has pointed out that this can not always be done if G is not compact. Whether or not this can always be done in the compact case is unknown. When G is compact and transitive on M it can always be done and this follows quite easily from the result of von Neumann already mentioned.

Hilbert's conjecture has been extended a bit by several people in a way which we shall now state. Let M be a manifold and let G be a compact effective transformation group of M. Then the extended conjecture is that G itself must be a manifold, but this also is unverified. If it were known to be true then we could return to Hilbert's question above and inquire whether or not analytic parameters could be introduced.

Combining these various statements we may sum up the spirit of Hilbert's conjecture on transformation groups for the compact case as follows; If G is a compact effective transformation group of a manifold M then the structure and action of G must be comparatively simple. With this simple statement before us it becomes clear that we are dealing with a profound and fundamental question which lies at the basis of geometry, and this was the point of view of Hilbert. Certain progess in the case where M is 3 dimensional has been made by Leo Zippin and the author. A very important case arises when G is finite, and in this case P. A. Smith has many beautiful theorems which give considerable support to the conjecture. Newman has an important paper on this case which points in the same direction.

Even granting that the compact effective group G acts analytically on a manifold M, the topology of the resulting situation in the large has been explored but little, and is a very interesting field for investigation. It appears likely that these and related questions will provide topology with a rich source of problems for some time to come.

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POLYNOMIALS WHOSE ZEROS HAVE NEGATIVE REAL PARTS*

H. S. WALL, Illinois Institute of Technology

1. Introduction. In the problem of stability of dynamical systems it is important to have simple conditions that a polynomial with real coefficients shall have only zeros with negative real parts. Although such conditions are known, it has seemed desirable to formulate and establish them in a simple way, and to make them more widely available. We have obtained the following theorem.

THEOREM A. *Let*

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \tag{1.1}$$

be a polynomial with real coefficients, and let

$$Q(z) = a_1 z^{n-1} + a_3 z^{n-3} + a_5 z^{n-5} + \cdots \tag{1.2}$$

be the polynomial obtained from $P(z)$ *by dropping out the first, third, fifth,* $\cdots$ *terms. Then all the zeros of* $P(z)$ *have negative real parts if and only if*

$$\frac{Q(z)}{P(z)} = \cfrac{1}{c_1 z + 1 + \cfrac{1}{c_2 z + \cfrac{1}{c_3 z + \cfrac{\ddots}{\ddots + \cfrac{1}{c_n z}}}}}, \tag{1.3}$$

where the coefficients $c_1, c_2, \cdots, c_n$ *are all positive.*

Thus, to test a given polynomial $P(z)$, one applies the euclidean algorithm for the greatest common divisor, to the polynomials $P(z)$ and $Q(z)$. If the successive quotients are of the form $c_1z+1, c_2z, \cdots, c_nz$, where the c_p are positive, then the zeros of $P(z)$ all have negative real parts. In the contrary event, there must be at least one zero with real part greater than or equal to zero. The proof of Theorem A is contained in §2.

In §3 we have given necessary and sufficient conditions for a polynomial (1.1) with complex coefficients to be the last denominator of a continued fraction (1.3), namely the conditions that all the determinants

$$D_1 = a_1,\ D_2 = \begin{vmatrix} a_1, & a_3 \\ 1, & a_2 \end{vmatrix}, \cdots, D_n = \begin{vmatrix} a_1, & a_3, & \cdots, & a_{2n-1} \\ 1, & a_2, & \cdots, & a_{2n-2} \\ 0, & a_1, & \cdots, & a_{2n-3} \\ \cdot & \cdot & \cdots & \cdot \\ & & & a_n \end{vmatrix}, \tag{1.4}$$

* Presented to the American Mathematical Society, April, 1945. This paper was written as the result of a conversation with Professor H. S. Snyder of the Department of Physics, Northwestern University.

($a_0=1$, $a_p=0$ for $p<0$ or $p>n$) shall be different from zero. By means of a known theorem on continued fractions we then give an upper bound for the moduli of the zeros of any polynomial $P(z)$ for which (1.3) holds, (§4).

In §5 we have shown that if $P(z)$ has real coefficients, and (1.3) holds with $c_p \neq 0$, $p=1, 2, \cdots, n$, then the number of zeros of $P(z)$ with positive real parts is equal to the number of negative terms in the sequence $c_1, c_2, \cdots, c_n$, and the number of zeros with negative real parts is equal to the number of positive terms in this sequence.

There is a numerical example at the end of §5.

Finally, we have given in §6 a brief survey of the literature of these and related problems.

2. Proof of Theorem A. We suppose first that (1.3) holds, where the coefficients c_p are all positive, and shall prove that all the zeros of $P(z)$ have negative real parts. It will be convenient to make the substitution

$$(2.1) \qquad b_0 = 1/c_1, \qquad b_p = 1/c_p c_{p+1}, \qquad p = 1, 2, \cdots, n-1,$$

so that the continued fraction takes the form

$$(2.2) \qquad \frac{Q(z)}{P(z)} = \cfrac{b_0}{z + b_0 + \cfrac{b_1}{z + \cfrac{b_2}{z + \cdots + \cfrac{b_{n-1}}{z}}}}.$$

This has the advantage that the coefficients of the highest powers of z in the successive denominators $P_1(z)=z+b_0$, $P_2(z)=z^2+b_0z+b_1, \cdots, P_n(z)\equiv P(z)$, are equal to unity. The b_p are evidently positive if and only if the c_p are positive. If we let $Q_1(z)=b_0$, $Q_2(z)=b_0z, \cdots, Q_n(z)=Q(z)$ be the successive numerators, then we have the determinant formula

$$(2.3) \qquad Q_m(z)P_{m-1}(z) - Q_{m-1}(z)P_m(z) = (-1)^{m-1}b_0b_1 \cdots b_{m-1},$$

$m=1, 2, \cdots, n$, ($Q_0=0$, $P_0=1$). If we put $m=n$ in this formula we conclude at once that $Q(z)/P(z)$ is irreducible.

Consider the linear fractional transformations*

$$t = \frac{b_0}{z + b_0 + w_1}, \quad w_1 = \frac{b_1}{z + w_2}, \cdots, w_{n-1} = \frac{b_{n-1}}{z + w_n},$$

in the variables $t, w_1, w_2, \cdots, w_n$, z being a parameter with nonnegative real part. The value of the rational fraction $Q(z)/P(z)$ is the image of the point

* This argument was used, for instance, in [2], pp. 105–107.

$w_n=0$ under the product of these transformations. Since, for $R(w_1)\geqq 0$, $R(z)\geqq 0$, we have

$$|t-\tfrac{1}{2}|\leqq \tfrac{1}{2}, \tag{2.4}$$

and since $R(w_p)\geqq 0$ implies $R(w_{p-1})\geqq 0$, $p=2, 3, \cdots, n$, it follows that if $R(w_n)\geqq 0$, then t satisfies (2.4). Hence, in particular,

$$\left|\frac{Q(z)}{P(z)}-\tfrac{1}{2}\right|\leqq \tfrac{1}{2}, \quad \text{if} \quad R(z)\geqq 0. \tag{2.5}$$

Thus, $Q(z)/P(z)$ is bounded for $R(z)\geqq 0$, and, since it is irreducible, the denominator $P(z)\neq 0$ for $R(z)\geqq 0$.

We now suppose, conversely, that $P(z)$ is an arbitrary polynomial of the form (1.1) with real coefficients, whose zeros all have negative real parts. If we regard $|P(z)|$ as the product of the lengths of the vectors from z to its zeros, we then see immediately that $|P(z)|\geqq|P(-z)|$ for $R(z)\geqq 0$, where inequality holds in both if it holds in either. Therefore,

$$|P(z)\pm P(-z)|\geqq|P(z)|-|P(-z)|>0 \quad \text{for} \quad R(z)>0. \tag{2.6}$$

Similarly, we have

$$|P(z)\mp P(-z)|\geqq|P(-z)|-|P(z)|>0 \quad \text{for} \quad R(z)<0. \tag{2.7}$$

Let $Q(z)$ be defined as $[P(z)+P(-z)]/2$ or $[P(z)-P(-z)]/2$, according as the degree n of $P(z)$ is odd or even, respectively. Then, by (2.6), (2.7), the zeros of $Q(z)$ all lie upon the axis of imaginaries.

Moreover, since

$$\left|\frac{P(-z)}{P(z)}\right|=\left|\frac{P(-z)}{P(z)}\pm 1\mp 1\right|\leqq 1 \quad \text{for} \quad R(z)\geqq 0,$$

it follows that (2.5) holds.

By division we now find that

$$\frac{Q(z)}{P(z)}=\frac{1}{c_1z+1+[C(z)/Q(z)]}, \qquad c_1=1/a_1>0, \tag{2.8}$$

where $C(z)/Q(z)$ is an irreducible rational fraction in which the denominator is of degree $n-1$, and the numerator is of lower degree than the denominator. Also, it follows from (2.5) and (2.8) that $R[C(z)/Q(z)]\geqq -c_1x$, where $x=R(z)\geqq 0$. Consequently

$$R[C(z)/Q(z)]\geqq 0 \quad \text{for} \quad R(z)\geqq 0. \tag{2.9}$$

In fact, if $R[C(z)/Q(z)]=-k$, $k>0$, for some point $z=z_0$ such that $R(z_0)>0$, we can contradict the theorem that a nonconstant harmonic function cannot take on its minimum value on the interior of a region in which it is harmonic. It suffices to take for the region the portion of the right half-plane exterior to

circles of radius $k/2c_1$ with centers at the zeros of $Q(z)$, and interior to a circle $|z| = r > |z_0|$, so large that $|C(z)/Q(z)| \leqq k/2$ on this circle.

From (2.9) we can conclude that the zeros of $Q(z)$ are simple, and that the residues of $C(z)/Q(z)$ are positive. Otherwise, one can choose a path of z in $R(z) > 0$ approaching a pole of $C(z)/Q(z)$ in such a way that the real part of the latter will become negative. We therefore have a partial fraction development of the form

$$C(z)/Q(z) = z \sum_{p=1}^{m} \frac{2M_p}{z^2 + t_p^2}, \qquad m = [n/2], \tag{2.10}$$

where the M_p are positive, and the t_p are distinct real numbers.

We now replace z by $-iz$ in (2.10), and then multiply both members by $-i$. This gives

$$\frac{-iC(-iz)}{Q(-iz)} = \sum_{p=1}^{[n/2]} \left(\frac{M_p}{z + t_p} + \frac{M_p}{z - t_p} \right). \tag{2.11}$$

Let $M(u)$ be a step-function, constant except at the points $\pm t_p$ where it has jumps equal to M_p. Then the function (2.11) may be written as the Stieltjes integral

$$\int_{-\infty}^{+\infty} \frac{dM(u)}{z + u}.$$

Let $\sum (-1)^p k_p / z^{p+1}$ be the expansion of this function in descending powers of z, so that

$$k_p = \int_{-\infty}^{+\infty} u^p dM(u), \qquad p = 0, 1, 2, \cdots.$$

Now, the quadratic form

$$F_p(X, X) = \int_{-\infty}^{+\infty} (X_0 + X_1 u + \cdots + X_{p-1} u^{p-1})^2 dM(u)$$

is positive definite for $p = 1, 2, 3, \cdots, n-1$. Consequently, the determinants

$$\Delta_p = \begin{vmatrix} k_0, & k_1, & \cdots, & k_p \\ k_1, & k_2, & \cdots, & k_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ k_p, & k_{p+1}, & \cdots, & k_{2p} \end{vmatrix}, \qquad p = 0, 1, \cdots, n-2,$$

are all positive. Therefore, the function (2.11) has a continued fraction expansion of the form*

* *Cf.* [10] pp. 421–422 and, for instance, [11].

$$\cfrac{p_0}{q_1 + z - \cfrac{p_1}{q_2 + z - \cdots - \cfrac{p_{n-2}}{q_{n-1} + z}}},$$

where $p_0, p_1, \cdots, p_{n-2}$ are positive. Inasmuch as (2.11) is an odd function of z, it follows that $q_m = 0$, $m = 1, 2, 3, \cdots, n-1$. On replacing z by iz, and changing the notation, we then have

$$\frac{C(z)}{Q(z)} = \cfrac{1}{c_2 z + \cfrac{1}{c_3 z + \cdots + \cfrac{1}{c_n z}}},$$

where the c_p are positive. On substituting this into (2.8) we then have (1.3), and the theorem is proved.

3. Condition for a polynomial to be the denominator of a continued fraction. Let $P_n(z)$ be the nth denominator of the continued fraction

$$\cfrac{b_0}{z + b_0 + \cfrac{b_1}{z + \cfrac{b_2}{z + \cdots}}}, \qquad b_p \neq 0, \tag{3.1}$$

so that

$$P_0(z) = 1,\ P_1(z) = z + b_0,\ P_{m+1}(z) = zP_m(z) + b_m P_{m-1}(z), \qquad m = 1, 2, \cdots. \tag{3.2}$$

Then $P_n(z)$ is a polynomial of degree n in which the coefficient of z^n is unity. One may show by mathematical induction that*

$$P_n(z) = z^n + b_0 z^{n-1} + S_{11} z^{n-2} + b_0 S_{21} z^{n-3} + S_{12} z^{n-4} + b_0 S_{22} z^{n-5} + \cdots, \tag{3.3}$$

where S_{ij} is the sum of all possible products of the form

$$b_{p_1} b_{p_2} \cdots b_{p_j}$$

under the condition that $i \leqq p_1 \leqq 2 + p_2 \leqq 4 + p_3 \leqq \cdots \leqq 2j - 2 + p_j \leqq n-1$. In

* *Cf.* [7] p. 301.

order for a polynomial $P(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n$ to be the nth denominator of (3.1) it is therefore necessary and sufficient that the system of nonlinear equations

$$b_0 = a_1,\ S_{11} = a_2,\ b_0S_{21} = a_3,\ S_{12} = a_4, \cdots, \tag{3.4}$$

have a solution $b_0, b_1, \cdots, b_{n-1}$, $b_p \neq 0$, $p=0, 1, \cdots, n-1$.

Let us suppose first that the a_p have values such that a solution exists. Then, one may show that the determinants D_p of (1.4) are different from zero:

$$D_p \neq 0, \qquad p = 1, 2, \cdots, n. \tag{3.5}$$

In fact, we have the formulas

$$\begin{aligned} D_{2k+1} &= b_0^{k+1} b_1^{k} b_2^{k} b_3^{k-1} b_4^{k-1} \cdots b_{2k-1}b_{2k}, & 2k+1 \leqq n,\\ D_{2k+2} &= b_0^{k+1} b_1^{k+1} b_2^{k} b_3^{k} b_4^{k-1} \cdots b_{2k}b_{2k+1}, & 2k+2 \leqq n. \end{aligned} \tag{3.6}$$

For, if $p=2k+1$ or $2k+2$, we have

$$D_p = \begin{vmatrix} b_0, & b_0S_{21}, & b_0S_{22}, & b_0S_{23}, & b_0S_{24}, & \cdots \\ 1, & S_{11}, & S_{12}, & S_{13}, & S_{14}, & \cdots \\ 0, & b_0, & b_0S_{21}, & b_0S_{22}, & b_0S_{23}, & \cdots \\ 0, & 1, & S_{11}, & S_{12}, & S_{13}, & \cdots \\ 0, & 0, & b_0, & b_0S_{21}, & b_0S_{22}, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= b_0^{k+1} \begin{vmatrix} b_1, & b_1S_{31}, & b_1S_{32}, & b_1S_{33}, & b_1S_{34}, & \cdots \\ 1, & S_{21}, & S_{22}, & S_{23}, & S_{24}, & \cdots \\ 0, & b_1, & b_1S_{31}, & b_1S_{32}, & b_1S_{33}, & \cdots \\ 0, & 1, & S_{21}, & S_{22}, & S_{23}, & \cdots \\ 0, & 0, & b_0, & b_0S_{31}, & b_0S_{32}, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

as may be seen by removing the factor b_0 from the first, third, fifth, $\cdots$ rows of D_p, and then subtracting the first row from the second, the third from the fourth, the fifth from the sixth, $\cdots$. The new determinant on the right is of the same form as the original, but of order $p-1$. Therefore, the validity of (3.6) readily follows by mathematical induction. Since the b_p are different from zero, we conclude from (3.6) that (3.5) holds. Furthermore, we see that the solution of (3.4) is unique and is given by the formulas

$$b_p = \frac{D_{p-1}D_{p+1}}{D_{p-1}D_p}, \qquad p = 0, 1, \cdots, n-1, \tag{3.7}$$

where we must set $D_{-2}=D_{-1}=D_0=1$.

We suppose now, conversely, that $a_1, a_2, \cdots, a_n$ are any given numbers such that (3.5) holds, and shall prove that the system (3.4) has a solution $b_p \neq 0$, $p=0, 1, \cdots, n-1$. The statement is obviously true if $n=1$. Assuming that it is true for $n-1$, $n>1$, we shall prove it for n. To that end, consider the system

$$b_1 = a_1', \quad S_{21} = a_2', \quad b_1 S_{31} = a_3', \quad S_{22} = a_4', \cdots, \tag{3.8}$$

where $a_1', a_2', \cdots, a_{n-1}'$ are given by

$$a_1' = a_2 - (a_3/a_1), \quad a_2' = a_3/a_1, \quad a_3' = a_4 - (a_5/a_1), \quad a_4' = a_5/a_1, \cdots.$$

Denote by D_p' the determinant D_p formed with the numbers a_p'. Then, it is easy to verify that

$$D_{2p}' = D_{2p+1}/D_1^{p+1}, \qquad D_{2p+1}' = D_{2p+2}/D_1^{p+1},$$

and consequently $D_p' \neq 0$, $p=1, 2, \cdots, n-1$. Therefore, by the hypothesis of the induction, the system (3.8) has a unique solution $b_1, b_2, \cdots, b_{n-1}$, and this is given by

$$b_{p+1} = D_{p-2}' D_{p+1}' / D_{p-1}' D_p' = D_{p-1} D_{p+2} / D_p D_{p+1}, \qquad p = 0, 1, \cdots, n-2.$$

If to these we adjoin the equation $b_0 = a_1$, it then follows that the system (3.4) has the unique solution (3.7).

We have completed the proof of the following theorem.

THEOREM B. *If $P_n(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ is a polynomial of degree n whose coefficients satisfy* (3.5), *then there is determined uniquely a continued fraction of the form* (2.2) *such that $P_n(z)$ is its last denominator. The b_p are given in terms of the a_p by* (3.7). *Conversely, the coefficients in the last denominator of a continued fraction of the form* (2.2) *satisfy* (3.5).

From Theorems A and B it follows that if $P(z)$ is a polynomial of the form (1.1) with real coefficients whose zeros have negative real parts, then there exist uniquely determined polynomials $P_0=1, P_1, \cdots, P_{n-1}$, of degrees $0, 1, \cdots, n-1$, having the same properties as $P(z)$ which are connected with $P(z)$ by the recurrence formula $P_{m+1} = zP_m + b_m P_{m-1}$, $m=1, 2, \cdots, n-1$, $P_n = P$.

We note that $P(z)$ can be written in the form

$$P(z) = \begin{vmatrix} b_0 + z, & -b_1, & 0, & 0, & 0, \cdots, 0 & & \\ 1, & z, & -b_2, & 0, & 0, \cdots, 0 & & \\ 0, & 1, & z, & -b_3, & 0, \cdots, 0 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0, \cdots, 1, & & z, & -b_{n-1} \\ 0, & 0, & 0, & 0, \cdots, 0, & & 1, & z \end{vmatrix}.$$

Conversely, if the b_p are positive, a polynomial of this form has all its zeros in the left half-plane, $R(z)<0$. From this expression for $P(z)$ as a determinant, it follows at once that

$$P(z) = G(z) + H(z), \tag{3.9}$$

where

$$\frac{G(z)}{H(z)} = \cfrac{b_0}{z + \cfrac{b_1}{z + \cdots \cfrac{b_{n-1}}{z}}}. \tag{3.10}$$

We shall make use of this remark in §5.

If the b_p are positive, the fraction (3.10) has the same properties as the fraction $C(z)/Q(z)$ of (2.11). In particular, the zeros of $G(z)$ and $H(z)$ lie on the axis of imaginaries; $R[G(z)/H(z)] \geqq 0$ for $R(z) \geqq 0$; and there is a partial fraction development of the form

$$\frac{G(z)}{H(z)} = z \sum_{p=1}^{[(n+1)/2]} \frac{2M_p}{z^2 + t_p^2}, \qquad M_p > 0,\ t_1 < t_2 < t_3 < \cdots. \tag{3.11}$$

Conversely, if $G(z)/H(z)$ is an irreducible rational fraction of the form (3.11), then $P(z) = G(z) + H(z)$ has only zeros with negative real parts.

There is at least one case where the vanishing of one of the determinants D_p has a simple interpretation in terms of the zeros of the polynomial $P(z)$. We have the following theorem.

THEOREM C. *Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial with complex coefficients. A necessary and sufficient condition for $P(z)$ and $P(-z)$ to have a zero in common is that the determinant D_n shall vanish.*

Proof. Since $a_n D_{n-1} = D_n$, so that $D_n = 0$ when $a_n = 0$, we see that when $P(z)$ and $P(-z)$ vanish for $z = 0$ then $D_n = 0$. Supposing that $a_n \neq 0$, we shall show that $P(z)$ and $P(-z)$ have a zero in common if and only if $D_{n-1} = 0$, and the theorem will be proved. Now, $P(z)$ and $P(-z)$ have a zero in common if and only if $P(z) + P(-z)$ and $P(z) - P(-z)$ have a zero in common, *i.e.*, if and only if the polynomials

$$G(z) = a_1 z^{n-1} + a_3 z^{n-3} + \cdots$$

and

$$H(z) = z^n + a_2 z^{n-2} + \cdots$$

have a zero in common. If n is even, $n = 2k$, then

$$G(z) = z(a_1 z^{2k-2} + a_3 z^{2k-4} + \cdots), \quad H(z) = z^{2k} + a_2 z^{2k-2} + \cdots,$$

and if n is odd, $n = 2k + 1$, then

$$G(z) = a_1 z^{2k} + a_3 z^{2k-2} + \cdots, \ H(z) = z(z^{2k} + a_2 z^{2k-2} + \cdots).$$

Since $G(0) \neq 0$ in the second case, and $H(0) \neq 0$ in the first, we conclude that

$G(z)$ and $H(z)$ have a zero in common, when $n=2k$, if and only if the polynomials $a_1w^{k-1}+a_3w^{k-2}+\cdots+a_{2k-1}$ and $w^k+a_2w^{k-1}+\cdots+a_{2k}$ have a zero in common, and, when $n=2k+1$, if and only if the polynomials $a_1w^k+a_3w^{k-1}+\cdots+a_{2k+1}$ and $w^k+a_2w^{k-1}+\cdots+a_{2k}$ have a zero in common. In either case the resultant of the two equations is D_{n-1}, and the theorem is therefore true.

4. Bounds for the moduli of the zeros of $P(z)$. Let us suppose that (1.3) holds with $c_p\neq 0$, $p=1, 2, \cdots, n$, the coefficients of $P(z)$ being complex numbers. We may write (1.3) in the form

$$\frac{Q(z)}{P(z)}=\cfrac{(c_1z+1)^{-1}}{1+\cfrac{h_1}{1+\cfrac{h_2}{1+\cfrac{\ddots}{+\cfrac{h_{n-1}}{1}}}}},$$

where

$$h_1=1/[c_2z(c_1z+1)],\ h_2=1/(c_2c_3z^2),\ h_3=1/(c_3c_4z^2),\cdots,\ h_{n-1}=1/(c_{n-1}c_nz^2).$$

Let $g_1, g_2, \cdots, g_{n-1}$ be any numbers such that $0<g_p<1$, $p=1, 2, \cdots, n-1$. Then if z satisfies the inequalities

$$\begin{gathered}|h_1|\leqq g_1,\ |h_2|\leqq(1-g_1)g_2,\ |h_3|\leqq(1-g_2)g_3,\cdots,\\ |h_{n-1}|\leqq(1-g_{n-2})g_{n-1},\end{gathered}\tag{4.1}$$

we are assured by a known theorem* that $P(z)\neq 0$.

For the polynomial $z^2+a_1z+a_2$, (4.1) gives the bound

$$|z|\geqq\frac{|a_1|+(|a_1|^2+[4|a_2|/g_1])^{1/2}}{2}.$$

Inasmuch as g_1 may be taken as near unity as desired, this bound cannot be improved upon.

On putting $g_p=1/2$, (4.1) gives the bound $|z|\geqq c$, where c is the largest of the numbers

$$\frac{(|c_2/c_1|)^{1/2}+(8+|c_2/c_1|)^{1/2}}{2|c_1c_2|^{1/2}},\ 2/|c_2c_3|^{1/2},\cdots,2/|c_{n-1}c_n|^{1/2}.\tag{4.2}$$

In particular, if c is the smallest of the numbers $|c_p|$, $p=1, 2, \cdots, n$, then $P(z)\neq 0$ for $|z|\geqq 2/c$.

5. Determination of the number of zeros of $P(z)$ in each of the half-planes $R(z)<0$ and $R(z)>0$. We suppose now that $P(z)$ is a polynomial with real coefficients of the form (1.1), such that the determinants (1.4) are different from zero.

* See [6] pp. 369–372.

Then, by Theorem B, we have the expansion (1.3) where the c_p are different from zero.

THEOREM D. *In the expansion* (1.3), *let k of the coefficients c_p be negative and the remaining $n-k$ be positive. Then k of the zeros of $P(z)$ have positive real parts and $n-k$ have negative real parts.*

Proof. We start with the expansion (3.10), where $G(z)$ and $H(z)$ are the polynomials in (3.9). It is easy to see that

$$G(z) = a_1 z^{n-1} + a_3 z^{n-3} + a_5 z^{n-5} + \cdots,$$
$$H(z) = z^n + a_2 z^{n-2} + a_4 z^{n-4} + \cdots.$$

If we replace z by $-iz$ and make the substitution (2.1) in (3.10), we obtain after some simple transformations:

$$\frac{G(-iz)}{iH(-iz)} = \cfrac{1}{c_1 z - \cfrac{1}{c_2 z - \cfrac{1}{c_3 z - \cdots - \cfrac{1}{c_n z}}}}. \tag{5.1}$$

Under the hypothesis that the expansion (1.3) exists, the fraction $Q(z)/P(z)$ is irreducible, and consequently $P(z)$ and $P(-z)$ have no zeros in common. Thus, the zeros of $P(z)$ have their real parts different from zero. For $R(z)=0$ we may therefore write $P(z)=re^{i\pi\theta}$, where $r>0$. If we regard $P(z)$ as the product of the vectors from its zeros to the point z, we then conclude immediately that as z ranges along the axis of imaginaries from $i\cdot\infty$ to $-i\cdot\infty$, θ decreases by the integral amount

$$N - P = \Delta, \tag{5.2}$$

where P and N are the numbers of zeros of $P(z)$ having positive and negative real parts, respectively. The same conclusion evidently results if we introduce an arbitrary constant factor $k\neq 0$, and consider $kP(z)=re^{i\pi\theta}$.

Now,

$$\begin{aligned} i^n P(-iz) &= (z^n - a_2 z^{n-2} + a_4 z^{n-4} - \cdots) + i(a_1 z^{n-1} - a_5 z^{n-3} + a_5 z^{n-5} - \cdots) \\ &= U(z) + iV(z), \end{aligned}$$

where $U(z)=z^n - a_2 z^{n-2} + \cdots$ and $V(z)=a_1 z^{n-1} - a_3 z^{n-3} + \cdots$ are real when z is real. From the preceding we then conclude that as z increases over the real axis from $-\infty$ to $+\infty$,

$$\theta = (1/\pi)\ \mathrm{arc\ tan}\ [V(z)/U(z)]$$

decreases by the amount Δ. Moreover, from (5.1) it follows that

$$\frac{V(z)}{U(z)} = \cfrac{1}{c_1 z - \cfrac{1}{c_2 z - \cfrac{\ddots}{\quad - \cfrac{1}{c_n z}}}}. \tag{5.3}$$

Let $x_1, x_2, \cdots, x_k$ denote the real distinct zeros of $U(z)$. Let $s_p = +1$, 0, or -1 according as $V(z)/U(z)$ increases from $-\infty$ to $+\infty$ does not change sign, or decreases from $+\infty$ to $-\infty$, respectively, as z increases through the value x_p. We must then have*

$$\Delta = \sum_{p=1}^{k} s_p. \tag{5.4}$$

To compute the number Δ, let the polynomials $f_0 = 1$, $f_1 = c_n z, \cdots, f_n$ be defined by the recurrence formula

$$f_{p+1} = c_{n-p} z f_p - f_{p-1}, \qquad p = 1, 2, \cdots, n-1, \tag{5.5}$$

and define polynomials $F_0 = 0$, $F_1 = 1, \cdots, F_n$ by the formulas

$$F_{p+1} = c_{n-p} z F_p - F_{p-1}, \qquad p = 1, 2, \cdots, n-1. \tag{5.6}$$

On multiplying (5.6) by f_p and (5.5) by $-F_p$, and adding, we get $F_{p+1} f_p - F_p f_{p+1} = F_p f_{p-1} - F_{p-1} f_p$, from which we conclude that

$$F_{p+1} f_p - F_p f_{p+1} = 1, \qquad p = 0, 1, \cdots, n-1. \tag{5.7}$$

Consider now the sequence

$$f_0, f_1, f_2, \cdots, f_n. \tag{5.8}$$

From (5.7) it follows that two successive members of this sequence cannot vanish simultaneously. From (5.5) it follows that when f_p vanishes, then f_{p-1} and f_{p+1} have opposite signs $(1 \leqq p \leqq n-1)$. Hence, as z increases through a real zero of f_p there can be no loss or gain in the number of variations in signs in the sequence (5.8). Therefore, as z increases through real values from $-\infty$ to $+\infty$, any change in the number of variations must be due to the vanishing of f_n. Moreover, there will be a loss or a gain in the number of variations according as the product $f_{n-1} f_n$ changes from negative to positive or from positive to negative, respectively, as z passes through a zero of f_n. But $f_{n-1}/f_n \equiv V(z)/U(z)$, and consequently the number Δ is precisely the net loss in the number of variations in signs in the sequence (5.8) as z ranges from $-\infty$ to $+\infty$ through real values.

Now when z is negative, the signs of the leading terms in (5.8) are the same as the signs of

* This is the *Cauchy index* of the fraction (5.3). *Cf.* §6.

$$(5.9) \qquad 1,\ -c_n,\ +c_{n-1}c_n,\ -c_{n-2}c_{n-1}c_n,\ \cdots,\ (-1)^n c_1c_2\cdots c_n,$$

while for positive z, the signs are those of

$$(5.10) \qquad 1,\ +c_n,\ +c_{n-1}c_n,\ +c_{n-2}c_{n-1}c_n,\ \cdots,\ +c_1c_2\cdots c_n.$$

If there are k variations in signs in (5.10), then there are $n-k$ variations in signs in (5.9). Therefore, $\Delta=(n-k)-k=n-2k$. By (5.2) and the relation $P+N=n$, it then follows that $P=k$ and $N=n-k$. Since k is clearly equal to the number of negative terms in the sequence $c_1, c_2, \cdots, c_n$, Theorem D is established.

Theorem D cannot be applied to a polynomial $P(z)$ in case it has a zero in common with $P(-z)$. This is not a serious difficulty inasmuch as the common factor may be determined by means of the euclidean algorithm. However, there are other cases where the theorem will not apply. For example, if

$$P(z) = z^4 - (9/4)z^3 + z^2 - (9/4)z + (5/2),$$

the determinant $D_2=0$, so that the expansion (1.3) does not exist. The zeros of this polynomial are

$$1,\ 2,\ [-3+i\sqrt{71}]/8,\ [-3-i\sqrt{71}]/8.$$

This difficulty may be avoided in certain cases by considering instead of $P(z)$ the polynomial $(z+s)P(z)$ for a suitable value of s. In the above example, if we take $s=+7/4$, we find that (1.3) exists with $c_1<0$, $c_2>0$, $c_3>0$, $c_4>0$, $c_5<0$.

We remark that if Theorem D is applied to $P(z+h)$ for suitable real values of h, the zeros of $P(z)$ may be separated into sets lying in vertical strips.

Example. Let $P(z)=z^5-3z^4-9z^3-27z^2-32z-30$. Then $Q(z)=-3z^4-27z^2-30$. The expansion (1.3) may be obtained by dividing $P(z)$ by $Q(z)$ until a remainder is obtained which is of lower degree than $Q(z)$; then $Q(z)$ is divided by this remainder, and so on. If we write only the coefficients, the computation may be arranged as follows.

```
                        -1/3+1
-3+0-27+0-30 | 1-3- 9-27-32-30
               1+0+ 9+ 0+10
               -3-18-27-42-30
               -3+ 0-27+ 0-30   1/6
                  -18+ 0-42+ 0 | -3+0-27+0-30
                                 -3+0- 7+0       9/10
                                    -20+0-30 | -18+0-42+0
                                               -18+0-27      4/3
                                                   -15+0 | -20+0-30
                                                           -20+0     1/2
                                                                -30 | -15
                                                                      -15
                                                                        0
```

Hence, $c_1=-1/3$, $c_2=1/6$, $c_3=9/10$, $c_4=4/3$, $c_5=1/2$. Therefore, there is one zero in the right half-plane and four in the left half-plane. We find that the zeros are $-1+i$, $-1-i$, 6.197, and the two zeros of the quadratic $z^2+1.92z+2.416$. The smallest modulus of the c_p is $c=1/6$, and therefore an upper bound for the moduli of the zeros is 12. A better bound (*cf.* (4.2)) is the largest of the numbers $3[1+17^{1/2}]/2$, $4(15)^{1/2}/3$, $30^{1/2}/3$, $6^{1/2}$, namely, the first, which is $7.68+$. By suitably adjusting the values of the parameters g_p in (4.1), one can obtain a bound very close to the least bound. For example, with $g_1=4/5$, $g_2=4/5$, $g_3=1/2$, $g_4=1/12$, we find that the largest of the numbers corresponding to (4.2) is 6.474.

6. Historical survey.* The problem of determining the number N of zeros of a polynomial $P(z)$ which lie in a given region was solved by Cauchy by means of his formula

$$N=\frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}\,dz.$$

Here C is the rectifiable boundary curve of the region, and it must be supposed that $P(z)\neq 0$ on C. If we write $P(z)=re^{2i\pi\theta}$, $r>0$ on C, then it suffices to consider only the real part of the integral, namely

$$N=\int_C d\theta.$$

Hence, N is the amount that θ increases as the point z traverses the curve C in the positive sense. If $P(z)=U(z)+iV(z)$, where $U(z)$ and $V(z)$ are real on C, then

$$\theta=\frac{1}{2\pi}\arctan\,[V(z)/U(z)].$$

These considerations led Cauchy [1] to introduce the notion of the *index* of a rational fraction. The number Δ of (5.4) is the index of the fraction (5.3) for the real axis. Cauchy developed formulas for the computation of the index. He also introduced the method, which we have used in §5, to compute the index by means of a Sturm's series. Kronecker [5] extended the notion of index to systems of functions.

On the basis of Cauchy's work, E. J. Routh [8] derived the following rule for testing a polynomial $P(z)=a_0z^n+z_1z^{n-1}+\cdots+a_n$, $a_0>0$, with real coefficients. Consider the array

$$\begin{array}{cccc}
a_0, & a_2, & a_4, & \cdots \\
a_1, & a_3, & a_\beta, & \cdots \\
\dfrac{a_1a_2-a_0a_3}{a_1}, & \dfrac{a_1a_4-a_0a_\beta}{a_1}, & & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{array}$$

* *Cf.* [9].

where the third row is obtained from the first two by cross-multiplication. The next row is obtained from the second and third by the same process. Thus, the first element in the fourth row is

$$\frac{a_3\left(\dfrac{a_1a_2 - a_0a_3}{a_1}\right) - (a_1a_4 - a_0a_\beta)}{\dfrac{a_1a_2 - a_0a_3}{a_1}}.$$

Each row has one fewer elements than the preceding row. Then the number of variations in signs in the sequence making up the first column of the array is equal to the number of zeros of $P(z)$ having positive real parts.

This is essentially Theorem D of §5. In fact, if one starts with the equations (3.4), he will readily see that the test sequence of Routh is, if $a_0 = 1$:

$$(6.1) \qquad 1,\ b_0,\ b_1,\ b_0b_2,\ b_1b_3,\ b_0b_2,\ \cdots,$$

which is equivalent to $1, c_1, c_1c_2 \cdots, c_1c_2 \cdots c_n$. The number of variations in signs in this sequence is equal to the number of negative terms in the sequence $c_1, c_2, \cdots, c_n$. The method of Routh fails in case division by zero is involved in his algorithm. Theorem A shows that the method cannot fail in the important case where the zeros all have negative real parts. In terms of the determinants D_p one sees by (3.7) and (6.1) that Routh's test sequence is $1, D_1, D_2/D_1, \cdots, D_n/D_{n-1}$.

Hermite [3] expressed Cauchy's index in terms of the signature of a certain quadratic form, and obtained necessary and sufficient conditions for the zeros of a polynomial with complex coefficients to lie in a half-plane. For the function (5.3), this quadratic form can be written, following A. Hurwitz [4], as

$$F_m(X, X) = \frac{1}{2\pi i}\int \frac{V(z)}{U(z)}\,[X_0 + X_1z + \cdots + X_{m-1}z^{m-1}]^2dz,$$

where the integral is taken around a curve containing all the zeros of $U(z)$. If we write $V(z)/U(z) = \sum_{p=0}^{\infty} k_p/z^{p+1}$, then an easy computation shows that

$$F_m(X, X) = \sum_{p,q=0}^{m-1} k_{p+q}X_pX_q.$$

Hence, if

$$\Delta_p = \begin{vmatrix} k_0, & k_1, \cdots, & k_{p-1} \\ k_1, & k_2, \cdots, & k_p \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ k_{p-1}, & k_p, \cdots, & k_{2p-2} \end{vmatrix} \neq 0, \qquad p = 1, 2, \cdots, n,$$

then

$$F_n(X, X) = \Delta_1 Y_0^2 + [\Delta_2/\Delta_1]Y_1^2 + \cdots + [\Delta_n/\Delta_{n-1}]Y_{n-1}^2,$$

where the Y_p are linear functions of the X_p. The Cauchy index Δ is the signature

of this quadratic form. Now, the coefficients c_p in the continued fraction (5.3) can be expressed in terms of the Δ_p by well-known formulas.* In this way one may obtain Theorem D without using the Sturm's series (5.8). This is essentially the method used by Hurwitz to prove that all the zeros of $P(z)$ have negative real parts if and only if the determinants (1.4) are all positive. This theorem is an immediate consequence of Theorems A and B.

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* *Cf.*, for instance, [11].

Mathematicians and philosophers. In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy—an opinion, moreover, which has been expressed by many philosophers.—Henri Lebesgue (1936).

The judgment of oratory. In the pure mathematics we contemplate absolute truths which existed in the divine mind before the morning stars sang together, and which will continue to exist there when the last of their radiant host shall have fallen from heaven.—Edward Everett (1870).

The judgment of reason. Everything else they have tried to fasten onto mathematics—its absoluteness and its complete exactitude, its generality and its independence, in a word its truth and everlastingness—all this (I may be pardoned the strong language), *is a pure superstition!*—Gerritt Mannoury (1934). *Contributed.*

NUMBERS GENERATED BY THE FUNCTION e^{e^x-1}

G. T. WILLIAMS, Harvard University

1. Introduction. The integers which are the subject of this paper have been discussed cursorily by several writers* who contented themselves with discovering Lemma 1 and Corollary 1, and computing the first few. They occur in combinatory analysis, being, in fact, the sum of the horizontal entries in the table of p. 169 of Netto's *Lehrbuch der Combinatorik*. Their interest is primarily number-theoretic. Indeed, from Minetola's† work, it is evident that G_n (as defined below) is the number of ways in which a product of n distinct primes can be factored. Thus, $p_1p_2p_3=(p_1p_2)p_3=(p_1p_3)p_2=(p_2p_3)p_1=(p_1)(p_2)(p_3)$, and so $G_3=5$.

It will be convenient to give an algebraic definition of G_n.

2. Definition and algebraic properties. We determine the sequence of G's (we shall only be interested in the case where n is a non-negative integer) by the following

DEFINITION.

$$G_ne = \sum_{r=0}^{\infty} \frac{r^n}{r!} \qquad (n = 0, 1, \cdots).$$

It is plain from this that

$$G_0 = G_1 = 1.$$

More general summation are expressible in terms of the G's; in fact

THEOREM 1.

$$\sum_{r=0}^{\infty} \frac{(ar+b)^n}{r!} = (aG+b)^n e,$$

where the right-hand member means

$$e\sum_{r=0}^{n}\binom{n}{r}a^rG_rb^{n-r}.‡$$

For, expanding the first member, we find it equals

* Wohlsentolme gave just this as a problem, on the Tripos: to prove Lem. 1, Cor. 1, and find G_8. See, Bromwich, Infinite Series, p. 197.

† Silvio Minetola, Principii di Analisi Combinatoria, Jior. di Mat., vol. 45, 1907, pp. 333–366, vol. 47, 1909, pp. 173–200.

‡ We shall make considerable use of this standard symbolic convention: whenever we write a "product" of G's, say,

$$G^jG^kG^m\cdots,$$

we mean the sum of the exponents to be taken as a subscript

$$G_{j+k+m+\cdots}.$$

$$\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{s=0}^{n}\binom{n}{s}a^s r^s b^{n-s}=\sum_{s=0}^{n}\binom{n}{s}a^s b^{n-s}\sum_{r=0}^{\infty}\frac{r^s}{r!}\cdot$$

LEMMA 1. $$G_{n+1}=(G+1)^n \qquad (n=0, 1, \cdots).$$

Write $a=b=1$ in Theorem 1 and multiply numerator and denominator of the summand by $(r+1)$ to reduce it to the form of the definition. This proposition, together with the fact that the first two G's are integers, insures that all the G's are positive integers.

The lemma gives us a ready method of computing the numbers. We find

$G_0 = 1$	$G_5 = 52$	$G_{10} = 115975$
$G_1 = 1$	$G_6 = 203$	$G_{11} = 678570$
$G_2 = 2$	$G_7 = 877$	$G_{12} = 4213597$
$G_3 = 5$	$G_8 = 4140$	$G_{13} = 27644437$
$G_4 = 15$	$G_9 = 21147$	$G_{14} = 190899322$

and so on.

It is evident that the G's increase very rapidly; this is reflected in

THEOREM 2. $$G_n \geqq k^n/k! \qquad (k=0, 1, \cdots).$$

For $k=0$ this is trivial; if it is true for some k, we have

$$G_n=\sum\binom{n-1}{r}G_r \geqq \sum\binom{n-1}{r}\frac{k^r}{k!}=\frac{(k+1)^n}{(k+1)!}\cdot$$

Unfortunately G_{n+1} is not a very good upper bound for the function $x^n/\Gamma(x)$; *e.g.*, the greatest value attained by the function $x^3/\Gamma(x)$ is approximately 13.56.

Lemma 1 generalizes inductively to

THEOREM 3.

$$\sum_{r=0}^{k}(-)^r\binom{k}{r}G_{n-r+1}=\sum_{r=0}^{n-k}\binom{n-k}{r}G_{n-r} \qquad (0 \leqq k \leqq n).$$

This is certainly valid for $k=0$. Now, let $F(n, k)$ denote either member of the equation which we assume true for this particular k. Then, by the well known properties of the binomial coefficients, we have

$$F(n, k) - F(n-1, k) = F(n, k+1),$$

which establishes the theorem, by induction on k.

The case $k=n$ is of some interest.

COROLLARY 1. $$G_n=G(G-1)^n.$$

We justify the title of the paper by

THEOREM 4. $$e^{e^x-1}=e^{Gx},$$

where, again, e^{Gx} *is the symbolic representation of*

$$\sum_{r=0}^{\infty} G_r x^r/r!.$$

For,

$$e^{e^x} = \sum_r \frac{1}{r!}\sum_s \frac{r^s x^s}{s!} = \sum_s \frac{x^s}{s!}\sum_r \frac{r^s}{r!}.$$

This theorem gives rise to a recurrence relation for the G's which is quite different from any we have yet stated.

THEOREM 5. $$G(G-1)\cdots(G-n+1)=1 \qquad (n=0, 1, \cdots).$$

We shall adopt the notation S_j^i to stand for the sum of all possible products of the numbers $1, 2, \cdots, (i-1)$, taken j at a time. Then, by a change of variable ($x=\log(1+u)$) in Theorem 4,

$$e^x = \sum \frac{G_r}{r!}\log^r(1+x) = \sum_r G_r \sum_n (-)^{n-r} S_{n-r}^n \frac{x^n}{n!}$$

$$= \sum_n \frac{x^n}{n!}\sum_r (-)^{n-r} S_{n-r}^n G_r,$$

whence,

$$\sum_{r=0}^{n} (-)^{n-r} S_{n-r}^n G_r = 1.$$

There is also an elegant symbolic proof of this.

$$e^x = e^{G\log(1+x)} = (1+x)^G = \sum_{n=0}^{\infty} G\cdots(G-n+1)\frac{x^n}{n!}$$

3. Number-theoretic properties. We are now in a position to discuss the curious arithmetic properties which these numbers possess.

LEMMA 2. $$G_p \equiv 2 \pmod{p},$$

where, as in all the following theorems, p *is a prime number.*

Write $n=p$ in Theorem 5. By Lagrange's proof of Fermat's Theorem, all the intermediate coefficients are congruent to zero, modulo p; the first to $+1$; and the last to -1.

This proposition is the starting point for a series of inductions, which culminates in Theorem 6.

LEMMA 3. $$G_{p+n} \equiv G_n + G_{n+1} \pmod{p}.$$

When $n=0$ this reduces to Lemma 2. Write $p+n-1$ for n, and p for k, in Theorem 3; then

$$G_{p+n} - G_n \equiv \sum \binom{n-1}{r} G_{p+n-r-1} \pmod{p}.$$

Now, assume that, for all integers $<n$ the statement is true. We then have

$$G_{p+n} - G_n \equiv \sum \binom{n-1}{r} G_{n-r-1} + \sum \binom{n-1}{r} G_{n-r} \pmod{p}$$
$$= \sum \binom{n}{r} G_{n-r} = G_{n+1}.$$

Letting $p=2$ we obtain the significant

COROLLARY 2. $\qquad G_n+G_{n+1}+G_{n+2}\equiv 0 \pmod{2}$.

LEMMA 4. $\qquad G_{kp^s+n}\equiv G^n(G+s)^k \pmod{p}$.

We shall show that its truth for fixed s, all n, and $k=1$, implies its truth for the same s, all n, and all k. For, assume it for some s and k, and all n; then

$$G_{(k+1)p^s+n} = G_{kp^s+(p^s+n)} \equiv G^{p^s+n}(G+s)^k \pmod{p}$$
$$\equiv G^n(G+s)(G+s)^k = G^n(G+s)^{k+1}.$$

Now, assume the theorem for all k, all n, and some s. Writing $k=p$,

$$G_{p^{s+1}+n} \equiv G_{p+n} + s^p G_n \equiv G_{n+1} + (s+1)G_n \pmod{p},$$

which, by our preliminary remark, implies its validity for all k, all n, and $s+1$. Since the statement is obviously correct for $s=0$, it is universally true by induction on s.

If the subscript of G is not of the form of the lemma, but is a polynomial in p, it can be reduced by considering everything after the leading term as n. Repeated application of the proposition yields

THEOREM 6. $\qquad G_{\Sigma k_r p^r}\equiv \Pi(G+r)^{k_r} \pmod{p}$,

where the limits of the summation and product are the same.

THEOREM 7. *The G's have a "congruence-period" of* $(p^p-1)/(p-1)$ *places; i.e.,*

$$G_{n+(p^p-1)/(p-1)} \equiv G_n \pmod{p} \qquad (n = 0, 1, \cdots).$$

For, by Theorem 6, we have

$$G_{p^{p-1}+\cdots+p+1+n} \equiv G^{n+1}(G+1)\cdots(G+p-1) \pmod{p}$$
$$\equiv G_{p+n} - G_{n+1} \equiv G_n.$$

We have shown therefore that the smallest period of the least residues of the G's is a divisor of $(p^p-1)/(p-1)$. Indeed, when $p=2$, 3, or 5, it is precisely this, although for $p=5$, either of the factors of 781, 11 and 71 might be a priori candidates. The author has succeeded in computing the least residue pattern in these

three cases, but since the latter is rather excessive, we give only the first two. For 2, it is

$$1,\ 1,\ 0$$

and for 3

$$1,\ 1,\ 2,\ 2,\ 0,\ 1,\ 2,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1.$$

For primes >5 the situation is unknown.

We close with what is, in view of Theorem 7, a natural generalization of Corollary 2. We are indebted to Dr. Irving Kaplansky for the proof.

THEOREM 8. *The sum of* $(p^p-1)/(p-1)$ *consecutive* G*'s is a multiple of* p.

We pass to the Galois field of p elements, so that congruence modulo p becomes equality. Solving the difference equation

$$G_{n+p} - G_{n+1} - G_n = 0$$

by standard methods, we find

$$G_n = a_1 x_1^n + a_2 x_2^n + \cdots + a_p x_p^n$$

where $x_1, \cdots, x_p$ are the (distinct) roots of

$$x^p - x - 1 = 0.$$

Now, precisely as in the case of the G's, we find

$$x^{p^n} = x + n,$$

so that

$$x^{(p^p-1)/(p-1)} = x(x+1)\cdots(x+p-1) = x^p - x = 1,$$

whence

$$1 + x + \cdots + x^{(p^p-1)/(p-1)-1} = \frac{1-1}{x-1} = 0$$

which proves the theorem.

DESCRIPTIVE GEOMETRY AS USED IN THE SLAUGHTER HOUSE

(As reported in the Kansas City Star to have been quoted from a 24-page booklet from the Office of Price Administration)

"Then all fat shall be removed which extends above a flat plane using the following two lines as guides for each edge of the plane: an imaginary line parallel with the full length of the protruding edge of the lumbar section of the chine bone which line extends one inch directly above such protruding edge; a line on the inside of the loin two inches from the flank edge, and running parallel with such edge for the full length of the loin."

Shades of Taurus—can it be that our classes soon will be filled with embryo butchers and meat-cutters, learning the rudiments of their trade via courses in descriptive geometry?

W. B. Campbell.

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

SOME SLIDE RULES

L. R. FORD, Illinois Institute of Technology

1. Introduction. The material to follow was presented twice during 1944 to mathematical clubs made up largely of high school teachers.* The rules that are described are, for the most part, applicable to problems in high school mathematics; and rules made by high school students were exhibited and used. Their design and construction serve to stimulate the interest of the student and to make demands upon his ingenuity.

The underlying theory of these devices is very simple. Unfortunately it is ordinarily not readily available in the students' text books. It is because of this situation, and at the request of several of the teachers, that this description is being written.

2. Scales. The theory of the slide rule is based on the concept of the scale. Familiar examples of scales are the common foot rules, thermometers, and the like, with the graduations marked upon them. These exemplify the scale as a line with certain marks upon it to which numbers are attached.

If we mark an origin O on the line and locate points of the line by their distances (with suitable sign) from O, then the distance X to the point marked x depends upon x. That is, X is a function of x,

$$X = f(x). \tag{1}$$

Conversely, we may assume some function $f(x)$ and construct a scale from it. Thus we take $x=0, 1, 2, \cdots$, or whatever values we are interested in, and compute the corresponding distances X. We lay these distances off from O, mark the points, and attach the values of x.

By taking various functions in (1) we can form the greatest variety of scales. The uniform scales of foot rule and thermometer arise from relations of the form $X=ax+b$. The scales of the logarithmic slide rule are based on the function $X=\log x$. The scales to be used in what follows will depend upon the diverse functions appearing in the problems to be solved.

3. The slide rule. The slide rule is a device for adding distances. Because the distances are those involved in scales the result is the addition of functions.

* Men's Mathematics Club, Chicago and Metropolitan Area, May 19, 1944. Women's Mathematics Club, November 4, 1944.

Consider two material objects, such as strips of wood or of pasteboard, which can be placed on opposite sides of a line along which they can slide freely. Place the objects together in an initial position and mark the common origin O upon the line. Measuring from O we construct upper and lower scales,

(2) $$X = f(x), \qquad X = F(y),$$

respectively.

Now, let one scale slide along the other to a new position, as in Figure 1. Let z in the upper scale coincide with y in the lower scale; and let u in the upper coincide with v in the lower. Since the distance from the mark z to the mark u is equal to the distance from the mark y to the mark v we have the equation

(3) $$f(u) - f(z) = F(v) - F(y).$$

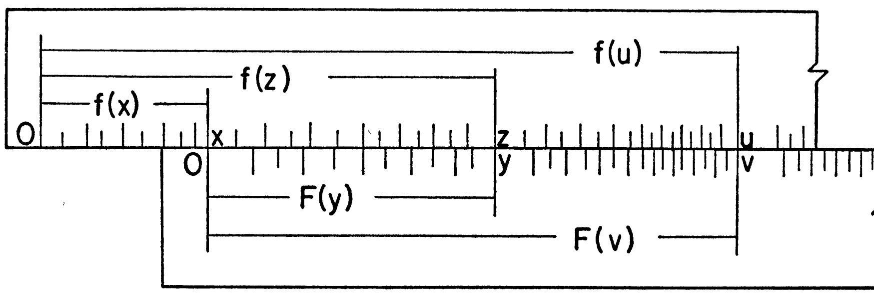

FIG. 1.

This rule enables us to solve equation (3) for any one of the four variables when the other three are given. For example, suppose u, v, z are given and it is required to find y. We bring u on the upper scale into coincidence with v on the lower scale; then opposite z on the upper scale we read the desired value of y on the lower scale. The problem is thus solved by one setting of the rule.

An equation in three variables results if the initial mark O, corresponding to a zero value of one of the functions, is used. Thus clearly, from the figure, we have

(4) $$f(z) = f(x) + F(y).$$

We can solve (4) for any of the three variables when the other two are given by one setting of the rule.

Many useful rules employ the same scale function for both scales. This will be true of the examples to be presently shown. If $f(x)$ is the function used then (3) takes the form

(5) $$f(y) + f(u) = f(z) + f(v);$$

and (4) becomes

(6) $$f(z) = f(x) + f(y).$$

4. The Pythagorean rule. The rule shown in Figure 2, which uses the scale function $X=x^2$, is one of the most instructive for the beginner. It is not difficult to construct and its uses are legion. We lay off from O distances, 1, 4, 9, 16, $\cdots$, 100, and mark them 1, 2, 3, $\cdots$, 10. We mark the fractional points of division but do not number them. Upper and lower scales are alike.

The figure shows how a folded piece of cardboard and a long strip sliding in the groove make up a fairly good mechanism. Each part carries a scale and the two scales slide freely along one another.

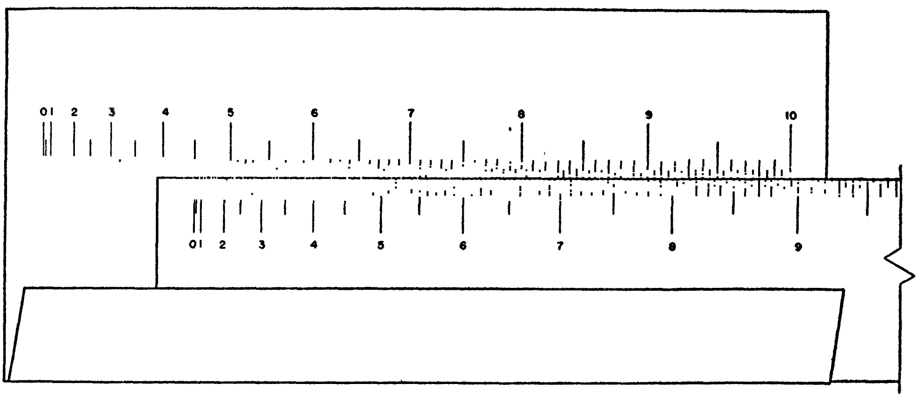

FIG. 2. $x^2+y^2=z^2$

Experience will suggest convenient ways of making the scales. Since the scales are alike they can be made by drawing markings across a line after which they are cut apart along this line and pasted in their places. Rough scales may be very rapidly made by drawing them on the white back of ruled paper held against a window. The rulings are visible while the scale is being made but are not visible when pasted on the rule. Good rules, however, take more care than this. In any case, a table of squares is very useful in the construction of this rule.

Equations (5) and (6) now become

$$y^2 + u^2 = z^2 + v^2, \tag{7}$$

and

$$z^2 = x^2 + y^2. \tag{8}$$

We can use the rule to solve for any one of the variables in either of these equations when the remaining variables are given.

Let us observe, first of all, that the numbers on the rule may all be multiplied by the same factor. Thus, 5, 6, 7, *etc.*, may be taken to represent 50, 60, 70, *etc.*, or, in general, $5k$, $6k$, $7k$, *etc.* This results from the fact that the equation satisfied by the variables is homogeneous in the variables. For the same reason, the numbers on each of the two rules described in the following sections may be

multiplied by a common factor. This change of scale is often resorted to in order to bring desired values within the range of the slide rule.

We give some problems which are solved by the rule as set in the figure.

What is the distance to the top of a tower 45 feet high from a point on the level ground 70 feet from the base?

We set 0 under 45 and read above 70 the answer, 83.2.

A force is the resultant of perpendicular components of 150 pounds and 160 pounds. How may it be broken into perpendicular components one of which is 120 pounds?

Here we divide by 2 to get convenient values. Setting 60 under 75 we read above 80 the value 91.8. The answer is twice this, or 183.6 pounds.

If $\sin\theta = .45$, what is the value of $\cos\theta$?

Since $\sin^2\theta + \cos^2\theta = 1$, we set 0 under .45 and read the answer under 1.00; namely, $\cos\theta = .894$.

Many other interesting applications will occur to the student. For example, knowing the radii of two circles and the distance between their centers he can read off the lengths of the internal and external tangents. He will be gratified by the rapidity with which he can get numerical answers based on applications of the theorem of Pythagoras.

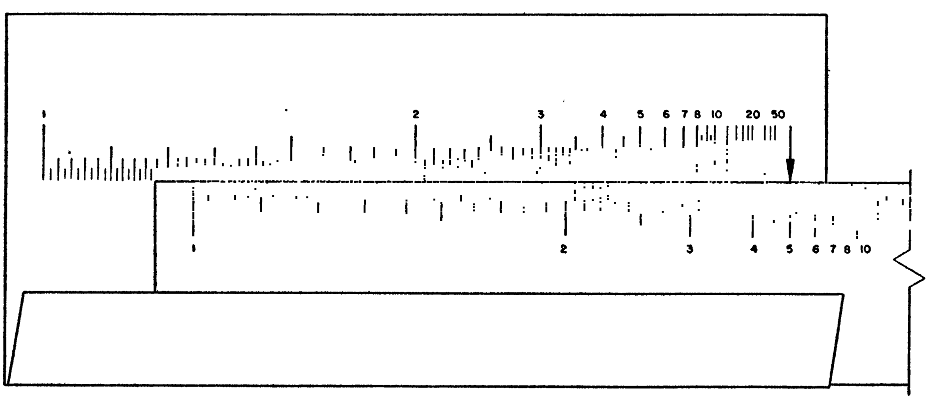

FIG. 3. $1/x+1/y=1/z$.

5. The Share-the-Work rule. If A can do a piece of work in x days and B can do it in y days, how long will it take them to do it working together?

Letting z be the required number of days, we add the fractional part of the whole that is done in a day by A to the part done by B to get the part done by both together:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}. \tag{9}$$

In Figure 3 is shown a rule for solving this problem. It is based on the scale

function, $X=1/x$, both above and below. The position marked O in earlier scales and from which distances are measured is marked here by an arrow. Distances are considered positive to the left. This was suggested by a high school student with the object of having the natural numbers run in their conventional order from left to right.

If A does the work in 5 days and B does it in 7 days we learn from the rule (reading just below 7) that both together will do it in 2.9 days.

This rule is useful in the study of lenses and mirrors. The distance p of an object from lens or mirror and the distance q of the image are connected with the focal length f by the equation

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}\cdot \tag{10}$$

For example, if an object is 5 cm. from a lens of focal length 4 cm., we read from the rule that the image will be at a distance 20 cm. from the lens.

If thin lenses of focal lengths f_1 and f_2 are in contact, the combination is equivalent to a single lens of focal length F, where

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{F}\cdot \tag{10'}$$

This is adapted to solution by the rule.

A similar formula arises in aviation. The radius of action R of a plane is the distance it can go and return in one hour. If the ground speeds out and in are V_0 and V_1 then

$$\frac{1}{V_0}+\frac{1}{V_1}=\frac{1}{R}\cdot \tag{11}$$

Another application appears in the case of resistances R_1 and R_2 in parallel. The total resistance R is given by

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{R}\cdot \tag{12}$$

Thus, resistances of 5 ohms and 3 ohms in parallel are equivalent (reading just below 3) to a total resistance of 1.87 ohms.

Finally, the capacitance of condensers connected in series combine according to the same law.

6. The Hypocycloid rule. If a line of length a moves with its ends on the x- and y-axes (Figure 4) the envelope is the hypocycloid of four cusps

$$x^{2/3}+y^{2/3}=a^{2/3}. \tag{13}$$

We shall assume this result from the calculus. It may be verified by finding the length of the tangent to (13) at a point (x_1, y_1),

$$x_1^{-1/3}x+y_1^{-1/3}y=a^{2/3}, \tag{14}$$

between the intercepts on the axes.

A rule for solving (13) is shown in Figure 7, the scale function being $X = x^{2/3}$ on both parts. This rule was described in the author's article on "Alignment Charts," Notre Dame Mathematical Lectures, No. 4, 1944. It solves various problems in the calculus.

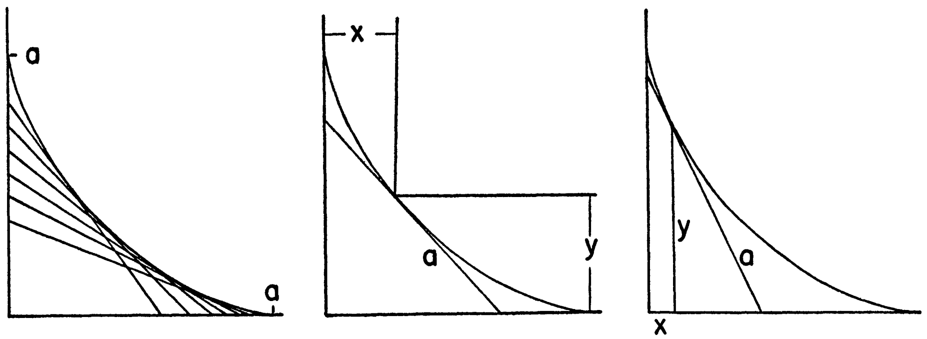

FIG. 4. FIG. 5. FIG. 6.

Find the length of the longest girder that can be moved from a corridor 6 feet wide into another 8 feet wide at right angles to it.

Figure 5 shows that we are led to the solution of (13). If the envelope of the line passes above the corner the line cannot make the turn; if it passes below there is room to spare. We are concerned therefore with the curve which passes through the corner. Setting 0 under 6 and reading above 8 we find 19.7 feet. The equation of the line in which the girder lies in the extreme position is given by (14).

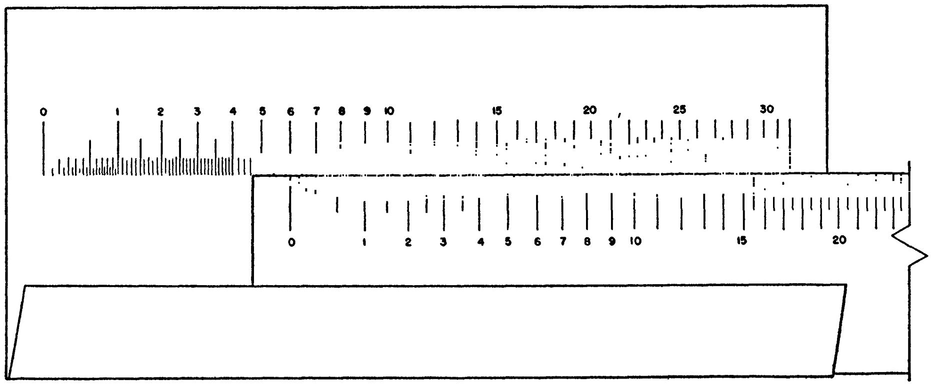

FIG. 7. $x^{2/3}+y^{2/3}=a^{2/3}$.

Find the length of the shortest ladder which will pass over a wall 6 feet high and lean against a wall 1.2 feet back of the first wall. Figure 6 shows that we have here the same sort of problem as before. Above 1.2 on the rule we read the answer 9.3 feet. The position of the ladder is found from (14).

This rule can be used to solve the problem of the shortest straight road across the corner of a rectangular field so as to pass a spring whose location is given.

Surfaces of similar solids are proportional to the two-thirds powers of the volumes. For example, what is the weight of a man whose skin area is equal to the combined skin areas of a boy of 60 pounds and a boy of 70 pounds? Calling a unit 10 pounds, the rule is set to give the answer, 184 pounds. This kind of problem is suggestive of basal metabolism.

7. Other slide rules. The student will think of many other possibilities. In trigonometry, the rule with the scales

$$X = \log x, \qquad X = \log \sin y$$

enables us to use the law of sines rapidly,

$$\log a - \log b = \log \sin A - \log \sin B.$$

A rule with the scales

$$X = \log x, \qquad X = \log \tan y$$

enables us to use the law of tangents; and so on.

Returning to Figure 1, it is clear that the scales where u and v are read may be made with different scale functions, $g(u)$ and $h(v)$. We thus have

$$g(u) - f(z) = h(v) - F(y).$$

The origins for the u- and v-scales should lie together initially, but need not be the same as the origins for the z- and y-scales. Further, they need not lie along the same line as before, but may slide along one another on some other part of the rule.

A confession. I have heard myself accused of being an opponent, an enemy of mathematics, which no one can value more highly than I, for it accomplishes the very thing [a theory of color to supplant Newton's] whose achievement has been denied me.—Goethe.

Academic applied mathematics. The tendency of the mathematician is to overrate the solidity of his theoretical structures, and to forget the narrowness of the experimental foundation upon which many of them rest.—Lord Rayleigh.

Litotes. The mind of the mathematician is subject to many disturbing causes, such as fatigue, loss of memory, and hasty conclusions; and it is found that, from these and other causes, mathematicians make mistakes.—James Clerk Maxwell.

Pythagoras returns. Nature is the numerable.—Spengler.

A geometer on algebra. Algebra which cannot be translated into good English and sound common sense is bad algebra.—W. K. Clifford. *Contributed.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE ALTITUDES OF A TRIANGLE AND OF A TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

THEOREM 1. *In a triangle ABC having altitudes AA', BB', CC', the perpendiculars drawn from the midpoints of $B'C'$, $C'A'$, $A'B'$ to the sides BC, CA, AB respectively, are concurrent at the radical center of the circles (A, AA'), (B, BB'), (C, CC') having centers A, B, C and radii AA', BB', CC' respectively.*

In fact, the circles (B, BB'), (C, CC') pass respectively through B', C', and the circle on BC as diameter passes through both B' and C'. The midpoint of $B'C'$ lies on the radical axis of the two circles, for if the line $B'C'$ meets these circles again in B'' and C'', the midpoint of $B'C'$ coincides with that of $B''C''$, and the theorem is established.

Remarks. The radical axes of the pairs of circles (B, BB') and (C, CC'), (C, CC') and (A, AA'), (A, AA') and (B, BB'), being parallel to AA', BB', CC', bisect three of the interior or exterior angles of the complementary† triangle of $A'B'C'$, so that *the radical center, Ω, of the three circles under consideration coincides with the center of one of the Taylor circles‡ of the fundamental triangle ABC.* Moreover, *the circle (Ω), orthogonal to the circles (A, AA'), (B, BB'), (C, CC'), belongs to the coaxal system determined by the circumcircle and the Lemoine line of triangle ABC.* Finally, Ω coincides with the radical center of the circles having centers A, B, C (or B, C, H) and tangent to the three sides of triangle $A'B'C'$, according as angle A is acute or obtuse, H being the orthocenter of triangle ABC.§

* Translated from the French by Howard Eves. The footnotes are due to the translator.

† Commonly called the *medial* triangle of $A'B'C'$; *i.e.*, the triangle formed by connecting the midpoints of the sides of $A'B'C'$.

‡ In fact, Ω coincides with the center of the circle well-known as *the* Taylor circle of the triangle ABC.

§ This last statement is in error and should read: *Finally, Ω coincides with the radical center of the three circles having centers A, B, C and tangent to the three sides of triangle $A'B'C'$.* When A, B, C are all acute these circles escribe $A'B'C'$; when A, B, or C is obtuse, two of these circles escribe $A'B'C'$ and one (that corresponding to the obtuse angle) inscribes $A'B'C'$. The Taylor circle of triangle ABC is, in either case, the radical circle of these three circles.

The literature dealing with Taylor's circle contains many confusing statements with regard to the acuteness and obtuseness of the given triangle. Thus, theorems 109 and 115 of Chapter I in Coolidge's A Treatise on the Circle and the Sphere (1916) are not true, as stated, if the fundamental triangle is obtuse.

THEOREM 2. *In a tetrahedron $ABCD$ having altitudes AA', BB', CC', DD', the planes perpendicular to the edges BC, CA, AB, DA, DB, DC and passing through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ respectively, are concurrent at the radical center of the spheres (A, AA'), (B, BB'), (C, CC'), (D, DD') having centers at A, B, C, D and radii AA', BB', CC', DD' respectively.*

In fact, the spheres (B, BB'), (C, CC') pass respectively through B', C', and the sphere on BC as diameter passes through both B' and C'. The midpoint of $B'C'$, then, lies on the radical plane of the two spheres, for if the line $B'C'$cuts these spheres again in B'', C'', the midpoints of $B'C'$ and $B''C''$ coincide, and the theorem is proved.

Remarks. The sphere (Ω), orthogonal to the four spheres (A, AA'), (B, BB'), (C, CC'), (D, DD'), belongs to the coaxal system determined by the circumsphere of the tetrahedron $ABCD$ and the harmonic plane of the point K for which the sum of the squares of the distances from the planes of the faces is a minimum (*the first Lemoine point*). When the tetrahedron $ABCD$ is equifacial,* the point Ω coincides with the radical center of the spheres inscribed in the truncs of the tetrahedron.

THE TRANSCENDENTAL CHARACTER OF COS X

R. W. HAMMING, University of Louisville

This note contains an entirely elementary proof of a theorem that does not seem to be well known.†

THEOREM. *The value of* $\cos (a^\circ\, b'\, c'')$, *where* a, b, *and* c *are integers, is an algebraic number.*

The proof is based on De Moivre's theorem:

$$(\cos x + i \sin x)^k = \cos kx + i \sin kx.$$

Taking real parts and noting that only even powers of sines occur we have

$$\cos kx = P_k(\cos x),$$

where $P_k(\cos x)$ is a polynomial in $\cos x$ with integral coefficients. Put $k = 90 \cdot 60^2$ and $x = 1''$. Then

$$\cos 90^\circ = 0 = P_k(\cos 1'').$$

Thus $\cos 1''$ is an algebraic number. Now using $k = 60^2a + 60b + c$ we have

$$\cos (a^\circ\, b'\, c'') = P_k(\cos 1'')$$

and hence it is also algebraic.

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* Such a tetrahedron is commonly referred to as an *isosceles* tetrahedron. See N. A. Court's Modern Pure Solid Geometry, art. 307, page 98.

† D. H. Lehmer, A note on trigonometric algebraic numbers, this MONTHLY, vol. 40, 1933, pp. 165–166.

The fact that the values of the trigonometric functions are algebraic numbers for all rational angles *in degrees* (the method of proof actually shows this) does not contradict the fact that the trigonometric functions are transcendental functions. The one statement says that at certain points the values are algebraic (of varying degrees), while the other says that no single algebraic equation $f(x, y)=0$ is satisfied by a trigonometric function for all x.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Nautical Mathematics and Marine Navigation. By S. A. Walling, J. C. Hill, and C. J. Rees. Cambridge, The University Press; New York, The Macmillan Company, 1944. 9+221 pages. $2.00.

This book is planned primarily for those who are anticipating some kind of sea service and find it advisable to review elementary mathematics and practice its application to problems of seamanship.

Part I, indicated by the title, includes a set of interesting and practical problems solvable by the four fundamental operations of arithmetic and simple geometric constructions. These problems range over a remarkably large number of subjects and their solution would make excellent preparation for nearly any work that would be encountered in port or at sea in any kind of ship.

Part II extends the use of the same elementary principles to the simpler problems of piloting but does not go into the field of celestial navigation.

It takes up such subjects as the history and description of the ships compass, the sextant and its use in measuring horizontal angles, variation, deviation, Mercator projection, lines of position, fixes, running fixes, and the use of vectors in determining the effects of currents. All these discussions are supplemented by well selected typical problems. The reading material is illustrated by sixty or more diagrams or drawings. The appendix, consisting of a table of square roots for use in visibility problems and a practice chart centering at Falmouth, England, is followed by a list of answers to all problems.

The Anglo-American authorship introduces a few terms and expressions which may not be familiar to all readers, but for those interested in ships in any way the study of this book will mean time spent in an entertaining and profitable way.

F. E. CARR

Aircraft Analytic Geometry. By J. J. Apalategui and L. J. Adams. New York, McGraw-Hill Book Company, Inc., 1944. 18+285 pages, $3.00.

As stated in the preface, this "book constitutes a new approach to a certain class of problems which arise in the engineering, lofting, and tooling of airplanes." The new approach consists in the application of plane and solid analytic geometry to certain aircraft problems, some of which are enumerated below. It is further stated that "the book is intended for use by men in the lofting, tool designing, and jig building departments and will also be very useful for men in the layout and development groups of the engineering department." Students of descriptive geometry will be interested in the methods given to simplify the checking of layouts made by descriptive geometry. We shall give a brief résumé of each chapter.

Chapter 1 treats of the elements of plane trigonometry, including fundamental identities, interpolation, functions of angles which differ by a multiple of 90°, a chart of 32 figures for solving right and oblique triangles, *etc.* The graphs of sin x and cos x on page 12 are rather inaccurate as measurement of sin 45° and the slope at the origin show.

Chapter 2 consists of the plane analytic geometry of the straight line. Some formulas and relations are derived while others are stated without proof. A few so-called derivations are meagre. For example, on page 27 the point-slope equation is derived by merely substituting x and y for x_2 and y_2 in the formula for the slope. Just a few words of explanation would clarify this, or else the "derivation" might well be omitted entirely.

Chapter 3, Cartesian Coördinates, is an introduction to three-dimensional rectangular coördinate systems as applied to the airplane and its parts. First a system of rigged axes (axes in flying position) is set up for the airplane as a whole. Two principal wing designs are described, and a system of coördinate axes is defined for each. Then, for convenience in analyzing, the airplane is divided into the fuselage, the wings, the nacelles, and the empennage. Axes are defined for each of these divisions, and their relations to the rigged axes are discussed.

Chapter 4, True Lengths and True Angles, comprises an introduction to the space geometry of the straight line, the direction ratios forming the basic concept. (Incidentally, it would seem to be a simplification in conception to speak merely of the direction numbers 2, 4, 5 of a line rather than of the direction ratios 2:4:5, since we need never be concerned with the ratios 2:4 or 2:5; most modern texts in analytic geometry appear to agree with this suggestion.) The formula for "true length" (distance) is derived, direction cosines introduced, and theorems on projections discussed and then applied to the angle between two lines.

Chapter 5 applies the principles of Chapter 4 to determine the direction numbers of various wing axes with respect to the rigged axes, as well as to find the true angle between a line and a plane or between two planes under various hypotheses.

Chapter 6, Equations of Planes, starts with the normal form and then verifies it by a projection theorem of Chapter 4. The usual forms of plane-equations are

given with more than usual emphasis on the determination of angles, this apparently being of great importance in aircraft construction.

Chapter 7, Equations of Lines, gives the standard forms and shows how to compute the distance from a line to a point and between two skew lines.

Chapter 8, Translation and Rotation of Axes, gives derivations of the rotation formulas for transforming from various sub-assembly axes to the rigged axes. The tabular form given in each case has the advantage of being immediately adaptable for transformation from either system of axes into the other.

Chapter 9 deals with various applications of the principles of the preceding chapters. Some of these include the checking of the solution of a problem solved by descriptive geometry and the determination of the angles between certain basic planes in the wing or in the fuselage. The preparation of basic data tables, the master diagram, and the loft layout are explained and illustrated, and various uses of these are given.

Chapter 10 gives a brief treatment of the projective geometry of conic sections. The Theorems of Pascal and Brianchon are stated and illustrated, and these are then applied to various problems in the construction of conics which are to satisfy certain initial conditions. The importance, and advantages of conic sections in aircraft design are pointed out, among these being increased production and interchangeability of parts resulting from mass-production methods.

Chapter 11 deals briefly with the analytic theory of conics, one application of which is the computation of "offsets" (ordinates) at various "stations" (abscissas) for lofting. Matching of conics—that is, the fitting together of two conics at a given point so that they will have a common tangent and the same curvature there—is discussed with some calculus. In this chapter as in the one preceding, results and methods are given for the most part without proofs.

The Appendix contains a summary of forty formulas from analytic geometry of which ten are for the rotation of various axes. There is also a 5-place table of the first four natural trigonometric functions for each minute.

Some features of the book might well be improved in a later revision. For instance, it seems to the reviewer that it would be highly desirable to include precise definitions of various aircraft terms (perhaps in the form of a glossary) such as root chord, tip chord, wing reference plane, common percent plane, *etc.*, which are, admittedly, shown on drawings. Although this is not a text in aeronautics, the lack of these definitions is confusing to the uninitiated reader because these terms are basic to many of the fundamental systems of axes discussed; it would perhaps not be too much to add that even the well initiated reader would derive some benefit from such definitions. Some explanations in the book are unnecessarily long; for example, the derivation of the expression for z_w on page 125 requires a quarter of the page, whereas it is an immediate consequence of eliminating y_w from the two equations immediately above. On page 127 there is the misleading statement that the equations of a line are three in number without mentioning the fact that only two are independent until the end of Example 4 on page 128.

As a whole, in spite of an opposite impression that the preceding paragraph might give, this book shows painstaking execution. Discussions are clear, and there is a copious supply of examples illustrating almost every section. These generally consist of some which are purely mathematical and others applied to aircraft. The reviewer checked through a few of the computations and found no errors. No actual misprints were detected. There are a few minor anomalies, such as the splitting of sin A at the end of line 13, page 66, and that of $y_1 \cos \beta$ near the bottom of page 135. There are many excellent drawings, which are neatly labeled and accurately done. The typography is all that could be desired. Exercises are provided in Chapters 1, 2, and 9.

This book should do much in promoting the use of analytic geometry in the more scientific development and construction of airplanes.

F. A. BUTTER, JR.

NEW BOOKS RECEIVED

College Algebra and Trigonometry. By F. H. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1945. 12+324 pages. $3.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 676. *Proposed by T. R. Running, University of Michigan*

Show that in using the Gregorian calendar the first day of a century cannot occur on a Sunday, a Wednesday, or a Friday.

E 677. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the altitudes of a tetrahedron $ABCD$ meet the circumsphere again in the points A', B', C', D'. Show that the volume of the solid $ABCDA'B'C'D'$ is three times that of the given tetrahedron.

E 678. *Proposed by H. D. Grossman, New York City*

Prove that the number of circular permutations of $n=\sum_{i=1}^{k} a_i$ objects of which a_1 are alike, a_2 are alike, *etc.*, is

$$\frac{1}{n}\sum_{d}\phi(d)\,\frac{(n/d)!}{(a_1/d)!(a_2/d)!\cdots(a_k/d)!},$$

where d ranges over all divisors of the g.c.d. of $a_1, a_2, \cdots, a_k$.

E 679. *Proposed by Marshall Naul, Cumberland, Md.*

Compute the length of the curve of intersection of the unit sphere around the origin with the right helicoid $z=\theta$.

E 680. *Proposed by Gordon Pall, McGill University*

Prove that a real determinant of order 6, with elements numerically not exceeding unity, cannot have a value greater than 160.

SOLUTIONS

A Duodenary Square Ending in Zero

E 642 [1944, 530]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the duodenary scale, a certain number does not end in 0, but its square does. Show that the square ends in 30, and the cube in 60.

Solution by Monte Dernham, San Francisco. Excluding 0, we find 6 to be the only digit whose square is divisible by twelve. Working in the duodenary scale, we have

$$(10N+6)^2 = 100N^2 + 100N + 30,$$

$$(10N+6)^3 = 1000N^3 + 1600N^2 + 900N + 160.$$

Thus the square of every duodenary number ending in 6 terminates in 30, and the cube in 60.

Also solved by Murray Barbour, Colin Blyth, Leon Bunyan, D. H. Browne, W. E. Buker, Frank Hawthorne, Irving Kaplansky, Samuel Kramer, W. R. Ransom, E. D. Schell, E. P. Starke, Albert Wilansky, and the proposer.

Positive Terms of a Determinant

E 644 [1944, 530]. *Proposed by A. W. Goodman, Republic Aviation Corporation*

Do there exist determinants of order $n>2$, such that all the $n!$ terms of the expansion are positive?

Solution by Irving Kaplansky, Columbia University. No. Any 2×2 minor would necessarily have three terms of like sign, *viz.*, either three $+$'s and a $-$ or three $-$'s and a $+$. But it is already impossible to arrange a 3×2 matrix whose three 2×2 minors all have this structure.

Also solved by Murray Barbour, D. H. Browne, Howard Eves, Daniel Finkel, J. B. Kelly, E. D. Schell, Lowell Schoenfeld, E. P. Starke, and the proposer.

Editorial Note. This naturally raises the question, What is the maximum difference between the numbers of positive and negative terms which a determinant of order m can give on expansion? Or, equivalently, What is the largest value a determinant can have, if every element is either 1 or -1? We know that this value cannot exceed $m^{m/2}$, which is the value for the determinant of a "U-matrix" in the sense of Paley, "On orthogonal matrices," *Journal of Mathematics and Physics* (M. I. T.), vol. 12 (1933), pp. 311–320. (A U-matrix becomes an orthogonal matrix, in the strict sense, when all its elements are divided by $m^{1/2}$. The determinant is then ± 1. Hence before division the determinant was $\pm m^{m/2}$.) It can hardly be doubted that the value $m^{m/2}$ is actually attained whenever m is a multiple of 4. It certainly is when $m=4, 8, 12, \cdots, 88$, and in infinitely many higher cases; but a U-matrix of order 92 has not yet been constructed. Nor is it yet known how far the maximum value for a determinant of ± 1's can fall short of $m^{m/2}$ for intermediate values of m; *e.g.*, such a determinant of order $m=6$ cannot exceed 160, although $m^{m/2}=216$.

A Quadrangle with Given Sides and Area

E 643 [1944, 530]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

The sides a, b, c, d of a plane quadrangle being given in order and also the area A, find the length of one of the diagonals.

Solution by H. N. Carleton, West Newbury, Massachusetts. In order to make the expression less cumbersome, we define

$$p = (d^2 + a^2)/4, \quad q = (c^2 + b^2)/4, \quad r = (d^2 - a^2)/4, \quad s = (c^2 - b^2)/4.$$

After some straightforward though voluminous work (which has been suppressed for the sake of brevity) we find that, if such a quadrangle exists, its diagonal from (a, b) to (c, d) may have either of the values

$$\sqrt{\frac{2\{(p+q)A^2+(p-q)(r^2-s^2)\}\pm 2A\sqrt{-A^4+2(2pq-r^2-s^2)A^2-2(p-q)(ps^2-qr^2)-(r^2-s^2)^2}}{A^2+(p-q)^2}}.$$

These are the positive roots of the equation

$$\{A^2 + (p-q)^2\}x^4 - 4\{(p+q)A^2 + (p-q)(r^2-s^2)\}x^2 + 4\{A^4 + 2(r^2+s^2)A^2 + (r^2-s^2)^2\} = 0.$$

Conjugate Sections of a Sphere

E 646 [1944, 586]. *Proposed by Orrin Frink, Jr., Xenia, Ohio*

Prove that any two conjugate planes through a secant of a sphere meet the sphere in orthogonal circles.

Solution by J. H. Butchart, Grinnell College. Let the two conjugate planes p and q intersect in a line which meets the sphere in the point L. Let P be the pole of p. Since P is the vertex of a right circular cone whose base is the circle cut from the sphere by p, it is clear that LP is orthogonal to this circle at L. Since q passes through LP, which is a tangent to the sphere, LP is a tangent to the

circle cut by q from the sphere, and the circles cut by p and q are orthogonal.

Also solved by Howard Eves and J. M. Feld. For a proof using coordinates, see Graustein, *Introduction to Higher Geometry* (New York, 1940), p. 456, Theorem 8.

Barycentric Coordinates with Respect to a Regular Tetrahedron

E 647 [1944, 586]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let (m_1, m_2, m_3, m_4) be the barycentric coordinates of a point G with respect to a regular tetrahedron $A_1A_2A_3A_4$ of edge a (so that G is the centroid of masses m_1, m_2, m_3, m_4 at A_1, A_2, A_3, A_4). Obtain an expression for the distance A_4G.

Solution by Howard Eves, Syracuse University. We shall employ the following theorem: For any point P,

$$\sum (m_i \cdot A_iP^2) = \sum (m_i \cdot A_iG^2) + \sum m_i \cdot GP^2. \tag{1}$$

(McClelland, *Geometry of the Circle*, Art. 55, gives a proof of this theorem for the plane. The same method can be used in three or more dimensions.) Now, if P coincides with A_1, A_2, A_3, A_4 in turn, the relation (1) yields

$$\begin{aligned} a^2(s - m_1) &= (s + m_1)A_1G^2 + m_2A_2G^2 + m_3A_3G^2 + m_4A_4G^2, \\ a^2(s - m_2) &= m_1A_1G^2 + (s + m_2)A_2G^2 + m_3A_3G^2 + m_4A_4G^2, \\ a^2(s - m_3) &= m_1A_1G^2 + m_2A_2G^2 + (s + m_3)A_3G^2 + m_4A_4G^2, \\ a^2(s - m_4) &= m_1A_1G^2 + m_2A_2G^2 + m_3A_3G^2 + (s + m_4)A_4G^2, \end{aligned}$$

where $s = m_1 + m_2 + m_3 + m_4$. Solving these equations for A_4G^2, we find

$$A_4G^2 = a^2(m_1^2 + m_2^2 + m_3^2 + m_2m_3 + m_3m_1 + m_1m_2)/(m_1 + m_2 + m_3 + m_4)^2.$$

This result is easily extended to the regular simplex in n-space.

Also solved by the proposer.

A Trigonometrical Determinant

E 648 [1944, 586]. *Proposed by Mary L. Boas and R. P. Boas, Jr., Tufts College and Harvard University*

Show that, when $x = 2 \cos \pi/(n+1)$, the n-rowed determinant

$$\begin{vmatrix} x & 1 & 0 & 0 \cdots 0 & 0 \\ 1 & x & 1 & 0 \cdots 0 & 0 \\ 0 & 1 & x & 1 \cdots 0 & 0 \\ 0 & 0 & 1 & x \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ 0 & 0 & 0 & 0 \cdots x & 1 \\ 0 & 0 & 0 & 0 \cdots 1 & x \end{vmatrix}$$

has the value zero.

Solution by J. B. Kelly, Langley Field, Va. Set $x=p+p^{-1}$, and let P_n denote the n-rowed determinant. Then

$$P_1 = x = p + p^{-1}, \qquad P_2 = x^2 - 1 = p^2 + 1 + p^{-2}.$$

Since $P_n = xP_{n-1} - P_{n-2}$ and $p^{n+1} - p^{-n-1} = (p+p^{-1})(p^n - p^{-n}) - (p^{n-1} - p^{-n+1})$, we can prove by induction that

$$P_n = p^n + p^{n-2} + \cdots + p^{-n+2} + p^{-n} = (p^{n+1} - p^{-n-1})/(p - p^{-1}).$$

Set $p = e^{\theta i}$, so that $x = 2\cos\theta$. Then

$$P_n = \frac{\sin (n+1)\theta}{\sin \theta},$$

which vanishes when

$$\theta = \frac{\pi}{n+1}.$$

Also solved by J. H. Butchart, Paul Civin, A. B. Farnell, N. J. Fine, Marguerite Z. and E. A. Hedberg, G. A. Hedlund, Irving Kaplansky, O. E. Lancaster, Leonard McFadden, M. F. Smiley, Alan Wayne, and the proposers.

Editorial Note. The following alternative solution is less significant, but rather entertaining. If $x = 2\cos\theta$, we have (identically)

$$\begin{array}{lr}
x \sin\theta - \sin 2\theta & = 0, \\
\sin\theta - x\sin 2\theta + \sin 3\theta & = 0, \\
- \sin 2\theta + x \sin 3\theta - \sin 4\theta & = 0, \\
\cdots\cdots\cdots\cdots & \\
\pm \sin (n-1)\theta \mp x \sin n\theta \pm \sin (n+1)\theta & = 0.
\end{array}$$

If $\theta = \pi/(n+1)$, the last term of the last equation vanishes, so we can formally eliminate $\sin\theta : \sin 2\theta : \cdots : \sin n\theta$ from the n equations (with that term omitted), obtaining the desired result immediately.

Hedlund remarks that

$$P_n = x^n - \binom{n-1}{1} x^{n-2} + \binom{n-2}{2} x^{n-4} - \cdots.$$

In other words, $P_n = U_n(x/2)$, where U_n is the Tschebyscheff polynomial of the second kind. (*Cf.* E 620 [1945, 44–46] and E 629 [1945, 100–101].) The determinantal expansion for $\sin (n+1)\theta$ was first given by Studnička in 1897.

An Almost Linear Set of Segments

E 649 [1944, 586]. *Proposed by L. A. Santaló, Rosario, Argentine Republic*

A set of parallel line segments will be called "linear" if all of them can be cut by one straight line. Show by an example that an infinite set of parallel segments in one plane may have the property that every subset of three is linear while the

whole set is not linear. (The segments are understood to be "open": not including their end points.)

I. *Solution by N. J. Fine, Lukas-Harold Laboratory, Indianapolis.* Let $f(x)$ be any real, positive-valued function such that $\lim_{|x|\to\infty} f(x)=0$. Then the set of ordinates $\{0<y<f(x)\}$ satisfies the required conditions. As a matter of fact, every finite subset is "linear."

A discrete set having the same property may be obtained by selecting those ordinates for which x is integral.

II. *Solution by Howard Eves, Syracuse University.* The segments

$$a_0,\ a_1,\ a_2,\ \cdots,\ \text{and } b,$$

defined as follows, are seen to satisfy the requirements.

$$a_i:\quad x = 2^{-i},\qquad 0 < y < 1;$$

$$b:\quad x = -1,\qquad 2 < y < 3.$$

Also solved by L. M. Kelly and the proposer. No *finite* set of segments can have this property; nor can any set of *intervals* (or "closed" segments). In other words, if every three of the intervals are linear, then the whole set is. See L. A. Santaló, "Complemento a la nota: sobre conjuntos de paralelepípedos de artistas paralelas," *Publicaciones del Instituto de Matemáticas*, vol. 3, no. 7 (1942); *Mathematical Reviews*, vol. 4 (1943), p. 112.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4161. *Proposed by R. E. Gaines, University of Richmond*

Along a straight road a miles long there are n persons. What is the probability that no two persons are less than b miles apart?

4162. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove, by the methods of real projective geometry, that if a projectivity $P \barwedge P'$ on a conic is not an involution, the envelope of PP' is a conic. (For a complex proof, see H. F. Baker, *Principles of Geometry*, Cambridge 1922 or 1930, p. 52.)

4163. *Proposed by W. H. Cullum, Washington Navy Yard*

In a vertical plane a heavy particle moves along a mathematical curve under the force of gravity, from a point A to a point B. The distance measured along the curve from A to B is the given length L. The initial velocity is V_0 while the velocity V at any point is such that $1/V-\lambda>0$, where λ is a positive quantity. Determine the time of passage from A to B such that it is a maximum.

4164. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a number of twelve digits $a_1a_2a_3a_4b_1b_2b_3b_4c_1c_2c_3c_4$ which is a perfect cube, and the cube root is the sum of the three numbers $a_1a_2a_3a_4$, $b_1b_2b_3b_4$, $c_1c_2c_3c_4$.

4165. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A and B be two fixed points of a given circle while C and D are two variable points of the same circle such that the arc length CD remains constant. The orthogonal projection of D on AC is P, of C on BD is Q. Show that, (1) The Simson lines of C and D for the triangles ABD and ABC have fixed directions. (2) The centers of the circles tritangent to DPQ (inscribed and escribed) describe two Pascal limaçons.

SOLUTIONS

Cube of Special Form

4114 [1944, 168]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are taken each once and only once to form three numbers N, n, r so that $N=n^3-r$. Find these three numbers.

Solution by Marion L. Gaines, Student, University of North Carolina. Since every number is congruent to the sum of its digits mod 9, we have

$$n + r + N \equiv \sum_{v=1}^{9} v \equiv 0 \pmod 9.$$

On the other hand

$$N \equiv n^3 - r \pmod 9.$$

Hence:

$$n + r + n^3 - r \equiv n(n^2 + 1) \equiv 0 \pmod 9.$$

Since -1 is a quadratic non-residue mod 3, n^2+1 is not divisible by 3 and n must be divisible by 9.

If $n>100$, then $n^3=N+r>100^3$. Then n contains at least 3 digits and N or r at least 7 digits. This is impossible. Therefore we have to consider only the cases $n=9a$ where a is $1, 2, \cdots, 9$.

The cases where $n=99$ or $n=90$ are impossible since n cannot contain either two 9's or a 0.

If $n=81$, then $n^3=531441$. Since n contains 2 digits, N and r must contain together 7 digits and one of them must contain exactly 6 digits in order that

the sum have 6 digits. The other number has only one digit and is therefore $\leqq 9$. The six digit number must contain the 1 at the fourth place from the right since by adding a number $\leqq 9$ we cannot raise the fourth place digit. This is impossible since 81 already contains the 1.

By the same reasoning the cases $n=72$, $n^3=373248$ and $n=54$, $n^3=157464$ are impossible since we would use 7 and 5 respectively twice.

For $n=36$, $n^3=46656$ it is evident that N and r contain 5 and 2 digits. Since the 6 has already been used we cannot obtain the 6 at the fourth place.

For $n=63$, $n^3=250047$. N or r must contain 0 at the fourth place. This is impossible.

For $n=45$, $n^3=91125$. N or r must have a 1 at the third and fourth place. This is impossible.

$n=9$ contains only one digit. Since $n^3=729$, the numbers N and r can have only 3 digits each. This is impossible.

For $n=18$, $n^3=5832$. We can use as last digits of N and r only the combinations 5, 7 and 3, 9. Since we have to carry over a 1 to the next place we can use only the same combinations in order to obtain this 3. Therefore at the last two places in both numbers, there are contained the odd numbers 3, 5, 7, 9. Since the 1 is already used we would have only even integers at the third place. Therefore we could not obtain the 8 since 1 has to be carried over.

Finally, we have to the case $n=27$, $n^3=19683$. It is easy to see that one of the numbers N or r must contain 5 digits, the other 2. The 5 digit number must start with 195 or 196. The 195 is impossible since none of the possible combinations gives us 3 for the last digit. For 196 we obtain the following four solutions:

19635 and 48; 19638 and 45; 19645 and 38; and 19648 and 35.

Solved also by Monte Dernham, W. W. Gandy, R. W. Hamming, Irving Kaplansky, J. B. Kelly, E. D. Schell, E. P. Starke, E. C. Varnum, R. H. Wilson, Jr., and the proposer.

A System of Tetrahedrons

4116 [1944, 234]. *Proposed by N. A. Court, University of Oklahoma*

Given the tetrahedron $(T_1)=SA_1B_1C_1$, the tangent place to its circumsphere at the diameter opposite of S meets the edges SA_1, SB_1, SC_1 in the points A_2, B_2, C_2. The tangent plane to the circumsphere of the tetrahedron $(T_2)=SA_2B_2C_2$ at the diametric opposite of S meets the edges of (T_2) through S in the points A_3, B_3, C_3 thus forming the tetrahedron $(T_3)=SA_3B_3C_3$, *etc.* Find the locus of the incenters of these tetrahedrons.

Solution by the Proposer. The tangent plane $A_2B_2C_2$ to the sphere $SA_1B_1C_1$ at the diametric opposite of S is parallel to the tangent plane to the same sphere at the point S, hence the plane $A_2B_2C_2$ is antiparallel to the plane $A_1B_1C_1$ for the trihedral angle $S-A_1B_1C_1$ (See the proposer's *Modern Pure Solid Geometry*, p. 247, art. 759). Similarly the tangent plane $A_3B_3C_3$ will be antiparallel to $A_2B_2C_2$ and therefore parallel to $A_1B_1C_1$.

Thus the tetrahedrons (T_1), (T_3), (T_5), $\cdots$ will be formed by a fixed trihedral angle $S-A_1B_1C_1$ and a series of parallel planes; hence these tetrahedrons will be homothetic. Consequently homologous points of these tetrahedrons, and in particular their incenters, will lie on a straight line passing through S.

Similarly for the tetrahedrons (T_2), (T_4), (T_6), $\cdots$.

Thus the required locus consists of two straight lines passing through the point S.

Note. The corresponding problem in the plane was considered in the *Journal de mathématiques élémentaires*, 1888, p. 233, Q. 250.

Editorial Note. This theorem may be extended to euclidean space of n dimensions. Let $T^i=(S, A_1^i, A_2^i, \cdots, A_n^i)$ be a non-degenerate simplex with the circumsphere (O^i) and diameter SO^iS^i; then $(SA_j^i)(SA_j^{i+1})=(SS^i)^2$, $(SA_j^{i+1})(SA^{i+2})=(SS^{i+1})^2$, and hence $(SA_j^i)/(SA_j^{i+2})=(SS^i)^2/(SS^{i+2})^1$, $1\leqq j\leqq n$. Therefore S is the center of similitude for T^i and T^{i+2}, and the theorem follows.

Commutative Transformations of a Polygon

4117 [1944, 234]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A polygon $A_1A_2 \cdots A_n$ may be transformed into a polygon $B_1B_2 \cdots B_n$ by locating the points B_i on the sides A_iA_{i+1} so that the ratio of A_iB_i to B_iA_{i+1} is equal to a constant r. Prove that if T_1 and T_2 are two such transformations for the ratios r_1 and r_2, then $T_1 \circ T_2 = T_2 \circ T_1$, and generalize.

Solution by R. C. Buck, Cambridge, Mass. Regard the vertices A_i as being determined by vectors, which need not lie in the same plane, or, in n-space, the same subspace. Then,

$$B_i = (1-r)A_i + rA_{i+1} = (1-r+rE)(A_i)$$

where $A_{n+1}=A_1$, and $E(A_i)=A_{i+1}$. Since E acts on the A_i only, and $T=1-r+rE$ is linear with *constant* coefficients, any collection of such operators commute. The same would be true for any transformation T of the form $T=P_0+P_1E+P_2E^2+\cdots+P_mE^m$, where the P_i are left unchanged by E and its powers.

Solved also by Howard Eves, Michael Goldberg, and the proposer.

Editorial Note. The solutions by Eves and Goldberg are similar and use vectors; Eves states in addition to the above extensions that $A_1, A_2, \cdots, A_n$ need not be points, but any elements (such as lines, circles, etc.) linear combinations of which yield the same kind of element. The proposer considered only plane polygons and used rectangular coordinates. The case was considered where the triangles $A_iA_{i+1}B_i$ are similar and constructed all exteriorly, or all interiorly, for a given transformation. In this case we may also use vectors or complex coordinates, but the expressions are much simpler for the latter, and the use of Buck's E adds additional simplicity. Thus $A_t=\rho_te^{i\theta_t}$, $\alpha=$angle $A_{t+1}A_tB_t$, $r=A_tB_t/A_tA_{t+1}$, $\lambda=re^{i\alpha}$; then $B_t=[1+\lambda\Delta]A_t$, $\Delta A_t=A_{t+1}-A_t$, and $T=1+\lambda\Delta$. The desired result is obvious.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The Jagelonski Uniwersytet in Cracow will shortly resume a full schedule of activities under the rectorship of Professor Ler-Splawinski. Thirteen of the professors died in the concentration camp at Oranienburg and five others shortly after returning to Cracow; Antoni Hoberski is known to have perished. Even during the occupation secret work was furnished by the university to eight hundred students directed by over two hundred professors and scientists.

Professor R. P. Agnew of Cornell University has been awarded the degree of Doctor of Science by Allegheny College.

Dr. Mary D. Clement of Wells College has been promoted to an assistant professorship.

Professors G. C. Evans and Jerzy Neyman of the University of California are on leaves of absence for work with the War Department and the N.D.R.C. respectively.

Professor Tomlinson Fort of Lehigh University has been appointed professor of mathematics and head of the department at the University of Georgia.

Assistant Professor W. C. Janes of Kansas State College has been granted a leave of absence for research work with the Beachcraft Corporation of Wichita, Kansas.

Dr. Charles Loewner of Brown University has been appointed to an associate professorship at Syracuse University.

Professor L. J. Mordell of the University of Manchester has been elected Sadleirian Professor of Mathematics at Cambridge University, succeeding Professor G. H. Hardy.

S. T. Parker of the University of Louisville has been promoted to an assistant professorship.

Associate Professor George Polya of Stanford University has been appointed to a visiting professorship at the University of California.

Professor O. H. Rechard of the University of Wyoming has been appointed dean of the liberal arts college.

Assistant Professor C. E. Rhodes of Union College has been appointed professor of mathematics and head of the department at Washington College, Chestertown, Maryland.

G. F. Rose has been appointed by George Washington University to serve as research associate in the Allegheny Ballistics Laboratory, Cumberland, Maryland.

Dr. Stanislaw Ulam of the University of Wisconsin has been appointed to an associate professorship at the University of Southern California.

Professor Oscar Zariski of the Johns Hopkins University has been appointed to a visiting professorship for one year at the University of São Paulo, Brazil.

Dean Emeritus T. F. Holgate of Northwestern University died April 10, 1945. He was a charter member of the Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE NATIONAL ROSTER

The National Roster of Scientific and Specialized Personnel is the division in the Bureau of Placement of the War Manpower Commission charged with the responsibility of obtaining the most effective utilization of professional and scientific personnel during the war period. When established in June, 1940, it was set up under the joint auspices of the United States Civil Service Commission and the National Resources Planning Board, because at that time it was believed that the Federal Government would be the principal user of the specialized personnel that would register with the Roster. It soon became apparent, however, that the need for these specialists would be greatest in industry and in the Armed Forces and, accordingly, when the President by Executive Order established the War Manpower Commission in April, 1942, he set forth in that order that the National Roster should be transferred from the Civil Service Commission to the War Manpower Commission. Dr. George A. Works is now Director of the National Roster; he replaced Dr. Leonard Carmichael on November 1, 1944. The address of the Roster is 1066 U Street, N.W., Washington 25, D. C.

The National Roster's principal task has been to register, recruit, and place professional and scientific personnel in those jobs in which they could best aid the war effort. At present, the Roster has registered and maintains in its records the education, experience, and training backgrounds of approximately half a million of the country's professionally qualified individuals. The data in these

records are kept current by the Roster's practice of recircularizing its registrants annually. The data obtained in this manner enable the Roster to carry on such important functions as: (1) the gathering and analysis of statistics to determine the characteristics and employment distribution of the various professions, (2) the making of studies of supply and demand in the important professions, (3) the publication of vocational guidance pamphlets in these professions, (4) the preparation of descriptions and definitions of the professions and their fields of specialization, (5) the provision of a free public placement service for the professional and scientific personnel of the country.

In performing its placement service, the National Roster carries on two types of activities: (1) the recruitment of personnel to fill specific jobs for which orders have been placed, and (2) positive placement designed to locate suitable positions for individuals who have indicated that they are or will soon become available for employment. The Roster receives job orders from private industry either directly or through the United States Employment Service, from federal agencies either directly or through the United States Civil Service Commission, from the Army and the Navy for military personnel usually to be considered for commissioning, from educational institutions, and from state and local governments. When a job order is received, a search is first made of papers maintained in a special file in which are contained brief statements of qualifications of individuals who have indicated that they are actively seeking employment or a change in employment. If the order can not be filled from this source, a machine run is made from the Roster's punch-card records, and qualified registrants are selected. The detailed records of these registrants are then examined to make certain that they possess the necessary qualifications and also to determine if the transfer would be in the interest of the war effort. As a result of this examination, a number of registrants are tentatively selected for referral. In most cases, these registrants will be employed in war jobs and it is important to determine their availability for transfer. The Roster's procedure is to write the registrant's employer, indicate the nature of the war job which has to be filled, and ask the employer whether he can grant his employee a release. Most employers are cooperative and if they believe the job to be filled is more important to the war effort than the job the employee is performing, they will agree to release him. The employer will give to his employee a form sent to him by the Roster on which there will be a brief description of the job in question and the employee will be asked to indicate on this form whether he is interested in being considered for transfer. In the event he notifies the Roster of his willingness to be considered for the position, his entire record is then referred to the establishment which has placed the job order with the Roster. This establishment will then deal directly with the registrant. Should the employer be a federal agency, the papers will be referred through the United States Civil Service Commission; referrals to industrial establishments are made through the United States Employment Service; military assignments are referred to the Army and Navy; and negotiations with colleges and universities are made directly.

Professionally qualified persons may write the Roster that they are seeking employment or they may register with the nearest local office of the United States Employment Service which will in turn notify the Roster. The Roster and the United States Employment Service have developed working relationships which make it possible for the local office to quickly refer job orders and job applicants to the Roster. Upon receipt of such notice, all current job orders in the Roster are checked, and if the individual is qualified, his papers are immediately referred for consideration. If there is no current job order for which he is qualified, a search is made through prior orders and other information which the Roster has on hand to locate establishments where individuals with such qualifications have been employed. When these have been located, a record of the registrant's qualifications is sent to such employers for consideration. In effect, the Roster solicits jobs for these people. If neither of the above steps produces results, a brief record of the individual's qualifications is transmitted to a few selected local United States Employment Service offices which are requested to inquire among employers in their respective areas for possible employment.

Professionally qualified members of the Armed Forces who are about to be discharged are invited to avail themselves of the Roster's services. At present, the Roster has a number of standing orders from large industrial concerns for engineers, chemists, and physicists. Orders have and are now being received specifying a desire for veterans. These positions offer opportunities for war work now and for permanent employment in the postwar period.

Registration with the Roster on December 31, 1944

	Registration			*Median Age*	*Education*	
	Total	*Male*	*Female*		*Doctors*	*Masters*
Total Registration	439,757	420,120	19,637	37.9	37,434	59,918
Mathematics	14,610	12,699	1,911	36.6	1,826	5,188
Actuarial Science	882	848	34	41.0	23	188
Statistics	2,435	1,978	457	37.2	282	634

AMERICAN-SOVIET SCIENCE COMMITTEE

The Science Panel of the Congress celebrating the tenth anniversary of U. S. recognition of the Soviet Union was held on November 7, 1943. Shortly thereafter a permanent Science Committee, affiliated with the National Council of American-Soviet Friendship, was formed with the purpose of facilitating scientific exchange in both directions between the U.S.A. and the U.S.S.R. The purposes of the permanent Committee were laid down as follows: "The primary purpose of the Science Committee of the National Council of American-Soviet Friendship is to bring the work in natural science in these two countries into closer and more continuous contact. It proposes to accomplish this purpose by facilitating the exchange between the U.S.A. and the Soviet Union of scientific

publications, of scientific personnel, and of students of natural science, and by acting as a clearing center for information from, and about science in the U.S.S.R. For the present, it proposes to restrict its activities to the natural sciences—astronomy, mathematics, physics, chemistry, biology, geology, and the basic sciences on which medicine, agriculture, and technology rest. Its most pressing immediate problem is the exchange and diffusion of scientific knowledge between the U.S.A. and the U.S.S.R., chiefly through publications and through exchange visits of scientists, both during and after the war." A large number of American scientists, including mathematicians, are members of the Science Committee. Headquarters of the Organization are located at 232 Madison Ave., New York 16, N. Y.

The Committee recently started the publication of a bulletin which may be of interest to many scientists. A feature of each number is a bibliography of books and manuscripts recently received from the U.S.S.R. The following quotations are from the first two issues of the bulletin.

"The Science Section of the USSR Society for Cultural Relations with Foreign Countries, generally known as VOKS, regularly communicates with our Committee and has sent us scientific journals and books, including those issued in 1944, together with scientific manuscripts which we are asked to prepare for publication in the U.S.A. We also receive their valuable periodical, *Science in the USSR*, which is published in English and contains scientific news and reports of a general nature. Scientific material received by the U. S. Department of State and by the Soviet Embassy is also sent to this Committee for abstracting, review, and distribution to American scientists. In return, reprints and gifts of books and periodicals for libraries damaged by the war are being sent from the U.S.A. to the Soviet Union."

"Any American scientist, interested in the work which his Soviet colleagues are now carrying on, is requested to write for information to the Science Committee. Members have the privilege of requesting specific material from the Soviet Union. A temporary repository for a number of the books, journals, and papers from the Soviet Union, has been arranged at the American-Russian Institute, 58 Park Avenue, New York City, for the convenience of Committee members. Material may be borrowed for a limited, but sufficient length of time, in return for a small fee to cover mailing charges. Requests should be addressed to the Science Committee."

"An arrangement has been made by which the Science Committee will turn over to *Mathematical Reviews* one copy of each mathematical journal received from the Soviet Union. Each paper will be reviewed promptly in *Mathematical Reviews*. A microfilm copy of the Russian original of each paper, together with the journal, will be deposited in the Brown University Library, which will furnish microfilm copies at the usual rates. Journals such as *Doklady*, which contain articles in other fields as well as mathematics, and physical and astronomical journals will also be sent to *Mathematical Reviews*, which will review those papers of interest to mathematicians. The tables of contents of such

'mixed' journals will be published in the Bulletin of the Science Committee. The journals will then be deposited with the Brown University Library, which, upon order, will furnish a microfilm or photostat copy of any paper at the usual rates. Correspondence concerning all papers so listed may be addressed to the Librarian, Brown University, Providence, R. I."

"A catalogue from the Soviet Union announces a new newspaper and magazine subscription service conducted by Kniga, Moscow. . . . Besides journals in economics, sociology, and political science, there are listed 11 in the physical sciences and mathematics; . . . . Two scientific publications contain articles in English, French, or German: *Acta Physiocochimica URSS* and *Journal of Physics*. Catalogues, with subscription prices, may be obtained from the Four Continent Book Corporation, 253 Fifth Avenue, New York City. Due to a United States post office ruling, Four Continents is, however, unable to enter any subscriptions to Russian journals. Persons wishing to subscribe must enter their own subscriptions by writing directly to Kniga in Moscow, enclosing a bank draft covering subscription costs."

CONGRESSIONAL STUDY OF HIGHER EDUCATION

House Resolution 63 of the 79th Congress authorized the appointment of an Advisory Committee of educators to conduct a study of the effect of the war upon higher education. This Committee recently made its report. Among the specific recommendations contained in the report are the following.

"The Committee recognizes the present serious shortage of manpower to meet the demands of war. It specifically calls attention, however, to the rapidly increasing shortage of men in professional fields essential to the national welfare, such as, medicine, dentistry, engineering, physics, chemistry, divinity and others. With each year of war the gap in the flow of young men into these essential fields becomes a more serious threat to the national health, safety or interest. The Committee recommends that: *A*. At the earliest possible time Selective Service reestablish student deferment, for those majoring in fields essential to the national welfare and for which extended periods of training are necessary. *B*. In plans for National Service, provision be made in the legislation for the exemption, for the period of their training, of students for fields essential to the national welfare and for which extended periods of training are necessary.

"The War Department has announced its plan of a point system as a basis for discharge of military personnel. In the plan as stated, no provision was made for preferential discharge of former students preparing for fields essential to the national welfare. Since such men have partly completed their education, preferential discharge will lessen the gap in the flow of trained men and women into such fields. The Committee recommends that: The War and Navy Departments include in their bases for discharge, consideration of the educational plans of those who have completed two or more years of college education in essential fields and who will continue their education in these fields after their discharge.

"It is essential in the national interest that higher educational institutions

continue to function at a high level of effectiveness. This can not be done if faculty members in areas of teacher shortage are inducted into the armed forces or national service. The Committee recommends that: *A*. Faculty members teaching in essential fields be deferred to meet the educational needs of veterans and others. *B*. In plans for National Service faculty members of higher educational institutions, teaching in necessary fields, be considered as being engaged in an essential activity.

"In certain fields of instruction in higher educational institutions there is now a serious shortage in teaching personnel. The Committee recommends that: Members of faculties of higher educational institutions whose services are requested by the institutions be given priority in release from military duty or other government positions.

"The war demonstrated the effectiveness of national coordination and cooperation in carrying forward research essential in the national welfare. The gains made through the successful operation of the Office of Scientific Research and Development should be retained and should be broadened to include research in the social sciences and humanities. Higher educational institutions are the principal agencies for the training of scientific and research workers of the future. It is therefore essential that as much scientific work as possible be widely distributed among the higher educational institutions in order (*a*) To accomplish definite objectives in specific instances of research; and, (*b*) To utilize effectively the staff at the institutions so as to extend and continue the effective training of a competent corps of research workers. The administration of such research programs might well be through a special agency operating somewhat as the existing Office of Scientific Research and Development. The Committee recommends that: A federal research agency be established or designated by the Congress and directed to use, on a contractual basis, higher educational institutions for developing and conducting of research and the training of research workers. Funds should be appropriated for such research to be used for specific programs when approved by the research agency.

"When World War II was declared, no plan had been prepared for the effective utilization of institutions of higher education. To avoid the repetition of such lack of planning, and in order that the universities and colleges may be utilized most effectively in the national interest during a period of declared emergency, the Committee recommends that: A committee representing the educational institutions and the armed services prepare a unified plan, which should be revised periodically, for using the colleges and universities in declared national emergencies."

THE ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The seventh annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco on Saturday, January 27, 1945. The Chairman of the Section, Professor Gabor Szegö, presided during the first part of the morning session, and Professor R. M. Robinson presided for the remainder of the morning session. Mr. S. A. Francis presided at the afternoon session.

The attendance was sixty, including the following twenty-two members of the Association: H. M. Bacon, G. A. Baker, H. W. Becker, S. A. Francis, F. T. Frank, Emma V. Hesse, D. H. Lehmer, Sophia Levy McDonald, E. D. Miller, F. R. Morris, W. H. Myers, George Polya, R. M. Robinson, S. A. Schaaf, Ethel Spearman, Pauline Sperry, Ruth G. Sumner, Gabor Szegö, K. J. Waider, R. K. Wakerling, Harriet A. Welch, A. R. Williams.

The following officers were elected for the coming year: Chairman, Pauline Sperry, University of California; Vice-Chairman, W. H. Myers, San Jose State College; Secretary-Treasurer, H. M. Bacon, Stanford University; representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor A. C. Schaeffer of Stanford University gave an hour's address during the morning session.

The following papers were presented:

1. *An application of the conchoid of Nicomedes*, by Professor F. R. Morris, Fresno State College.

It was pointed out that the problem of deciding whether a board can be moved from one corridor into another at right angles to the first is, under certain assumptions, equivalent to that of finding the maximum distance of a point on the loop of the conchoid from the axis of the loop. This principle was used to illustrate three methods of solving the ladder problem.

2. *Heat conduction problems and the Laplace transform*, by Dr. S. A. Schaaf, University of California.

The speaker stated that in recent years Carslaw, Churchill, and others have developed a method for solving boundary value problems by use of the Laplace transform. The method is particularly well adapted to problems in heat conduction, where its application has yielded solutions of many difficult problems. The method was described in general terms, and several simple illustrative examples were given.

3. *Differentiable functions*, by Professor A. C. Schaeffer, Stanford University, introduced by the Secretary.

Professor Schaeffer gave an elementary proof of a classical theorem of Serge Bernstein which states that if in an interval $a \leqq x < b$ all the derivatives of a function $f(x)$ exist and are non-negative, then the function is represented by its Taylor's expansion in powers of $x-a$ in the interval $a \leqq x < b$. This theorem fills a definite need in elementary calculus courses, and its proof is not beyond the range of beginning students. It is customary in introductory calculus courses, when expanding tan x or sec x in Maclaurin's series, to find only the first few terms, and omit any discussion of convergence. The theorem of Bernstein reveals that the Maclaurin series for tan x converges to tan x when $0 \leqq x < \pi/2$. It is then quite easy to prove that the series represents tan x when x lies between $-\pi/2$ and zero. A similar procedure may be used to find the interval of convergence of the Maclaurin's series for sec x, or of the Taylor's series in powers of $x-(\pi/2)$ for cot x and csc x. The theorem of Bernstein has motivated a conderable amount of interesting research in recent years. There are several closely related problems which are still open questions.

4. *Combinatory analysis in electronics*, by H. W. Becker, Mare Island Fire Control School, United States Naval Training School.

The speaker remarked that electronic components may be represented by two types of symbols, namely, the schematic and the mathematical. The schematic diagram is best for the mechanic or student. But by assigning mathematical symbols to components as operands and connections as operators, one can represent and solve a circuit in the same line of type. This is the approach of Riordan, Shannon, Kron, and Becker. The procedure can be applied to pararithmetic, network severances, selector switch algebra, splices in Latin space, Bell's numbers, and operational equations.

5. *Constructions of triangles under certain conditions*, by Professor A. R. Williams, University of California.

It was explained that the title of this paper was intentionally non-committal. Professor Williams remarked that a triangle with given sides may be constructed on the sphere by use of a graduated great circle, and a triangle with given angles by use of the polar triangle. The principal purpose of the paper was to give some idea of the value of the study of non-euclidean geometry. The construction of a triangle on the hyperbolic plane when three angles (whose sum is less than two right angles) are given was explained. The construction of Liebmann by use of a series of related right triangles was given.

6. *Teacher's organizations, their place in the school program*, by Mrs. Ruth G. Sumner, Oakland High School.

Mrs. Sumner spoke of the various organizations whose membership is open to teachers, and called attention to those from whose activities the teacher of mathematics receives impetus for professional growth. She described briefly the

studies which are being carried out by some of the mathematical associations, and pointed out the desirability of coordinating the efforts of all the committees now studying problems which are more or less identical.

7. *Recursive functions*, by Professor R. M. Robinson, University of California.

The speaker considered only functions with natural numbers for arguments and functional values. A recursive function was defined to be one which can be obtained from the constant, identity, and successor functions by repeated substitutions and recursions. A typical recursion schema is

$$F(x, 0) = G(x), \qquad F(x, y+1) = H[x, y, F(x, y)],$$

which defines $F(x, y)$ in terms of $G(x)$ and $H(x, y, z)$. It was shown that $x+y$, xy, x^y, $x!$, $|x-y|$, and $[y/x]$ are recursive functions. For example, the recursive definition of x^y is $x^0=1$, $x^{y+1}=x^y \cdot x$. It was shown that to define $[y/x]$ we may take

$$G(x) = 0, \qquad H(x, y, z) = z + 0^{|(y+1)-x(z+1)|}.$$

H. M. BACON, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley January 26, 1946
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

BOOK NEWS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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AN OBJECTIVE IN EDUCATION*

C. C. MacDUFFEE, University of Wisconsin

1. A philosophy. Early in February a mimeographed letter was sent to more than a hundred national and sectional officers of the Mathematical Association of America asking for suggestions in formulating the policy of the Association with respect to post-war educational problems. The first and primary question was, "Do you believe that the Association should, through the action of its committees, keep in close touch with developments in the educational world and take an active part in moulding opinion?" The reply to this question was an emphatic and unanimous "yes." Since those replying undoubtedly constitute a fair sample of the total membership of the Association, the officers feel that they have a mandate for action.

In these days when education seems to have so many aims and objectives, it was interesting to note that the members of the Association are pretty thoroughly agreed upon a fundamental philosophy of education. This is not, as one might perhaps have believed, that all students be required to take mathematics. It is, on the contrary, the belief that education should be tailored to the capacity of the individual. In this belief the members of the Association seem to have the support of many educators in other fields.

While this talk derived much inspiration from the replies to the above mentioned letter, I wish to assume full responsibility for the remarks that follow. The Association is in no degree committed to agree with them.

2. A problem. The philosophy of an education fitted to the capacity of the student has, up to the present, been largely developed in the interests of the weaker students. This is of course an essential step in the unique experiment in universal education for which the schools of the United States are now a vast laboratory. It is an experiment which would be attempted only by a great democracy, and it is an experiment which must be successfully completed if democracies are to survive and flourish.

But in solving the problem of the poorer student, difficulties have been placed in the path of the superior student which in some instances make it difficult for him to obtain the education of which he is capable. Because the superior students are in the minority, and because they do not become problem children, they are often woefully neglected. Because advanced subjects in a small high school have small registration, they are often abandoned and the time of the teacher is "more economically employed" with large classes of "future citizens."

It seems evident that the people of the United States do not see with sufficient clarity that the education of leaders is an objective compared with which the education of the masses is of lesser importance. This may sound like heresy in modern America, but it is an historical fact. Nations have become prominent

* An address delivered before a joint meeting of the Wisconsin Section of the Mathematical Association of America, the Mathematics Section of the Wisconsin Education Association and the Mathematics Club of Milwaukee at the Milwaukee State Teachers College on May 5, 1945.

and prosperous without universal education, but no nation has ever been able to maintain its position as a civilized community without a group of talented and educated leaders. America can and must solve both problems.

Let us recall that the history of the world is largely the history of individual men, many of whom are remembered only through their works. Modern civilization has been developed and is being maintained by relatively few persons. For each branch of science, the arts, philosophy, government, and military science it would be possible to name a hundred men without whose contributions the world would be appreciably poorer. Where would American technology now be if every person listed in the biographical book *American Men of Science* had failed to attend college? It is interesting to speculate on the present state of our civilization if one man in a thousand, since 10,000 B.C., properly selected, had failed to mature intellectually. It is quite possible that we should still be roaming the woods in skin clothing. Whether we would be happier in that condition than we now are is beside the point.

It is pertinent to note that the present apex of American military might was achieved without compulsory peace-time military training. A vitally important factor in our success was the work of American scientists in developing new weapons of offense and defense. Indeed, this was probably the decisive factor in the war—a fact which will some day be clearer than it is now to the general public. If compulsory military training has the effect of delaying or in any way interfering with the development of our skilled scientists, the act will be as disastrous as the destruction of the proverbial goose with the talent for laying golden eggs.

Even if we suppose that the gifted student eventually reaches the university, the years of time which he has lost are never recovered. In no direction is American wastefulness more costly than in the waste of precious years in the schooling of the superior child. Contrary to popular superstition, it does not injure the brain of a child to allow him to absorb knowledge at his natural rate of speed, even though this be five times the rate of his slightly retarded classmates. On the other hand, to allow a superior child to coast along without the exercise of effort is distinctly detrimental.

The fact that Europe has produced men of great genius more abundantly than has America cannot be due to differences in native ability, for we are of the same stock. It is clearly due to the fact that the better European schools are geared to the pace of students of higher average ability, and about two years are saved in their elementary education. These two precious years which American students lose are never recovered, and are subtracted from their years of greatest productivity.

There are some who, with a befuddled conception of the meaning of democracy, argue that a special education for the gifted means the production of an intellectual aristocracy and is therefore undemocratic. The effect of such thinking is to limit all opportunities of leadership to those men whose families are sufficiently wealthy to send their sons to private schools and colleges, a plan

which is wasteful of precious talent and is the very antithesis of democracy. Genius does frequently appear among the children of the poor, and a truly democratic educational system will provide adequate education for the development of such genius.

In advocating the double-track curriculum in the secondary schools, we must be careful not to call the express track the "college preparatory" track. A superior student in high school has a right to a superior education even if he is financially unable to attend college. The fact that his financial situation is more subject to alteration than is his mental equipment should impel a teacher to be very cautious about advising a capable student to elect a vocational course. Many of life's little tragedies occur when students discover that their high school work is of no use to them in preparation for their college work.

3. **A solution.** It is far easier to state a problem than to solve it, and I have to approach this problem cautiously. The City of New York has reached a satisfactory solution with its highly trained and adequately paid teachers, its special schools for the gifted as well as for the handicapped, and its system of city colleges which admit only those graduates of the local high schools who have attained high grades in a substantial college-preparatory course. But the small high schools of northern Wisconsin, for instance, present a problem which cannot be solved in the same manner.

One possibly satisfactory answer to this problem would be in the establishment of a few centralized schools throughout the state which would take the better student at the tenth grade level and fit him for college. This idea is not new; it is, to take one instance, the motivation of the establishment of the college at the University of Chicago. President Hutchins sees very clearly that the last two years of high school are frequently very wasteful of a talented student's precious years of youth, and he has provided an answer.

A means of accomplishing this end might be the junior college offering four years of work from the eleventh to the fourteenth grades inclusive. The junior college movement had considerable impetus a few years ago and this has not been entirely lost. But if the junior college is merely to offer a diluted form of college education for the student who is too weak for the university, it will not serve the purpose with which we are now concerned. Such an institution would merely extend the student's high school education by two years. If the junior college is to be the solution of the problem of the superior student, it must have high entrance requirements, a substantial curriculum from which the unessential courses have been eliminated, and a competent faculty who are able to stimulate and enthuse the students.

At the beginning of the eleventh grade the tone of the school should change from the juvenile to the adult. The student by this time should be made to realize that scholarship is a dignified career, fully as worthy of masculine attention as football or class politics. That the scholars are the people who keep the sacred fire of culture burning is a fact which merits as much time for indoctrina-

tion as the nutritional value of cheese.* The age in which we live is a serious age. Competition between men and nations is deadly. The transition which our eighteen-year-olds had to make from the childish atmosphere of the school to the brutality of army life might have been easier if the schools had not for years pretended that realities need not be faced, that difficult subjects may be skipped, that students who fail must nevertheless be passed along for fear of consequences to the personality. The epoch of such nonsense is past, and I believe that the American people are now in a mood to face facts, and to demand that their children be prepared in the schools to face life as it is, not a fairy-land life in which one can avoid the consequences of stupidity and laziness.

4. A liberal education. For many years a college education meant the traditional course in liberal arts with Latin and Greek and the Roman and Hellenic cultures as a background. It was and still is a most excellent course, demanding that the student give his best efforts and rewarding him with a stereoscopic view of our language, history and civilization. The universities must preserve this course for those who will take it.

In the last two hundred years, our civilization has completely changed. Sciences which were unknown in 1700 now determine the fate of nations and guide the daily lives of men. While the traditional liberal arts course is still important, it is not sufficient as it once was to interpret to the individual the world in which he lives. If the universities would reexamine and restate their objectives, perhaps the high schools would be able to see more clearly what their procedure should be.

I believe that from now on the central course of study in the colleges should be a course in liberal science not leading to any specialization but serving to interpret to the student the marvelous civilization of which we all are conscious, but which is incomprehensible at least in part to most of us. I can see the raising of eyebrows by my colleagues in the humanities, the familiar charge that the mere scientist is a technician, not a truly educated man who feels the finer things of life. I am not impressed. Unless the humanitarian understands the forces which keep the earth in its orbit, he is not a truly educated man. The pure sciences, taught with the objective of explaining basic principles, have as much aesthetic value as the humanities, and they have a grandeur all their own.

This course in liberal science should include work in physics and chemistry, and at least an introduction to astronomy, geology, biology and physiology. These should not be of the I-can't-tell-you-why-because-you-wouldn't-understand-it type of course, but courses where facts are proved. Astronomy, that now neglected gem of the sciences, should be highly recommended. There is no course which the humanitarians can offer which has quite the same power to orient a student with respect to his environment and reduce him to a position of humility as astronomy. Mathematics through differential equations is, of course, essential for all the physical sciences.

* There is an uninforced law in Wisconsin that all pupils shall be instructed each week in the salubrity of dairy products.

It goes without saying that the course should include English and a modern language and history, but possibly all of these courses can be given the emphasis demanded by their position in a course in liberal science. Composition and speech remain important, but the detailed study of English literature may have to be sacrificed as was the literature of classic times. Similarly language study should give more attention to the scientific literature than to archaic forms. History should include a description of modern research in pre-history and much emphasis on the history of science. In fact, a course in the senior year in the history of science given to students who are familiar with the sciences should be one of the high spots which make the course a success. Have you a pet course which should be included? There is room for electives.

An engineering course of two years' duration following a course in liberal science such as I have described would produce a man well-rounded and generally educated, and technologically superior. The investment of an extra two years of study would be well worth while and as a matter of fact the double-track scheme of elementary and high school education could save for the superior child the two extra years required.

The vogue of education-by-stuffing is definitely on the wane. The great corporations who now take so many of the graduates of our engineering schools have come to desire the students of superior ability who have done well in the basic science courses. Courses in engineering practice are of some importance, but many corporations prefer to initiate the junior engineer into their own methods and techniques. Mathematics and the basic sciences are the subjects which count. All this argues in favor of the course in liberal science.

5. Mathematics. I am not going to put forth any arguments in favor of the teaching of mathematics in our public schools. If there is anyone present who does not realize that we are living in the age of science, and that science rests on a mathematical foundation, I know of no language by which I can reach him. There are in our schools students who cannot learn mathematics. It is doubtful if they are capable of fundamental thinking in any direction. They are destined to be drawers of water and hewers of wood in our civilization. They should be given generous and sympathetic attention so that they may become useful citizens. They will not become leaders.

There are other students who have in some way acquired a distaste for mathematics and a conviction that they are unable to master it. They are the casualties of poor teaching. Under normal conditions they pass through life firm in the conviction that they are not mathematically minded. During the days of the ASTP I encountered numerous instances which showed exactly where the trouble lay. These boys have many times assured me that they hated mathematics, had no ability for it, and that only the power of the Army of the United States could persuade them to take it. But in spite of the fact that the ASTP curriculum was an educational monstrosity, many of these boys became mathematical fans. Recently I received two letters from the Pacific theatre from boys who had been in my ASTP class in calculus. One stated that he wanted to return

to Wisconsin and take all the courses in mathematics and chemistry that were given. The other asked for the title of a book on non-euclidean geometry.

If I were asked for suggestions for the improvement of the teaching of mathematics in high school, I should say that we need more teachers who have a burning enthusiasm for the subject. If they love the subject for its own sake, they will continue to study it during their teaching years, and they will be able to communicate their enthusiasm to the student. Certainly mathematics is important and useful, but more than that, it is fun. This is the motive which, more often than any other, determines a person's choice of vocation. Many majors in the university are determined by the personality of a favorite teacher.

Those of us who appreciate mathematics should insist that more mathematics, not less, be offered in the post-war schools and that all students who can profit by it be encouraged to partake. There is strong evidence that the war has awakened America from its world of fancy and that the country is ready to face realities. Mathematics has been belittled by some educational experimenters in the last twenty years, but the demands of the war have brought confusion upon them. As Kipling stated it,

"It's 'Tommy' this, and 'Tommy' that,
And 'Throw him out, the brute!'
But it's 'Savior of his country'
When the guns begin to shoot."

6. Looking ahead. The four years of mathematics which high schools that wish to be designated as college-preparatory must offer should prepare a student to begin calculus upon entering college. In no other way can the student who takes physics in his freshman year avoid the loss of a year. Physics taught without calculus is a lean and ineffectual thing, and I sometimes believe that it does more harm than good in fixing rationalizations rather than fundamental methods in the student's mind. The synchronization of courses in physics and mathematics is one of the problems now crying for solution. This was done to some extent in the Army and Navy programs and was, I believe, one of the high spots in the programs.

Analytics is now being taught in the fourth year in some of our large city high schools, in many preparatory schools, and in the schools of Ontario. It is not an untried experiment. It supplants the traditional course in advanced algebra, accomplishes the same objective as regards manipulative skill, and by furnishing a motive, increases the interest of the student.

There is one objection which is brought up against the teaching of analytic geometry in high school, and that is, as one man expressed it, "The first persons to study it would have to be the teachers." If that is true, I think I am in favor of putting analytic geometry into all the high schools, for I am sure that all high school teachers of mathematics should know analytics. If teachers were recruited from the ranks of those who had taken the course in liberal science which I described, they would be thoroughly qualified in technical skill, breadth of vision and, I believe, enthusiasm.

7. Summary. It appears that more and more educators are abandoning the thesis that all students should be forced to go as far as they can through the college preparatory high school courses. The principle of the course tailored to fit the capacity of the student seems to be attracting more and more adherents, not only among high school teachers, but also among college teachers. This plan of a double-track curriculum promises excellent results provided the express track is not neglected in favor of the freight track, and provided that students of ability are persuaded to purchase Pullman tickets and not bills of lading.

There can be no real permanent threat to the teaching of mathematics in our schools, since our culture is essentially a scientific culture resting upon a mathematical base. The best way that mathematics teachers can circumvent ill-considered attacks upon mathematics is to support sound educational principles in general without too much emphasis upon mathematics in particular. If collegiate instruction were built around a non-specialized course designed to interpret modern civilization to the student, such a course would necessarily contain courses in science and mathematics. It would require in preparation a full four years of mathematics in high school, and would provide an ideal training for prospective high school teachers of science and mathematics.

Irrespective of how our problems are eventually to be solved, it seems desirable that teachers of mathematics at all levels should now work together in the interests of sound education, and should join forces with persons of similar ideals and objectives in other branches of learning. United behind well-defined objectives, the educational societies of America will be able to keep education in this country on a rational basis.

CONVERGENCE OF MULTIPLY-INFINITE SERIES

I. M. SHEFFER, Pennsylvania State College

1. Introduction. There are various definitions of convergence for multiply-infinite series. Generally speaking, a definition first defines a class P of permissible partial sums (a partial sum always consisting of a finite number of terms). Then the series is said to converge to the sum A if to every $\epsilon>0$ there corresponds a suitable subclass P_ϵ of partial sums taken from P, such that the inequality

$$|A-s|<\epsilon \tag{1.1}$$

holds for every partial sum s belonging to P_ϵ.

The class of series that converge according to a given definition will vary with the class P of permissible partial sums, and as class P is more and more restricted, the class of corresponding convergent series will be more and more inclusive.

An example of a widely used definition (the *Pringsheim* definition) is that

where P consists of all *rectangular sums*. Let the series be $\sum a_{ij}$, where double series are used for simplicity. By R_{pq} is meant the rectangular sum

$$R_{pq} = \sum a_{ij} \qquad (0 \leqq i \leqq p; 0 \leqq j \leqq q); \tag{1.2}$$

and $\sum a_{ij}$ is said to converge to A if to every $\epsilon > 0$ there correspond indices p, q such that

$$|A - R_{rs}| < \epsilon \tag{1.3}$$

for all $r > p$, $s > q$.

If a series converges according to this definition, the simple series forming the terms of a given row or column need not converge. For example, the definition gives convergence (to the sum 0) to the series whose first two rows are

$$\begin{gathered} 0! + 1! + 2! + \cdots + n! + \cdots \\ -(0!) - (1!) - (2!) - \cdots - (n!) - \cdots \end{gathered}$$

and where all other terms are zero. Each of the first two rows is highly divergent, which is undesirable for some purposes.

At the other extreme is the following definition: Series $\sum a_{ij}$ converges if no matter how the series is arranged to form a simple series, the resulting simple series converges, and to a sum that is independent of the order in which the terms are taken. This definition permits P to be the class of *all* possible partial sums, and the condition for convergence can be phrased as follows: Series $\sum a_{ij}$ converges to A if to every $\epsilon > 0$ correspond indices p, q such that $|A - s| < \epsilon$ for every partial sum s that includes R_{pq}. This definition is highly restrictive; it is, in fact, a definition that implies absolute convergence.

It is our purpose to give a definition of convergence that lies between the two above-mentioned definitions, one that possesses properties analogous to those of simple series; and to develop some of these properties. Where proofs are especially direct and simple we shall often omit them. For simplicity in notation, the definition of convergence, and the theorems and proofs that follow, will be stated for the case of double series. The extension to the general case of k-tuply infinite series is usually immediate; where this is not the case, further details will be found in §III.

2. Definition and properties of σ-sum convergence. Let the series $\sum a_{ij}$ be given, where i, j run independently from 0 to infinity.

DEFINITION. *A partial sum is said to be a sigma-sum (σ-sum) if it has the following property: If it contains a_{pq} then it also contains the rectangular sum*

$$R_{pq} = \sum a_{ij} \qquad (0 \leqq i \leqq p, 0 \leqq j \leqq q). \tag{2.1}$$

The reader can exhibit the nature of a σ-sum (for double series) as follows: Let all the squares occupied by the terms of a σ-sum be shaded. Then the boundary of the shaded region is a "staircase," in general irregular, *that rises to the right.*

We take the class P to be the class of all σ-sums.

DEFINITION. *Given indices p, q, a σ-sum is said to be a sigma-sum relative to (p, q) [written σ-sum(p, q)] if it contains a_{pq}.*

COROLLARY. *A σ-sum(p, q) contains R_{pq}.*

DEFINITION. *Series $\sum a_{ij}$ converges to the sum A if to every $\epsilon>0$ there correspond indices (p, q) such that*

$$|A - \sigma| < \epsilon \tag{2.2}$$

for every σ that is a σ-sum(p, q).

COROLLARY. *If $\sum a_{ij}$ converges, it has a unique sum.*

Since rectangular sums are also σ-sums, we conclude that convergence by rectangular sums is at least as inclusive as is the present definition. Accordingly we have

LEMMA 1. *If a series converges by σ-sums, then it converges also by rectangular sums, and to the same sum.*

We shall see that the converse does not hold.

LEMMA 2. *If $\sum a_{ij}$ converges, so does $\sum \lambda a_{ij}$; and*

$$\sum \lambda a_{ij} = \lambda \sum a_{ij}. \tag{2.3}$$

THEOREM 1 (*Cauchy criterion*). *A necessary and sufficient condition that series $\sum a_{ij}$ converges is that to every $\epsilon>0$ there correspond indices (p, q) such that*

$$|\sigma' - \sigma''| < \epsilon \tag{2.4}$$

for every pair σ', σ'' of σ-sums(p, q).

First suppose that the series converges, to the sum A. Then given $\epsilon>0$, indices (p, q) exist so that $|A-\sigma'|<\epsilon/2$, $|A-\sigma''|<\epsilon/2$ whenever σ', σ'' are σ-sums(p, q). Hence (2.4) holds. Now suppose (2.4) holds; to show that the series converges. If σ' is fixed, (2.4) shows that the set $\{\sigma''\}$ of all σ-sums(p, q) is bounded, and therefore has at least one limit point A. Accordingly, there is a σ-sum(p, q), say σ''', such that $|A-\sigma'''|<\epsilon$; and from (2.4), $|A-\sigma|<2\epsilon$ for every σ-sum(p, q). From the arbitrariness of ϵ we conclude that $\sum a_{ij}$ converges to A.

THEOREM 2. *If $\sum a_{ij}$ converges, then to every $\epsilon>0$ correspond indices (p, q) such that*

$$|a_{ij}| < \epsilon \tag{2.5}$$

for every a_{ij} not in R_{pq}.

Let (p, q) be chosen so that $|A-\sigma|<\epsilon/2$ for every σ-sum(p, q), A being the sum of the series. If a_{ij} is not in R_{pq}, it can be written as the difference of two σ-sums(p, q):

$$a_{ij} = \sigma' - \sigma''.$$

In fact, σ' can be chosen as R_{ij} augmented by all the terms of R_{pq} not already in R_{ij}; and σ'' is then obtained from σ' by depriving it of the term a_{ij}. That σ', σ'' as so chosen are σ-sums(p, q) is readily ascertained. Then,

$$|a_{ij}| = |\sigma' - \sigma''| \leqq |A - \sigma'| + |A - \sigma''| < \epsilon,$$

which is (2.5).

COROLLARY. *If $\sum a_{ij}$ converges, then*

$$\lim a_{ij} = 0 \qquad (i + j \to \infty). \tag{2.6}$$

From the example of §1 we conclude the further

COROLLARY. *Rectangular sum convergence does not imply σ-sum convergence.*

THEOREM 3. *If $\sum a_{ij}$, $\sum b_{ij}$ converge, then also $\sum(a_{ij}+b_{ij})$ converges, and*

$$\sum a_{ij} + \sum b_{ij} = \sum (a_{ij} + b_{ij}). \tag{2.7}$$

THEOREM 4 (*Comparison theorem*). *If $0\leqq a_{ij}\leqq b_{ij}$, and if $\sum b_{ij}$ converges, so does $\sum a_{ij}$, and $\sum a_{ij}\leqq\sum b_{ij}$.*

For let $B=\sum b_{ij}$. To $\epsilon>0$ correspond indices (p, q) so that $|B-\sigma'|<\epsilon$ where σ' is any σ-sum(p, q). If σ is the corresponding sum in $\sum a_{ij}$, then $\sigma\leqq\sigma'$. Since, as is easily seen, $\sigma'\leqq B$, therefore $\sigma\leqq B$. The set $\{\sigma\}$ is consequently bounded, and has a least upper bound A. We shall show that $\sum a_{ij}$ converges to A.

A σ_1 exists so that $0\leqq A-\sigma_1<\epsilon$. If indices (r, s) are then chosen so that σ_1 is contained in R_{rs}, it follows that for every σ that is a σ-sum(r, s), we have $\sigma_1\leqq\sigma$ and therefore $0\leqq A-\sigma<\epsilon$. Hence $\sum a_{ij}$ does converge to A; and, of course, $A\leqq B$.

COROLLARY. *If $0\leqq b_{ij}\leqq a_{ij}$, and if $\sum b_{ij}$ diverges, then also $\sum a_{ij}$ diverges.*

THEOREM 5. *Let $a_{ij}=a'_{ij}+\sqrt{-1}a''_{ij}$, where a'_{ij} and a''_{ij} are real. Then $\sum a_{ij}$ converges if and only if $\sum a'_{ij}$, $\sum a''_{ij}$ converge; and in this case,*

$$\sum a_{ij} = \sum a'_{ij} + \sqrt{-1}\sum a''_{ij}. \tag{2.8}$$

DEFINITION. *Series $\sum a_{ij}$ is said to converge absolutely if $\sum|a_{ij}|$ converges.*

THEOREM 6. *If $\sum a_{ij}$ converges absolutely, then it converges.*

Case 1. *a_{ij} real.* Let $\sum a'_{ij}$ be the series obtained from $\sum a_{ij}$ by replacing all negative terms by 0, and $\sum a''_{ij}$ by replacing all non-negative terms by 0 and changing the sign of the remaining terms. By Theorem 4, $\sum a'_{ij}$, $\sum a''_{ij}$ both converge; and by Lemma 2 and Theorem 3, so does $\sum(a'_{ij}-a''_{ij})$, which is to say, $\sum a_{ij}$.

Case 2. *a_{ij} complex.* Let $a_{ij}=a'_{ij}+\sqrt{-1}a''_{ij}$. Then $|a'_{ij}|\leqq|a_{ij}|$, $|a''_{ij}|\leqq|a_{ij}|$, so that $\sum a'_{ij}$, $\sum a''_{ij}$ both converge absolutely. By Case 1, they also converge, whence by Theorem 5, $\sum a_{ij}$ converges.

LEMMA 3. *If $\sum a_{ij}$ converges absolutely either by the rectangular sum definition or by the σ-sum definition, it converges absolutely by the other.*

Lemma 1 establishes the assertion in the direction of σ-sum to rectangular sum. Now suppose that $\sum|a_{ij}|$ converges to A^* by rectangular sums. To $\epsilon>0$ corresponds an R_{pq} so that $0\leqq A^*-R_{pq}<\epsilon$; hence for every σ-sum(p, q) for series $\sum|a_{ij}|$, $0\leqq A^*-\sigma<\epsilon$. That is, $\sum|a_{ij}|$ converges by the σ-sum definition.

LEMMA 4. *Convergence does not imply absolute convergence.*

This is shown by the following double series. Let the first row be the series

$$1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{(-1)^n}{n+1}+\cdots$$

while all other terms are zero. As a double series this converges to the same sum as the simple series in the first row; but it is not absolutely convergent.

THEOREM 7. *Let $\sum a_{ij}$ converge to the sum A. If i (if j) is fixed, the resulting row series (column series) converges; and if its sum is denoted by A_{i*} (by A_{*j}), then also the series $\sum A_{i*}$ (series $\sum A_{*j}$) converges, and its sum is A.*

In other words, all rows and all columns converge, and the series can be summed by summing either the rows or the columns. To prove this we shall work with the rows for definiteness.

Let $i=0$; that is, consider the first row. To each $\epsilon>0$ there correspond indices (p, q) such that $|\sigma_1-\sigma_2|<\epsilon$ for every pair of σ-sums(p, q). Choose σ_1 as the rectangular sum R_{pq}, and σ_2 as R_{pq} augmented by the sum $\sum_{q+1}^{n}a_{0j}$; then $|\sum_{q+1}^{n}a_{0j}|<\epsilon$. Since n may be chosen as any integer exceeding q, it follows that $|\sum_{m}^{n}a_{0j}|<2\epsilon$ for all m, n for which $q<m<n$; and this implies the convergence of $\sum a_{0j}$.

Now let $i=1$. Let σ_1 again be R_{pq}, and now let σ_2 be R_{pq} to which have been added the terms $\sum_{q+1}^{n}(a_{0j}+a_{1j})$. Then the last sum is in magnitude less than ϵ. Since $|\sum_{q+1}^{n}a_{0j}|<\epsilon$, as already shown, therefore $|\sum_{q+1}^{n}a_{1j}|<2\epsilon$, and $|\sum_{m}^{n}a_{1j}|<4\epsilon$; that is, $\sum a_{1j}$ converges. And so on for every row: all rows converge.

There remains to show that $\sum A_{i*}$ converges, and to the sum A. Let $\epsilon>0$ be given. Then indices (r, s) exist such that

$$\text{(a)}\qquad |A-\sigma|<\frac{\epsilon}{2}$$

for all σ-sums(r, s). Let p be an integer at least as great as r. Since all rows converge, there is an index $q>s$ (q depending on p) such that

(b) $$\left| A_{m*} - \sum_{j=0}^{t} a_{mj} \right| < \frac{\epsilon}{2(p+1)} \qquad \begin{pmatrix} m = 0, 1, \cdots, p; \\ t = q, q+1, q+2, \cdots \end{pmatrix},$$

hence

(c) $$\left| \sum_{0}^{p} A_{m*} - \sigma \right| \lessdot \frac{\epsilon}{2}$$

where $\sigma = \sum_{m=0}^{p} \sum_{j=0}^{t} a_{mj}$ is a σ-sum(r, s). In conjunction with (a), this leads to

(d) $$\left| A - \sum_{0}^{p} A_{m*} \right| < \epsilon.$$

As p is arbitrary (subject to the condition $p \geqq r$), we conclude that $\sum A_{m*}$ converges to the sum A.

In §1 an example of a double series was given which converges by the rectangular sum definition, but for which a row is divergent. Hence Theorem 7 is not true for rectangular sum convergence.

Consider the double series

$$\begin{array}{l} 1 - 1 + 0 + 0 + \cdots \\ + 0 + 1 - 1 + 0 + 0 + \cdots \\ + 0 + 0 + 1 - 1 + 0 + 0 + \cdots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

where in each row the only two non-zero terms are 1, -1. Each row converges to the sum 0, so $A_{i*} = 0$, $\sum A_{i*} = 0$. The first column sums to 1, and the others to 0, giving to $\sum A_{*j}$ the sum 1. Therefore by Theorem 7 the series does not converge. This result may be explained by the fact that not all series involved are absolutely convergent. This is illustrated by

THEOREM 8. *Given series* $\sum a_{ij}$. *If for each fixed i the simple series* $A_{i*} = \sum_j |a_{ij}|$ *converges, and if* $\sum A_{i*}$ *converges, then the original series* $\sum a_{ij}$ *converges absolutely; and correspondingly for columns.*

It suffices to consider the double series $\sum |a_{ij}|$. Let σ be any σ-sum (for $\sum |a_{ij}|$). Clearly, $\sigma \leqq \sum A_{i*}$, so that the set of all σ-sums is bounded, and therefore has a least upper bound, say L. Let $\epsilon > 0$ be given. There is a σ-sum, call it σ_1, such that $0 \leqq L - \sigma_1 < \epsilon$. Choose (p, q) so that σ_1 is contained in R_{pq}. Then for every σ-sum(p, q), $0 \leqq L - \sigma < \epsilon$. In other words, $\sum |a_{ij}|$ converges to the sum L.

DEFINITION. *If the first p rows and q columns are deleted from the series* $\sum a_{ij}$, *the new series, which has* a_{pq} *as its leading term, will be termed the truncate of order* (p, q) *relative to the original series.*

THEOREM 9. *If series* $\sum a_{ij}$ *converges, so does every truncate series.*

The truncate (0, 0) converges since it is the original series. Consider, then, the truncate (p, q), where either $p\neq 0$ or $q\neq 0$; and for definiteness suppose $p>0$. It is a straightforward matter, that we leave to the reader, to show that truncate (1, 0) converges. It then follows by applying the same argument to series (1, 0), that (2, 0) converges; and so on until the convergence of $(p, 0)$ is obtained. Now turning to the second index, the same argument, repeated, shows that the series $(p, 1)$, $(p, 2)$, $\cdots$, and finally (p, q), converge.

As is to be anticipated, the property of Theorem 9 is false for rectangular sum convergence. The double series example of §1 shows this, since the truncate (1, 0) is clearly divergent.

THEOREM 10. *Let the double power series $\sum a_{ij}u_1^iu_2^j$ converge for $u_s=\xi_s$, $s=1, 2$. Then it converges absolutely for every (u_1, u_2) for which $|u_s|<|\xi_s|$, $s=1, 2$, and converges uniformly in every closed region therein, thus representing an analytic function of the variables u_1, u_2 in $|u_s|<|\xi_s|$, $s=1, 2$.*

The theorem is vacuously true if $\xi_s=0$ for at least one s. We therefore assume that $|\xi_s|>0$, $s=1, 2$. Choose r_s, $s=1, 2$ so that $0<r_s<|\xi_s|$, and restrict u_1, u_2 to the region $|u_s|\leqq r_s$. By Theorem 2, an M exists so that $|a_{ij}\xi_1^i\xi_2^j|\leqq M$. Hence

$$|a_{ij}u_1^iu_2^j|\leqq M\left(\frac{r_1}{|\xi_1|}\right)^i\cdot\left(\frac{r_2}{|\xi_2|}\right)^j.$$

The right-hand member is the general term of a convergent series. In fact if we sum on one index at a time (Theorem 8), we arrive at the sum $M\cdot\prod_{s=1}^2\{1-r_s/|\xi_s|\}^{-1}$. The desired result now follows from Theorem 4.

COROLLARY. *In Theorem 10 the hypothesis that there is convergence for $u_s=\xi_s$, $s=1, 2$ can be replaced by the weaker condition that the set of numbers $\{a_{ij}\xi_1^i\xi_2^j\}$ is bounded.*

It is well known that Theorem 10 is no longer true if we use rectangular sum convergence. Consider for example the double power series

$$\begin{gathered}0!+(1!)u_2+(2!)u_2^2+\cdots+(n!)u_2^n+\cdots\\-(0!)u_1-(1!)u_1u_2-(2!)u_1u_2^2-\cdots-(n!)u_1u_2^n-\cdots\end{gathered}$$

with all other terms zero. Using the rectangular sum definition, this series converges for $u_1=u_2=1$, as the example of §1 shows. It also converges for $u_1=1$, u_2 arbitrary, and for u_1 arbitrary, $u_2=0$; but for no other values. This is seen from the following expression for the rectangular sum R_{sn}, $s\geqq 2$:

$$R_{sn}=(1-u_1)\{0!+1!u_2+\cdots+n!u_2^n\}.$$

The reason for the failure of Theorem 10 for rectangular sum convergence can again be laid to non-absolute convergence. If $\sum a_{ij}u_1^iu_2^j$ converges *absolutely* for $u_s=\xi_s$ (by either definition), then the conclusion of Theorem 10 holds even for the rectangular sum definition.

An important power series derived from $\sum a_{ij}u_1^i u_2^j$ is obtained by taking all the u's equal. We thus obtain a *multiple power series in one variable*: $\sum a_{ij}t^{i+j}$. As in Theorem 10, we have

THEOREM 11. *If the multiple power series $\sum a_{ij}t^{i+j}$ in the variable t converges for $t=\xi$, (or if the set of numbers $\{a_{ij}\xi^{i+j}\}$ is bounded), then the series converges absolutely for all t in $|t|<|\xi|$, and converges uniformly in every closed region therein, thus representing an analytic function of t in $|t|<|\xi|$.*

Theorem 11 fails for convergence by rectangular sums.

There is another definition of convergence that is apropos for multiple power series in one variable. It is convergence by *circular sums*. A circular sum for the series $\sum a_{ij}$ is one of the form

$$C_n \equiv \sum a_{ij} \qquad (i+j \leqq n). \tag{2.9}$$

For the series $\sum a_{ij}t^{i+j}$,

$$C_n(t) = \sum_{s=0}^{n} b_s t^s \tag{2.10}$$

where

$$b_s = \sum a_{ij} \qquad (i+j=s). \tag{2.11}$$

In other words, the nth circular sum is precisely the nth partial sum for the *simple* power series to which $\sum a_{ij}t^{i+j}$ can be reduced.

According to Theorem 11, the region of convergence of the double power series $\sum a_{ij}t^{i+j}$ in the variable t is a circle. It is possible to express its radius in terms of the coefficients of the series. Let

$$\rho = \limsup |a_{ij}|^{1/(i+j)} \qquad (i+j \to \infty). \tag{2.12}$$

We then have

THEOREM 12. *The radius of convergence R of the series $\sum a_{ij}t^{i+j}$ is given by*

$$R = 1/\rho. \tag{2.13}$$

Case 1. $\rho<\infty$. Let R' be any positive number for which $R'<1/\rho$. Choose $\epsilon>0$ small enough so that $\theta\equiv R'(\rho+\epsilon)<1$. If $|t|\leqq R'$, then for all i, j with $i+j>N=N_\epsilon$,

$$|a_{ij}t^{i+j}| \leqq \theta^{i+j}, \qquad 0<\theta<1.$$

Now $\sum\theta^{i+j}$ converges (to the sum $\{1-\theta\}^{-2}$); whence by Theorem 4, series $\sum a_{ij}t^{i+j}$ converges absolutely for $|t|<R'$. But R' can be chosen arbitrarily in its given range. It follows that $R\geqq 1/\rho$.

Now suppose $|t|=\lambda>1/\rho$. On choosing $\epsilon>0$ small enough so that $\lambda(\rho-\epsilon)>1$, there will exist infinitely many sets (i, j) for which

$$|a_{ij}t^{i+j}| > [\lambda(\rho-\epsilon)]^{i+j}.$$

The series must therefore diverge, in virtue of Theorem 2. That is, $R \leqq 1/\rho$. Consequently, $R = 1/\rho$.

Case 2. $\rho = \infty$. It is readily shown by the method of Case 1 that the series diverges for every $t \neq 0$, so that $R = 0$.

Turning back to circular sums, we can assert that the radius of convergence of $\sum a_{ij} t^{i+j}$ by the circular sum definition is *at least as great as* R. For by the uniform convergence guaranteed by Theorem 11, the series can be reduced to a simple series, precisely the series arrived at by circular sums, convergent in $|t| < R$ at least. The following two examples show that the radius of convergence of the simple series *may* exceed R, but will not *always* exceed R.

Example 1. If $\sum a_{ij} t^{i+j}$ is the series whose first two rows are

$$1 + t + t^2 + \cdots + t^n + \cdots$$
$$+ t + t^2 + \cdots + t^{n+1} + \cdots$$

while all other terms are 0, then the circular sum definition gives the radius of convergence as 1, which is also the value of R.

Example 2. If $\sum a_{ij} t^{i+j}$ is the series having

$$1 + t + t^2 + \cdots + t^n + \cdots$$

as its first row and

$$1 - t - t^2 - \cdots - t^n - \cdots$$

as its first column, while all other terms are 0, then by circular sums the series converges to the sum 1 for all t, whereas $R = 1$.

For the series $\sum a_{ij} u_1^i u_2^j$ there is no result as precise as Theorem 12. We can, however, state

THEOREM 13. *Let ρ be given by* (2.12). *The series $\sum a_{ij} u_1^i u_2^j$ converges absolutely for all u_1, u_2 for which $|u_s| < 1/\rho$, $s = 1, 2$, and diverges for all u_1, u_2 for which $|u_s| > 1/\rho$, $s = 1, 2$.*

COROLLARY. *If $\rho = \infty$, every point (u_1, u_2) of convergence has at least one $u_s = 0$.*

COROLLARY. *Series $\sum a_{ij} u_1^i u_2^j$ converges absolutely for every u_1, u_2 for which*

$$\limsup \left| a_{ij} u_1^i u_2^j \right|^{1/(i+j)} < 1 \qquad (i + j \to \infty);$$

and diverges for every u_1, u_2 for which the above superior limit exceeds one.

The proof follows the line of demonstration of Theorem 12. Let

$$A(u_1, u_2) = \sum a_{ij} u_1^i u_2^j, \qquad B(u_1, u_2) = \sum b_{ij} u_1^i u_2^j \tag{2.14}$$

be two power series. The formal product is given by

$$C(u_1, u_2) = \sum c_{ij} u_1^i u_2^j \tag{2.15}$$

where

$$c_{mn} = \sum a_{ij} b_{rs}, \tag{2.16}$$

the sum in (2.16) being over all indices i, j, r, s for which, simultaneously,

$$i + r = m, \qquad j + s = n. \tag{2.17}$$

This suggests the following

DEFINITION. *By the Cauchy product series derived from the two series* $\sum a_{ij}$, $\sum b_{ij}$ *is meant the series* $\sum c_{ij}$ *where* c_{ij} *is given by* (2.16).

THEOREM 14. *If* $\sum a_{ij}$, $\sum b_{ij}$ *are absolutely convergent, then so is* $\sum c_{ij}$, *and*

$$\sum c_{ij} = (\sum a_{ij})(\sum b_{ij}). \tag{2.18}$$

Case 1. $a_{ij} \geqq 0$, $b_{ij} \geqq 0$. Let $A = \sum a_{ij}$, $B = \sum b_{ij}$. If σ is any σ-sum in the space of (m, n) (so that σ is a sum of form $\sum c_{mn}$), then $\sigma \leqq AB$. Hence $\{\sigma\}$ has a least upper bound C, and it is easy to see that $\sum c_{ij}$ converges to C and that $C = AB$. The theorem is thus true in this case.

Case 2. a_{ij}, b_{ij} *real*. Write $a_{ij} = a'_{ij} - a''_{ij}$ where $a'_{ij} = a_{ij}$, $a''_{ij} = 0$ if $a_{ij} \geqq 0$, and $a'_{ij} = 0$, $a''_{ij} = -a_{ij}$ if $a_{ij} < 0$; and similarly for the b's. Then (2.16) can be written

$$c_{mn} = \sum_{s=1}^{2} c_{mn}^{(s)} - \sum_{s=3}^{4} c_{mn}^{(s)}$$

where

$$c_{mn}^{(1)} = \sum a'_{ij} b'_{ij}, \qquad c_{mn}^{(2)} = \sum a''_{ij} b''_{ij}, \qquad c_{mn}^{(3)} = \sum a'_{ij} b''_{ij}, \qquad c_{mn}^{(4)} = \sum a''_{ij} b'_{ij}.$$

From Case 1 we now can write

$$C = C^{(1)} + C^{(2)} - C^{(3)} - C^{(4)},$$

where each $C^{(s)}$ is the sum of a convergent series of positive terms; and it is readily seen again that $C = AB$.

Case 3. a_{ij}, b_{ij} *complex*. We can split a_{ij} and b_{ij} into their real and imaginary components, and then apply the method of Case 2 to obtain the desired result.

3. Some remarks on k-tuply infinite series. It was stated in §1 that most of the work of §2 extends readily to the general case of the k-tuply infinite series $\sum a_{i_1 \cdots i_k}$ (or $\sum a_{(i)}$ for short). There are, however, certain points that may need further statement, and these we consider here.

In the definition of σ-sum, the rectangular sum R_{pq} of (2.1) is to be replaced by the k-dimensional rectangular sum

$$R_{p_1 \cdots p_k} = \sum a_{(i)} \qquad (0 \leqq i_s \leqq p_s,\ s = 1, \cdots, k); \tag{3.1}$$

and a σ-sum relative to the indices $(p_1, \cdots, p_k)$ will be written σ-sum $(p_1 \cdots, p_k)$. To extend Theorem 7 to k dimensions we use the following definition and lemma.

DEFINITION. *By a subseries of the k-tuple series* $\sum a_{(i)}$ *is meant any series obtained from* $\sum a_{(i)}$ *by fixing one or more of the k indices* $i_1, \cdots, i_k$.

LEMMA 5. *Let σ be a σ-sum for the series $\sum a_{(i)}$. Let r be any integer in the range $0<r<k$, and divide the indices $i_1, \cdots, i_k$ into two sets, one consisting of the indices $i_{m_1}, \cdots, i_{m_r}$, the other having the remaining $k-r$ indices. Write σ in the form*

$$\sigma = \sum_{(r)} \left\{ \sum_{(k-r)} a_{(i)} \right\}, \tag{3.2}$$

where the outer $\sum$ is summed over $i_{m_1}, \cdots, i_{m_r}$, and the inner $\sum$ over the remaining $k-r$ indices. Then the inner sum is a σ-sum for the $(k-r)$-tuple series obtained by fixing $i_{m_1}, \cdots, i_{m_r}$ in $\sum a_{(i)}$; and if its sum is denoted by $\sigma_{i_{m_1}, \cdots, i_{m_r}}$, then $\sum_{(r)} \sigma_{i_{m_1} \cdots i_{m_r}}$ is a σ-sum in the "space" of the indices $i_{m_1}, \cdots, i_{m_r}$.

This lemma is an immediate consequence of the meaning of "σ-sum."

We now state the extension of Theorem 7.

THEOREM 7'. *Let $\sum a_{(i)}$ be convergent to the sum A. Let r be any integer in the range $0<r<k$. If any r indices, say $i_{m_1}, \cdots, i_{m_r}$, are kept fixed the resulting $(k-r)$-tuple subseries converges; and if its sum is $A_{i_{m_1} \cdots i_{m_r}}$, then also the r-tuple series $\sum A_{i_{m_1} \cdots i_{m_r}}$ converges, and its sum is A.*

As this is the central theorem of the present work and as its proof is not too obviously an extension of the proof of Theorem 7, we now provide a proof.

The theorem is true for $k=1$ (vacuously) and for $k=2$ (Theorem 7). Assume it true through $k-1$; we shall prove it true for k. First suppose $r=1$. Since the σ-sum definition of convergence has no *preferred* index of summation, we may fix i_1.

Let $i_1=0$. To each $\epsilon>0$ there correspond indices $(p_1, \cdots, p_k)$ such that $|\sigma_1-\sigma_2|<\epsilon$ for every pair of σ-sums $(p_1, \cdots, p_k)$. Now let $\sigma_1^{(0)}, \sigma_2^{(0)}$ be any two σ-sums $(p_2, \cdots, p_k)$, taken from the $(k-1)$-tuple subseries for which $i_1=0$. If to each of $\sigma_1^{(0)}, \sigma_2^{(0)}$ we add those terms of $R_{p_1 \cdots p_k}$ (the same for both) not already present in them, we obtain σ-sums $(p_1, \cdots, p_k)$; call them σ_1, σ_2. From $|\sigma_1-\sigma_2|<\epsilon$ we get the relation $|\sigma_1^{(0)}-\sigma_2^{(0)}|<\epsilon$. Hence A_{i_1} exists for $i_1=0$. If $i_1=1$, let $\sigma_1^{(1)}, \sigma_2^{(1)}$ be any two σ-sums $(p_2, \cdots, p_k)$, whose terms are the terms of $\sigma_1^{(0)}, \sigma_2^{(0)}$ plus the corresponding terms in the subseries for which $i_1=1$. We again get $|\sigma_1^{(1)}-\sigma_2^{(1)}|<\epsilon$. On setting $\sigma_j^{(1)}=\sigma_j^{(0)}+\sigma_j'^{(1)}$, $j=1, 2$, and recalling that $|\sigma_1^{(0)}-\sigma_2^{(0)}|<\epsilon$, we obtain the inequality $|\sigma_1'^{(1)}-\sigma_2'^{(1)}|<2\epsilon$. This shows that A_{i_1} exists for $i_1=1$.

And so for every i_1. Hence, since any other i_s could have been used, all $(k-1)$-tuple series converge; and by our induction assumption, every subseries converges, since a subseries of $\sum a_{(i)}$ of fewer than $k-1$ indices is also a subseries of some $(k-1)$-tuple subseries.

There remains to establish the quantitative result of the theorem. Let $\epsilon>0$ be given. We are to show that indices $(p_1, \cdots, p_r)$ exist such that $|A-\sigma^*|<\epsilon$ for every σ^* that is a σ-sum $(p_1, \cdots, p_r)$ in the space of the indices $i_{m_1}, \cdots, i_{m_r}$; that is, σ^* is a σ-sum of form $\sum A_{i_{m_1} \cdots i_{m_r}}$, containing the rectangular sum $R_{p_1 \cdots p_r}=\sum A_{i_{m_1}, \cdots, i_{m_r}}$ $(0 \leqq i_{m_s} \leqq p_s,\ s=1, \cdots, r)$.

Now indices $(q_1, \cdots, q_k)$ exist so that

(a) $$|A - \sigma| < \frac{\epsilon}{2}$$

for every σ-sum $(q_1, \cdots, q_k)$ in the original space of $i_1, \cdots, i_k$. Choose the p's to have the values $p_s = q_{m_s}$, $s = 1, \cdots, r$. Consider a fixed σ^* (a σ-sum $(p_1, \cdots, p_r)$). It will consist of a finite number, say g, of terms $A_{i_{m_1}\cdots i_{m_r}}$. For each such term there is a set of indices $(t_1, \cdots, t_{k-r})$ such that

(b) $$|A_{i_{m_1}\cdots i_{m_r}} - \sigma_{i_{m_1}\cdots i_{m_r}}| < \frac{\epsilon}{2g}$$

for every $\sigma_{i_{m_1}\cdots i_{m_r}}$ that is a σ-sum $(t_1, \cdots, t_{k-r})$. (This is so since each $A_{i_{m_1}\cdots i_{m_r}}$ is the sum of a convergent $(k-r)$-fold series.)

If we choose $u_s \geqq \max t_s$, $s = 1, \cdots, k-r$, taken over all g sets $(t_1, \cdots, t_{k-r})$, then (b) holds uniformly for all g sets $(i_{m_1}, \cdots, i_{m_r})$ provided that each $\sigma_{i_{m_1}\cdots i_{m_r}}$ is a σ-sum $(u_1, \cdots, u_{k-r})$. We now impose on $u_1, \cdots, u_{k-r}$ the further conditions that each is at least as great as the corresponding index taken from $q_1, \cdots, q_k$ (minus, of course, $q_{m_1}, \cdots, q_{m_r}$). We have now arranged matters so that when suitably ordered, the k indices $u_1, \cdots, u_{k-r}, p_1, \cdots, p_r$ form a set $j_1, \cdots, j_k$ with $j_s \geqq q_s$, $s = 1, \cdots, k$.

We may (and do) in particular, further restrict the $\sigma_{i_{m_1}\cdots i_{m_r}}$'s that appear in (b) to be rectangular sums (in the space of the $k-r$ indices other than $i_{m_1}, \cdots, i_{m_r}$), and all of them to have the same maximum indices $u_1, \cdots, u_{k-r}$ in that $(k-r)$-fold space. In other words two such rectangular sums are indistinguishable if one looks only at the $k-r$ indices of summation.

From (b) we obtain on summing,

$$\left|\sum_{(g)} A_{i_{m_1}\cdots i_{m_r}} - \sum_{(g)} \sigma_{i_{m_1}\cdots i_{m_r}}\right| < \frac{\epsilon}{2},$$

which is to say,

(c) $$\left|\sigma^* - \sum_{(g)} \sigma_{i_{m_1}\cdots i_{m_r}}\right| < \frac{\epsilon}{2}\cdot$$

Now the conditions imposed on the $\sigma_{i_{m_1}\cdots i_{m_r}}$'s insure us, as is easily seen, that $\sum_{(g)}\sigma_{i_{m_1}\cdots i_{m_r}}$, when expressed as a sum of terms $a_{i_1\cdots i_k}$ of the original series, is a σ-sum of that series; and the choice of $u_1, \cdots, u_{k-r}, p_1, \cdots, p_r$ further guarantees that it is a σ-sum $(q_1, \cdots, q_k)$. Hence

(d) $$\left|A - \sum_{(g)} \sigma_{i_{m_1}\cdots i_{m_r}}\right| < \frac{\epsilon}{2};$$

and this, in conjunction with (c), leads to

(e) $$|A - \sigma^*| < \epsilon.$$

As the choice of $(p_1, \cdots, p_r)$ depends only on ϵ, (e) asserts that $\sum A_{i_{m_1}\cdots i_{m_r}}$ converges to A, which is the desired conclusion.

MEAN DEVIATION OF THE BINOMIAL DISTRIBUTION

J. S. FRAME, Michigan State College

The binomial distribution is one of the topics usually studied in an elementary course in statistics. It is known that if in a single experiment the probability of success of an event is p and the probability of failure is $q=1-p$, then the probability of r successes and $n-r$ failures in n repetitions of the same experiment is the $(r+1)$th term in the binomial expansion of $(q+p)^n$, which we write as y_r^n:

$$y_r^n = \frac{n!}{r!(n-r)!} p^r q^{n-r}. \tag{1}$$

Now it is well known and easy to prove that the mean of this distribution is $\bar{x}=np$, and the standard deviation is $\sigma_n=\sqrt{npq}$, but a simple expression for the *mean deviation* (sometimes called "average deviation") *from the mean* seems to be not well known, and was not found in a search of elementary statistics books.

The mean deviation (MD_n) from the mean $\bar{x}$ is given by the equations:

$$MD_n = \sum_{r=0}^{n} | r - \bar{x} | y_r^n = \sum_{r=0}^{m} (np - r) y_r^n + \sum_{r=m+1}^{n} (r - np) y_r^n. \tag{2}$$

$$MD_n = 2\sum_{r=0}^{m} (np - r) y_r^n, \tag{3}$$

where $m=[np]$ is the largest integer not exceeding np. Experimental evidence shows that the values of MD_n seem to be related to the values of the largest term Y_n in the expansion of $(q+p)^n$.

For the symmetric case $p=q=1/2$, we have $\bar{x}=n/2$. Computation yields the following values for MD_n and Y_n:

$p=\frac{1}{2}$, $q=\frac{1}{2}$	n	1	2	3	4	5	6	7	8	9	10
	MD_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{15}{16}$	$\frac{15}{16}$	$\frac{35}{32}$	$\frac{35}{32}$	$\frac{315}{256}$	$\frac{315}{256}$
	Y_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{35}{128}$	$\frac{35}{128}$	$\frac{63}{256}$	$\frac{63}{256}$

We may verify for these values of n that

$$MD_n = \begin{cases} (n+1)Y_n/2 & \text{for } n \text{ odd}, \quad (p = q = 1/2) \\ n \quad Y_n/2 & \text{for } n \text{ even}, \quad (p = q = 1/2), \end{cases} \tag{4}$$

but no simple formula seems to hold for both odd and even values of n.

For the case $p=2/3$, $q=1/3$, we have $\bar{x}=2n/3$. Computation yields the following values of MD_n and Y_n.

$p = \frac{2}{3}$, $q = \frac{1}{3}$

n	1	2	3	4	5	6	7	8	9	10
MD_n	$\frac{4}{9}$	$\frac{16}{27}$	$\frac{16}{27}$	$\frac{64}{81}$	$\frac{640}{729}$	$\frac{640}{729}$	$\frac{2240}{2187}$	$\frac{7168}{6561}$	$\frac{7168}{6561}$	$\frac{71680}{59049}$
Y_n	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{32}{81}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{672}{2187}$	$\frac{1792}{6561}$	$\frac{1792}{6561}$	$\frac{15360}{59049}$

We may verify for these values of n that

$$
(5) \qquad MD_n = \begin{cases} 2(2n+1)Y_n/9 & \text{for } n = 3k+1, \quad (p = 2/3,\ q = 1/3) \\ 2(2n+2)Y_n/9 & \text{for } n = 3k+2, \quad (p = 2/3,\ q = 1/3) \\ 2(2n)Y_n/9 & \text{for } n = 3k, \quad (p = 2/3,\ q = 1/3), \end{cases}
$$

but again we do not have a single simple formula for all values of n.

However, if we express MD_n as a multiple of Y_{n-1} instead of Y_n, then all cases come under a single simple formula:

$$
(6) \qquad MD_n = 2npqY_{n-1},
$$

where Y_{n-1} is the largest term in the expansion of $(q+p)^{n-1}$.

To prove formula (6) we shall first express each of the terms in (3) as the product of npq times the difference of two terms in the expansion of $(q+p)^{n-1}$. From the definition (1) we have

$$
(7) \qquad (n-r)y_r^n = \frac{n!}{r!(n-1-r)!} p^r q^{n-r} = nqy_r^{n-1}, \qquad r \geqq 0.
$$

$$
(8) \qquad ry_r^n = \frac{n!}{(r-1)!(n-r)!} p^r q^{n-r} = npy_{r-1}^{n-1}, \qquad r \geqq 1.
$$

From (7) and (8), using $p+q=1$, we have

$$
(9) \qquad (np-r)y_r^n = p(n-r)y_r^n - qry_r^n = npq(y_r^{n-1} - y_{r-1}^{n-1}), \qquad r \geqq 1,
$$

$$
(9') \qquad (np-0)y_0^n = npqy_0^{n-1}.
$$

Both sides of the equalities in (9) are positive for $r < np$, and negative for $r > np$, so the largest term Y_{n-1} in the expansion of $(q+p)^{n-1}$ is y_m^{n-1}, where $m = [np]$. (There are two equal largest terms if and only if np is an integer.) The sum of the members on the left side of (9) and (9′) for all r less than np is $MD_n/2$, by equation (3). But in evaluating the corresponding sum of the right hand members of (9) and (9′), we have a telescoping series in which all terms cancel except the single term $npq\ (y_m^{n-1})$, which equals $npqY_{n-1}$. This completes the proof of (6).

By using Stirling's formula for factorial n, namely

$$
(10) \qquad n! = \sqrt{2\pi n}\,(n/e)^n e^{R(n)}, \qquad R(n) = \frac{1}{12n} - \frac{1}{360n^3} + \cdots,
$$

it can be shown for large n that Y_{n-1} is approximately equal to $(2\pi npq)^{-1/2}$, or $1/\sqrt{2\pi}\sigma_n$. Thus from (6) we obtain the approximation

$$MD_n \sim \sqrt{2npq/\pi} = \sqrt{2/\pi}\,\sigma_n = 0.79788\sigma_n. \tag{11}$$

More exact computation, using the remainder terms in Stirling's formula, yields the better approximation

$$\frac{\pi}{2}(MD_n)^2 = npq + (np - [np])(nq - [nq]) - (1 - pq)/6 + E_n/24n, \tag{12}$$

where the error coefficient E_n becomes numerically less than or equal to unity as n becomes infinite, for all choices of np between 1 and $n-1$; and $[np]$ and $[nq]$ denote the greatest integers not exceeding np and nq respectively.

A SET OF EIGHT NUMBERS

ARTHUR PORGES, Western Military Academy

1. Introduction. In this paper the operation of adding the squared digits of any natural number A a finite number of times is proved to transform A either to unity or to one of a set of eight natural numbers closed under the operation.

2. Definitions. We use the expression *natural number* to denote a member of the set $1, 2, 3, \cdots$ of positive integers. Zero has not been adjoined to this set and is not to be included in the definition.

The operator G is defined by the equation

$$G(A) = \sum_{i=1}^{R} X_i^2, \tag{1}$$

where A is a natural number of R digits given by

$$A = \sum_{i=1}^{R} X_i 10^{i-1}. \tag{2}$$

Since A has R digits, $X_R \neq 0$.

We note that $G(0)=0$, and $G(1)=1$.

Using the customary notation, we write $G^n(A)$, where $n>1$, for n successive applications of the operator G to A.

G is not a linear operator since, in general, $G(A_1+A_2) \neq G(A_1)+G(A_2)$.

The set of numbers

$$\begin{aligned} a_1 &= 4, & a_5 &= 89,\\ a_2 &= 16, & a_6 &= 145,\\ a_3 &= 37, & a_7 &= 42,\\ a_4 &= 58, & a_8 &= 20, \end{aligned} \tag{3}$$

is closed under the operation defined by (1). We call (3) *Set K*, and use the symbol a' to denote any non-specified element of the set. The equation

(4) $$G^8(a') = a'$$

is easily verified.

Numbers of the form 10^n, $13\cdot 10^n$, $10^{n+1}+3$, where n is a positive integer or zero, and others not specified here, satisfy the equation

(5) $$G^r(A) = 1$$

for some integer $r>0$. Any natural number satisfying (5) will be denoted by the symbol b'.

3. Preliminary Lemmas. In what follows, the symbols A and B always represent finite natural numbers in the denary system of notation.

LEMMA 1. *Any natural number A of R digits, where $R\geqq 4$, satisfies the inequality*

(6) $$G(A) < A.$$

It is evident that $G(A)\leqq 81R$, and that $A\geqq 10^{R-1}$. The inequality

(7) $$81R < 10^{R-1}$$

becomes, upon taking the common logarithm of each member and transposing,

(8) $$\log_{10} R < R - 2.9085,$$

an inequality valid for $R\geqq 4$.

LEMMA 2. *For any natural number A there exists a positive integer n such that*

(9) $$G^n(A) \leqq 162.$$

For $R\geqq 4$, Lemma 1 establishes the inequality (6). As a direct consequence of (6), the operator G applied to A a finite number of times must result in a natural number of less than four digits, since for $R=4$, $G(A)\leqq 324$.

For $R<4$, the following inequalities are readily established.

(10) $$G(A) \leqq 243,$$

(11) $$G^2(A) \leqq G(199) = 163,$$

(12) $$G^3(A) \leqq G(99) = 162.$$

Since $G(A)$, where A is a three digit number, cannot exceed $3\cdot 81=243$, (10) is obviously valid. Also, since $G(199)\geqq G(B)$ for any $B\leqq 243$, (11) holds. Finally, since $G(99)\geqq G(P)$ for any $P\leqq 163$, (12) is proved.

The inequalities (10), (11), and (12) complete the proof of Lemma 2.

4. Convergence of $G^n(A)$. The following theorem is the main result of this paper.

THEOREM 1. *For every natural number A there exists either a positive integer n such that* (5) *holds for all $r \geqq n$, or a positive integer m such that*

$$G^r(A) = a' \tag{13}$$

for all $r \geqq m$, where a' is some element of Set K.

From Lemma 2 it is evident we need prove the theorem only for $A \leqq 162$. The writer was unable to find a simple indirect proof sufficiently superior to the following direct one of selective verification to justify its inclusion here.

We consider two cases.

Case 1. $100 \leqq A \leqq 162$.

For A thus restricted, it is apparent that $G(A) \leqq G(159) = 107$. Direct application of the operator G to A over the range 100 to 107 gives

$$\begin{aligned} G(100) &= 1, & G^6(104) &= a' = 89,\\ G^2(101) &= a' = 4, & G^3(105) &= a' = 16,\\ G^5(102) &= a' = 89, & G(106) &= a' = 37,\\ G^2(103) &= 1, & G^5(107) &= a' = 89, \end{aligned} \tag{14}$$

thus completing the proof of the theorem for Case 1.

Case 2. $0 < A < 100$.

For $A = 10X + Y$, where $0 \leqq X \leqq 9$, and $0 \leqq Y \leqq 9$, the following identity is valid.

$$G(10X + Y) \equiv G(10Y + X). \tag{15}$$

Further, if $G^n(A) = a'$, and $G^m(B) = A$, it follows that there exists a number $h = n + m$ such that $G^h(B) = a'$.

By means of these considerations, it is possible to verify Theorem 1 numerically for all $A < 100$ by actual computation of $G^n(A)$ for 30 values of $A < 100$, thus completing the proof of the theorem.

The writer is aware of the inelegance of such a proof, and would like very much to see a simple indirect one. However, proving the non-existence of another set like (3), which seems a necessary step, is quite difficult because of the non-linear character of G.

COROLLARY. *For every natural number A there exists either a positive integer n such that $G^n(A) = 1$, or a positive integer m such that $G^m(A) = 4$.*

The corollary follows directly from Theorem 1 and the nature of Set K. Since every natural number is transformed either into unity or into an element of Set K by the operator G, we need only note that for every $a' \neq 4$, there exists a positive integer $r \leqq 7$ such that $G^r(a') = 4$.

THEOREM 2. *The number of digits N in $G(A)$, where A has R digits, satisfies the inequality*

$$N \leqq 2.9 + \log_{10} R. \tag{16}$$

This theorem is a simple consequence of the inequality $G(A) \leqq 81R$. We have

$$G(A) \leqq 10^{1.9} + \log_{10} R \tag{17}$$

a number of N digits, where $N \leqq 2.9 + \mathrm{Log}_{10} R$.

THEOREM 3. *The only solutions in natural numbers of*

$$G^n(A) = A,$$

where $n \geqq 1$, *are*

$$A = 1, \qquad n = J, \tag{19}$$

$$A = a', \qquad n = 8, \tag{20}$$

where J is any natural number.

If we assume the existence of a natural number $A > 1$ and different from a' such that $G^n(A) = A$ for some $n \geqq 1$, it follows that A would not be transformed into either unity or an element of Set K by a finite number of applications of the operator G to A. But this is a direct contradiction of Theorem 1, and hence the assumption is false.

5. Concluding Remarks. A problem suggested by the one just discussed is that of repeatedly summing the *cubed* digits of a natural number. A complication occurs, however, since there is more than one number A such that $H(A) = A$, where H is the operator analogous to (1) given by

$$H(A) = \sum_{i=1}^{R} X_i^3. \tag{21}$$

For example, $H(153) = 153$, $H(407) = 407$, and $H(371) = 371$. This destroys the factor of uniqueness, since $H(A)$ may be unity as when $A = 100$; or A may be transformed into a number A' like 153.

It is interesting to note that since for any number A transformed into some element of Set K by a finite number of applications of G we can construct a number $B = 10^A$ such that $G(B) = 1$, there are at least "as many" numbers satisfying (5) as (13). This intuitionally unsatisfying conclusion results from the comparison of two infinite sets.

Leibniz discovers the obvious. I have made some observations on prime numbers which, in my opinion, are of consequence for the perfection of the science of numbers . . . . If the sequence [of primes] were well known, it would enable us to uncover the mystery of numbers in general; but up till now it has seemed so bizarre that nobody has succeeded in finding any affirmative characteristic or property . . . . I believe I have found the right road for penetrating their [primes'] nature: but not having had the leisure to pursue it, I shall give you here a positive property, which seems to me curious and useful.—Leibniz, in a letter to the editor of the *Journal des Savans*, 1678. The discovery: a prime is necessarily of one or other of the forms $6n+1$, $6n+5$.—*Contributed.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON PASCAL'S THEOREM

J. H. M. WEDDERBURN, Princeton University

Let $f(x, y, z)=0$ be the equation of a plane curve of degree n and P_i (x_i, y_i, z_i), $i=1, 2, \cdots, m$, the vertices of a closed polygon. The equation whose roots are the position ratios of the points in which the curve cuts the side P_iP_{i+1} has the form

$$t^n f(x_{i+1}, y_{i+1}, z_{i+1}) + \cdots + f(x_i, y_i, z_i) = 0$$

and the product of these roots is $(-1)^n f(x_i, y_i, z_i)/f(x_{i+1}, y_{i+1}, z_{i+1})$. Hence the product of the position ratios of all the points, real or imaginary, taken cyclically round the polygon is $(-1)^{mn}$; if the curve passes through a vertex, this is still true in the limit.

If we take $n=1$, that is, if the curve is a straight line, we see that *the cyclic product of the position ratios of the points in which a line cuts the sides of a closed polygon with m vertices is* $(-1)^m$. For the triangle this gives the theorem of Menelaus.

In preparation for the proof of Pascal's Theorem we take $m=3$, $n=2$ and the result may be stated as follows. *If a conic cuts the sides BC, CA, AB of a triangle in P_1, Q_1, P_2, Q_2, P_3, Q_3 and if the position ratios of these points relatively to the vertices of the sides on which they lie, taken in cyclic order, are p_1, q_1, p_2, q_2, p_3, q_3, then*

$$p_1q_1p_2q_2p_3q_3 = 1 \tag{1}$$

and conversely. The converse is immediate since a conic is uniquely determined by five points and (1) gives the position of the remaining point.

Let P_3Q_2, P_1Q_3, P_2Q_1 meet BC, CA, AB in R_1, R_2, R_3, respectively, and let the corresponding position ratios be r_1, r_2, r_3. By Menelaus' Theorem $r_1p_3q_2=-1$, $r_2p_1q_3=-1$, $r_3p_2q_1=-1$; multiplying corresponding sides of these equations together and using (1) we have $r_1r_2r_3=-1$ and hence R_1, R_2, R_3 are collinear, as required by Pascal's Theorem.

The following particular case is of some interest. *If AP_1, BP_2, CP_3 meet in a point S, then AQ_1, BQ_2, CQ_3 also meet in a point T.* For $p_1p_2p_3=1$ by Ceva's Theorem and hence also $q_1q_2q_3=1$. The point T can be readily constructed and a repetition of the construction, starting from T, leads back to S. If the conic is restricted to pass through two fixed points, this gives an interesting case of a Cremona transformation.

A metric proof of the lemma used in proving Pascal's Theorem is easily made for real intersections. If the conic is a circle, we have

$$AP_3 \cdot AQ_3 = AP_2 \cdot AQ_2, \qquad BP_1 \cdot BQ_1 = BP_3 \cdot BQ_3, \qquad CP_2 \cdot CQ_2 = CP_1 \cdot CQ_1$$

and multiplying corresponding sides of these equations together shows that the lemma is true for the circle; projection then gives it for any conic.

The proofs usually given for Menelaus' Theorem are metric, but it is easily shown that a change of projective coordinates merely multiplies the position ratio by a constant which depends on the given line and the base points and that for three lines forming the triangle of reference the product of the three constants is unity; and hence the theorem is projective.

SOME PROPERTIES OF THE LIMAÇON AND CARDIOID

J. H. BUTCHART, Grinnell College

Problem E 526 [1942, 404] of this MONTHLY, proposed by R. C. Yates, asked for the locus of a point P if the angles formed by the tangents from P to two fixed circles are equal. Yates thus called attention to a property of the circle of similitude.* This problem suggested the more difficult question: if tangents from P, one to each of two fixed circles, are allowed to vary while intersecting at a constant angle, what is the locus of P? The answer may be stated as follows:

THEOREM 1. *If two lines intersecting at a fixed angle are moved continuously tangent to two given circles, their intersection traces a limaçon whose double point lies on the circle of similitude of the two given circles.*

This locus is not new, for it is given as an exercise in Williamson's *Differential Calculus*, p. 372. However, since Williamson does not call attention to the location of the double point on the circle of similitude (S) of the given circles (O) and (O'), the following proof is believed to be new.

We might begin by assuming the variable tangents in an initial position, but it seems more convenient to use an indirect approach. Let C be a point on (S). Let Q be any point on the circle through $O'CO$, and let TP, $T'P'$ be tangents to (O), (O') parallel respectively to OQ, $O'Q$ and meeting CQ in P, P' such that these points are either both within CQ or else both without in the same direction, as in Fig. 1. Then since angles CPT and $CP'T'$ are constant, as they are equal to CQO and CQO' respectively, the segments QP and QP' are constant. When Q is in the special position diametrically opposite C, P and P' coincide, since then $CO:CT::CO':CT'$. Thus P and P' are identical in all positions and P describes a limaçon with double point at C.

That this discussion covers all possibilities may be seen by noting that the two tangents to (O) parallel to OQ meet the two tangents to (O') parallel to $O'Q$ in four points two of which lie on CQ and the other two on $C'Q$, where C' is the other point common to (S) and (OQO'). The two limaçons whose double points

* R. A. Johnson, Modern Geometry, p. 26.

are C and C' are evidently not in general congruent. If we were asking for the locus of all points from which tangents to the respective fixed circles can be drawn intersecting at a given angle, we should have to include two more limaçons symmetrical to these with respect to the line of centers. For completeness we should mention the cases where (S) is not a proper circle. If (O) and (O') are concentric, the locus degenerates to other circles concentric with them. When (O) and (O') are equal, (S) becomes their radical axis, but otherwise the discussion is unchanged.

The constant segment QP is less than, equal to, or greater than the diameter of (OQO') according as C lies without, on, or within both (O) and (O'). These conditions serve to classify the limaçons. The condition for a cardioid is equivalent to the requirement that the given angle formed by the tangents be the same as that at which the circles meet.

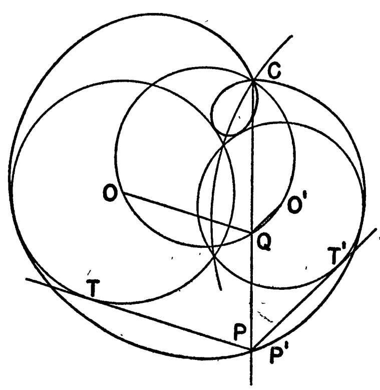


FIG. 1.

It is of interest to note that the triangle TCT' has a fixed shape. The truth of this follows easily from the evident similarity of triangles COT and $CO'T'$. It is well known that if a triangle, fixed in shape, has one vertex C fixed while another vertex T describes a circle, then the third vertex T' describes a circle.* We now note that the tangents to the two circles at T and T' meet on a limaçon with double point C.

Another comment which seems appropriate is that the intersection of tangents to two circles, the points of contact being collinear with one of the points common to the circles, traces a cardioid, for it is clear that the tangents meet at the same angle as the circles.

For the purpose of establishing the orthoptic† and some related properties of the cardioid synthetically, we may consider it an epicycloid generated by rolling the circle (R) on the fixed circle (F). This and the corresponding theorem for the general limaçon are given as an exercise in Williamson's *Differential Calculus*, p. 350. The reader may be interested in the following simple proof that

* N. A. Court, College Geometry, 1925, p. 48.

† R. C. Archibald, Encyclopedia Britannica, 14th edition, vol. 6, p. 897.

the limaçon can be generated as an epitrochoid. Let (R) carry P, the point which describes the curve. Let (R) start rolling with RP extended towards F, and take O on this fixed diameter of (F) so that FO equals PR. When (R) has rolled to a general position, draw PO and extend it to meet at Q the circle with center F and radius FO. Then $FRPQ$ is a parallelogram and QP is constant, thus showing that P moves along a limaçon with double point O. If P lies on (R), it will trace the cardioid. A definitive property of the limaçon is that as FR turns through an arbitrary angle, RP turns through an angle twice as large. It is also clear in this connection that the normal passes through the point of contact between (F) and (R) and that the line joining this point C to P will, for the cardioid, turn through an angle half again as great as the angle described by FC. It is well known that the evolute of the cardioid is another cardioid homothetic to it with respect to the center in the ratio $-1:3$.* We shall now establish the following:

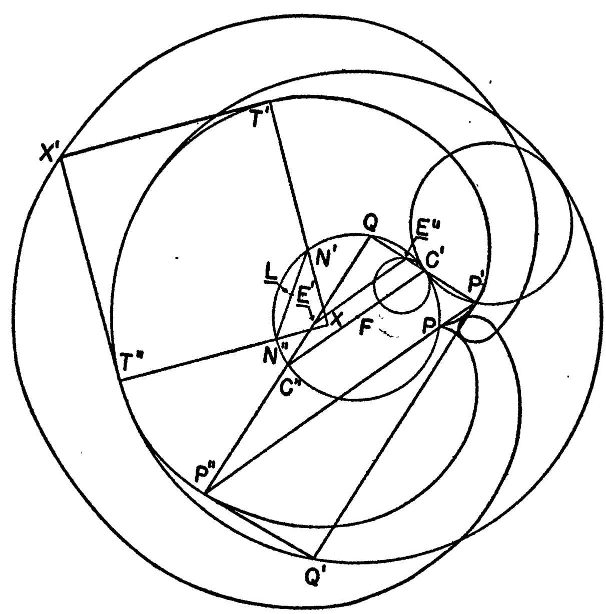


FIG. 2.

THEOREM 2. *The perpendicular tangents to a cardioid at the ends of a cuspidal chord meet on a circle concentric with* (F) *having a radius three times that of* (F), *and the normals at these points meet on* (F) *itself.*

Consider the rolling circle in two positions touching (F) at C' and C'' which are ends of a diameter of (F). (See Fig. 2.) Then the corresponding points P' and P'' of the cardioid are the reflections of the cusp P across the tangents at C' and C''. Since $C'P'$ and $C''P''$ are also reflections of $C'P$ and $C''P$ across $C'C''$, they meet orthogonally at Q on (F). This establishes the part of the theo-

* For a synthetic proof of this, see my solution of problem E 600 [1944, 587], this MONTHLY.

rem concerning the normals. By the argument establishing the evolute, we see that $C'P'$ and $C''P''$ touch the evolute at E'' and E' respectively such that $C'P'$ is three times $E''C'$ and $C''P''$ is three times $E'C''$. Thus from Q as homothetic center, three times $E''E'$ equals $P'P''$, and $E'E''$ is homothetic to $P'P''$ in the ratio $-1:3$ with respect to the center F. Thus $E'E''$ passes through the cusp of the evolute, and by homothetic figures the first part of the theorem is proved.

A right angle may also have its sides tangent to a cardioid without the points of contact T', T'' being collinear with the cusp. A characteristic feature of this situation is that the normals at T', T'' cut (F) at points N', N'' which are 60° apart. This is clear when we note that as the point of contact N between (R) and (F) moves through the arc θ, NT turns through the angle $3\theta/2$. Thus, in order for the normals to meet at 90° or 270°, it is necessary for the points of contact between (F) and the two positions of (R) to be 60° or 180° apart. The latter has been discussed when we were considering C', C'' above, so we know that N' and N'' are 60° apart. We shall proceed to the proof of the following:

THEOREM 3. *Perpendicular normals to a cardioid at points not collinear with the cusp meet on a nodal limaçon, and the perpendicular tangents at the same points meet on a similar limaçon three times as large.*

We see at once that $N'T'$ and $N''T''$ meet at a point X which is on the circle (L) constructed on $N'N''$ as diameter. As L turns through the angle θ about F, it may be shown that LX turns through the angle 2θ. For while N' is describing the arc θ on (F), since $FN'N''$ is equilateral, it is also moving to $\overline{N}$ on (L) through the arc θ. At the same time X is moving on (L) through the arc 2θ to $\overline{X}$, for $\overline{NX}$ makes the angle $3\theta/2$ with a parallel through $\overline{N}$ to $N'X$. Since LX turns through the angle 2θ while FL is turning through θ, X traces a limaçon. That it is nodal appears from the fact that LX is greater than $LF/2$, the radius of the circle centered at L rolling on an equal circle centered at F so as to carry X. The second part of the theorem follows from the homothetic relationship between the evolute and the cardioid itself.

Mathematics for/by the layman. It was with this bold axiom that the pamphlet began, and written in the form of a geometrical theorem it proceeded to prove the impossibility of the existence of God. It ended triumphantly with the three letters Q.E.D., quod erat demonstrandum. To Shelley, who understood nothing of mathematics, this formula had always seemed like a magician's spell for the evocation of Truth—André Maurois, *Ariel*, p. 34.

I could prove God statistically. Take the human body—the chance that all functions of the individual would just happen is a statistical monstrosity.—George Gallup, *Reader's Digest*, October, 1943.

The improbable is also a part of probability—Aristotle.—*E. D. Schell.*

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944-45

Junior Mathematical Club, University of Chicago

During the past year the following papers were presented by students:

Statistics, by Herman Rubin

One and two-dimensional intuitive topology, by R. B. Leipnik

Symbolic logic, by R. P. Brady

A class of definite boundary value problems, presented by Hyman Zimmerberg to the Senior Mathematical Club. As a prize for the best student paper in content and presentation, Mr. Zimmerberg was awarded a copy of Ince's *Ordinary Differential Equations.*

In addition to the student papers, the following talks were presented.

A history of mathematics in America before 1900, by Professor R. G. Sanger

Elliptic integrals, by Dr. J. E. Wilkins, Metallurgical Laboratory, University of Chicago

An application of tensor analysis to moving axes in Newtonian mechanics, by George Platzman, Institute of Meteorology, University of Chicago

Mathematical aspects of the structure of music, by Professor Emeritus A. C. Lunn

Fundamental aspects of valuation theory, by Professor O. F. G. Schilling

Mathematical biophysics of visual aesthetics, by Professor N. Rashevsky, Department of Mathematical Biophysics, University of Chicago

Polynomials with real coefficients whose zeros have negative real parts, by Professor H. S. Wall, Illinois Institute of Technology

Rational approximations to irrational numbers, by Professor L. R. Ford, Illinois Institute of Technology

A functional equation of differential type, by Professor M. L. Hartung

Some applications of the Poisson summation formula, by Dr. W. Karush, Metallurgical Laboratory, University of Chicago.

In addition to the teas given before the talks, the club sponsored parties in both the Autumn and Winter Quarters and a picnic in the Spring Quarter. The officers for the year 1944–45 were: President, Charles Nichols, who was succeeded by Hyman J. Zimmerberg on January 10, 1945; Social Chairman, Marie A. Wurster; Treasurer: Betty Alexander; Committee: Verna La Mantia, Herman Rubin, Harley Flanders. The officers elected for 1945–46 are: President, Marie A. Wurster; Social Chairman, Sally Springer; Treasurer: Flora Dinkines; Committee: August Newlander, Taffee Tanimoto, Dorothy Strayhorne.

Mathematics Club, Oberlin College

During the year from November through May, the following talks were given:

Unsolved mathematical problems, by Professor E. P. Vance at an informal meeting at the home of Professor Carr.

Three unsolvable problems in geometry, by Jeremiah Howald, Donald Taub, and Robert Stuckert

Calculating machines, by Sarah-Lou Lotz

History of mathematics clubs, by Mary Kinsman

A solution of a problem in the MONTHLY, by John Hofmann

The algebra of sets, by Charlotte Peters

Absolute values, by Helen Cutler and Joanne Benton

Logical paradoxes, by Professor Garvin of the Philosophy Department

The principle of duality, by Rodney Hood, who won a prize for this talk

Quaternions, by Ruth Cheney

The rule of double false position, by Margaret Waugh

A problem in limits, by Paul Meier.

We had a Christmas Party at which we had mathematical relays and games. We ended the year with a banquet at the Oberlin Inn, followed by a talk entitled:

The calculus of variations, by Professor E. J. Mickle of Ohio State University.

The officers for 1944–45 have been: President, Ruth Cheney; Vice-President, Rodney Hood; Secretary-Treasurer, Mary Kinsman; Social Chairman, Sarah-Lou Lotz; Publicity Chairman, Norman Weinstein; and Faculty Adviser, Professor E. P. Vance. The officers for 1945–46 are: President, Rodney Hood; Vice-President, Mary Kinsman; Secretary-Treasurer, Charlotte Peters; Social Chairman, Margaret Waugh; Publicity Chairman, Jeremiah Howald; and Faculty Adviser, Professor John Randolph.

Mathematics Club, College of Wooster

The Mathematics Club opened its meetings this year with a picnic, and plans to close them with another picnic.

During the course of the year at bi-monthly meetings invited speakers from various departments of the college told in what ways the study of mathematics was important to their respective fields. Their titles were as follows:

Mathematics in geology, by Dr. Karl Ver Steeg

Use of mathematical principles in physics, by Dr. Earl Ford

Logical aspects of mathematics, by Dr. John Hutchison

Roman engineering, by Dr. Frank Cowles

Mathematics and poets, by Dr. Lowell Coolidge

Mathematical education in Scotland, by Mrs. Haldean Lindsey

Mathematics and music, by Professor Daniel Parmelee.

The officers arranging the program this year are: President, Anne Widener; Secretary-Treasurer, Lois Hayenga. The Faculty Adviser is Dr. C. O. Williamson.

Kappa Mu Epsilon, Albion College

Even under the present circumstances the activities of this year have not diverged considerably from those of normal times. The six new members initiated in March were Phyllis Beckman, Corinne Calkins, Ruth Helzer, Audrey Schuett, Mary Shattuck, and Amy Thomas.

Responses such as algebraic formulae, early mathematicians, geometric theorems, and unusual applications of mathematics, were given in answer to roll call.

Papers presented during the year included:

Lesser known application of mathematics and applications of determinants, by Virginia Tripp

Isolation and approximation of roots, by Audrey McPherson

Mathematics in economics, by George Kawano

Mathematics as a recreation, by Harriette Leonard

Trisecting an angle, by Fusajiro Aburano

Teaching of secondary mathematics, by Janis Barker and Marion Bunte

Applications of elementary mathematics, by Charles Parkhurst

Probabilities, by Beryll Voelker

Fourier series, by Susanne Porter

Application of mathematics to statistics, by Mrs. MariAnn Jones.

Officers for the year 1944–45 were: President, Virginia Tripp; Vice-President, Fusajiro Aburano; Secretary-Treasurer, Marion Bunte, and an additional member of the Program Committee, Harriette Leonard. Faculty Advisers were Professor E. R. Sleight and Assistant Professor E. E. Ingalls.

Kappa Mu Epsilon, Upsala College

Nine meetings were held. Three of these featured papers by students, namely:

Approximation of roots by short-cut methods, by Mrs. Mary McKim

Mathematics in art, by Audrey Richter

A comparison between the Ptolemaic and Copernican systems of the universe, by Betty Rudebock.

Two meetings were devoted to applications as discussed by two engineers from the Wright Aeronautical Corporation:

Gears, by Mr. Jos. Rice, and

Numerical calculations by aid of trigonometry, by Mr. J. Rudebock.

At our initiation banquet, Professor Virgil Mallory of Montclair State Teachers College discussed

The prospects of mathematics in the post-war era.

The officers for the coming year are: President, Audrey Richter; Vice-President, Mrs. Mary McKim; Secretary, Natalie Manno; Treasurer, Marion Larsen; Historian, Betty Rudebock; Faculty Adviser and Corresponding Secretary, Professor M. A. Nordgaard.

RECENT PUBLICATIONS

Edited by H. P. Evans, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Empirical Equations and Nomography. By D. S. Davis. New York and London, McGraw-Hill Book Company, Inc., 1943. 200 pages. $2.50.

The two parts of the book, Part I, Empirical Equations, and Part II, Nomography, may be read independently. The mathematical background required from the reader does not go beyond elementary algebra and geometry but requires some mathematical maturity. Part I deals mainly with the problem of fitting formulas to empirically known relationships between two variables. One chapter is devoted to the same problem for three variables. In view of the elementary approach, no attempt is made to treat the problem statistically, and only one method is presented in detail: a number of observed pairs of values are plotted and, if the relationship suggested by the graph does not appear linear, a change of variables is found which transforms this graph into a linear one; then a straight line is fitted by the method of averages. The method of least squares for fitting a straight line is briefly mentioned, while the possibility of fitting a curve to the original data, without previous rectification, is not considered. Interpolation by the Lagrange-formula is described and illustrated in some detail; no mention is made of interpolation by differences.

Part II contains the derivations and very careful descriptions of procedures used to construct nomographs for relationships of the types (A) $f(x)=F(y)+\phi(z)$, (A′) $1/f(x)=1/F(y)+1/\phi(z)$, (B) $f(x)=F(y)\cdot\phi(z)$ (logarithmic charts), (C) $f(x)=\psi(x)\cdot F(y)+\phi(z)$, (D) $f(x)=F(y)/\phi(z)$ (non-logarithmic charts), and for various combinations of those types. A concluding chapter describes the construction of slide rules for special purposes.

The most impressive feature of the book is the great number of practical examples illustrating a great variety of possible situations, as well as the wealth of problems at the end of each chapter. Most of the examples and problems are taken from the field of industrial chemistry, which may make the book particularly interesting to readers engaged in this field. Without attempting an exhaustive mathematical treatment of either of the topics presented in Part I and Part II, the author has succeeded in giving a clear and interesting presentation of a considerable number of elementary methods which should prove useful in the hands of practical men, mainly in the fields of engineering and industrial management.

Z. W. Birnbaum

Nomography-Probability-Complex Functions. Three lectures by L. R. Ford, A. H. Copeland, Emil Artin. Notre Dame, Indiana (Notre Dame Mathematical Lectures, Number 4), University Press, Notre Dame (lithoprinted), 1944. 70 pages. \$1.25.

While this booklet is listed under the above title in the Notre Dame series, it actually consists of four papers, whose full titles and numbers of pages are given below. The paper by Menger on probability and statistics is an appendix to Copeland's lecture. The three lectures together with a lecture by Menger on the algebra of analysis (which in expanded form has been published as Number 3 of the series) constituted a symposium on the presentation of certain topics of advanced college mathematics, arranged by the University of Notre Dame in April, 1943.

Alignment charts. By L. R. Ford. 28 pages. The exposition is so skillful that it will satisfy the mathematician's desire for generality and yet be intelligible to an undergraduate. The theory is divided into three parts with the italicized titles: (1) *Scales.* A scale is "a curve with certain marks upon it to which numbers are attached." If the variable t denotes the scale-reading, it is convenient to introduce rectangular coordinates and take the equations of the curve in the parametric form $x=f(t)$, $y=g(t)$. (2) *Third order determinants.* An alignment chart consists of three scales, say $x_i=f_i(t_i)$, $y_i=g_i(t_i)$, $i=1, 2, 3$, across which a straight line is laid, its intercepts on the three scales determining a functional relation among t_1, t_2, t_3. Necessary and sufficient conditions that a given functional relation among three variables be reproducible by an alignment chart are given in terms of the vanishing of certain third order determinants. (3) *Transformation theory.* The general projective transformation of the plane is introduced analytically as a tool for transforming alignment charts. As an illustration it is shown how any desired part of an alignment chart lying in a convex quadrilateral may be transformed so as to occupy a given rectangle.

The teaching of the calculus of probability. By A. H. Copeland. 12 pages. This outline of the calculus of probability begins with sequences and expected values rather than measure and probabilities. It is assumed that if measurements $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ are made on a physical quantity, then these are the first n terms of an infinite sequence $x=\{x^{(i)}\}$ having certain properties, one of which is that

$$\lim_{n\to\infty} \sum_{i=1}^{n} x^{(i)}/n$$

exists. This is called the expected value of the quantity measured and is written $p(x)$. Given a (real-valued) function of real variables, a corresponding (sequence-valued) function of sequences is defined, for example, the function $f(x, y)$ of sequences $x=\{x^{(i)}\}$ and $y=\{y^{(i)}\}$ is defined to be the sequence $\{f(x^{(i)}, y^{(i)})\}$. Constant functions c are identified with sequences $c, c, c, \cdots$. From this it

follows that the expected value operator p is a linear operator. Probabilities are then introduced as expected values in sequences of ones and zeros corresponding respectively to successes and failures of an event. It is indicated how from this basis may be developed the usual tools of probability theory——distribution functions, Stieltjes integrals, *etc.* The lecture concludes by raising the following important problem of application of the theory: From a knowledge of a finite number of terms $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ of a sequence x of zeros and ones, representing the outcome of n trials of an event, how can the hypothesis that the probability $p(x)$ of the event has a given value, say $\frac{1}{2}$, be verified? The reviewer found the author's treatment of this well-known problem of statistical inference obscure, but the difficulty may lie in the highly condensed nature of the exposition.

The author's approach to probability theory by way of sequences of observations on physical quantities has the pedagogical advantage of seeming close to the material to which the theory is to be applied. The danger of this approach in its present form is that it leaves the impression that every series of physical measurements is in "statistical control" and hence a fit subject for statistical inferences valid for random samples. We owe to Shewhart our skepticism of this assumption. The usual way of avoiding the difficulty is to develop a model more independently of series of physical measurements and then to emphasize that not all series of measurements fit the model.

On the relation between calculus of probability and statistics. By Karl Menger. 10 pages. The editor considers the problem mentioned above of testing the hypothesis that the probability of an event has a given value. He uses this example to illustrate some of the basic concepts of the Neyman-Pearson theory of testing statistical hypotheses. The reader is gracefully introduced to these basic ideas without being burdened with the specialized terminology of the theory.

On the theory of complex functions. By Emil Artin. 14 pages. The purpose of the lecture is to indicate how a course in complex variables may be simplified by a heavier use than customary of certain simple topological concepts. A function $V(A, \zeta)$ of arcs A and points ζ is defined for pairs A, ζ with ζ not on A as the increase of the argument of the vector from ζ to a point z on the arc A as z sweeps out the (oriented) arc. Chains C of arcs are introduced and $V(C, \zeta)$ for ρ not on C is defined additively over the arcs of the chain. It is noted that for a closed chain, $V(C, \zeta)/(2\pi)$ is an integer-valued function, constant for ζ on any connected open set disjoint from C, and this function is called the *winding number* $W(C, \zeta)$. It is used extensively in developing the theory; for example, Cauchy's theorem reads as follows: If $f(z)$ is analytic in the open set D, and if C is a closed chain of rectifiable arcs in D for which the winding number $W(C, \zeta)$ vanishes for every ζ in the complement of D, then $\int_C f(z)dz = 0$. The developments extend through Cauchy's integral formula to some of the more elementary theorems concerning zeros and isolated singularities of analytic functions.

HENRY SCHEFFÉ

NEW BOOKS RECEIVED

Cambridge Joint Advisory Committee for Mathematics: Syllabuses for Examinations Taken by Sixth Form Pupils. Cambridge University Press, 1945. 12 pages. 6 d.

Collected Works. Vol. 1. By P. L. Tchebychev. Moscow-Leningrad, Academy of Sciences of the U.S.S.R., 1944. 342 pages.

Curso de Análisis Matemático. Vol. 1. By Cristóbal de Losada y Puga. Lima, Universidad Catolica del Peru, 1945. 15+632 pages.

Elementary Statistics. By Hyman Levy and E. E. Preidel. New York, Ronald Press Co., 1945. 7+184 pages. $2.25.

Engineering Preview. An Introduction to Engineering Including the Necessary Review of Science and Mathematics. By L. E. Grinter, H. N. Holmes, H. C. Spencer, Rufus Oldenburger, Charles Harris, R. G. Kloeffler and V. M. Faires, New York, Macmillan Co., 1945. 10+581 pages. $4.50.

Fermagoric Triangles. By Pedro Pizá. (Publication No. 1 of the Polytechnic Institute of Puerto Rico.) San Germán, P.R. 153 pages. $3.00.

How to Solve It. By G. Polya. Princeton, University Press, 1945. 15+204 pages. $2.50.

Lobachevsky. By V. F. Kagan. Moscow-Leningrad, Academy of Sciences of the U.S.S.R., 1944. 348 pages (In Russian).

Mathematics of Finance. By T. E. Raiford. Boston, Ginn and Co., 1945. Without Tables: 3+176 pages, $2.50; with tables: 3+176+305 pages, $3.60.

Plane and Spherical Trigonometry. By F. M. Morgan. New York, American Book Co., 1945. 15+257+72 pages. $2.50.

Select Topics of Plane Analytic Geometry for Scientific and Technical Workers. By M. W. Rosanoff. Brooklyn, Long Island University Press, 1944. 8+76 pages. $1.25.

The Meaning of Relativity. Second Edition. By Albert Einstein. Princeton, University Press, 1945. 4+135 pages. $2.00.

The Simple Calculation of Electrical Transients. By G. W. Carter. Cambridge, University Press; New York, Macmillan Co., 1945. 8+120 pages. $1.75.

Theory of Functions. Part 1. By Konrad Knopp. (Translated by Frederick Bogehmihl from the fifth German edition.) New York, Dover, 1945. 7+146 pages. $1.25.

Precision, morals, torches, and algebra. My thoughts were directed to the problem of how the moral sciences may be reduced to mathematical precision.—Christian Wolff (1716).

I, at any rate, have not the same lively hope as Condorcet, or even Edgeworth, "to illuminate the moral and political Sciences by the torch of Algebra."—J. M. Keynes (1921).—*Contributed.*

Send interesting short notes to Professor E. T. Bell, California Institute of Technology, Pasadena 4, California.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 681. *Proposed by W. B. Campbell, Philadelphia*

From *Mrs. Miniver*: "She saw every relationship as a pair of intersecting circles. The more they overlapped, it would seem at first glance, the better the relationship; but this is not so. Beyond a certain point, the law of diminishing returns sets in, and there are not enough private resources left on either side to enrich the life that is shared. Probably perfection is reached when the area of the two outer crescents, added together, is exactly equal to that of the leaf shaped piece in the middle. On paper there must be some neat mathematical formula for arriving at this; in life, none."

Discuss the possibility of a unique solution for circles of given radii.

E 682. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find three positive integers x, y, z, such that the sum of ratios

$$\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$$

is an integer. (Hint: suppose $x+y+z=23$.)

E 683. *Proposed by Irving Kaplansky, Columbia University*

A person tosses a coin n times. For every unbroken run of i heads he will receive 2^i dollars. What is his expectation?

E 684. *Proposed by Paul Erdös, Stanford University*

Prove that $n!^{(n-1)!}$ divides $(n!)!$; and that if n is not a power of a prime, $n!^{n(n-2)!}$ divides $(n!)!$.

E 685. *Proposed by Peter Scherk, University of Saskatchewan*

Prove that the equation $x^r-1=p^n$, where p is prime while r and n are integers greater than 1, has only one solution. (*Cf.* E 663 [1945, 159].)

SOLUTIONS

Four Equal Spheres in a Tetrahedron

E 637 [1944, 472]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Locate a plane which touches four equal spheres inscribed in the trihedra at the respective vertices of a given tetrahedron.

Solution by Howard Eves, College of Puget Sound. There are four planes of the required type, one parallel to each face of the tetrahedron. A representative one of these planes may be found as follows. Let h be an altitude and r the inradius of the tetrahedron. By simple considerations of similarity, the required plane parallel to the face corresponding to h is located at distance h' from the opposite vertex, here $h':h=h:(h+2r)$, so that $h'=h^2/(h+2r)$.

The analogous problem for a triangle is solved similarly.

A Property of Central Conics

E 645 [1944, 530]. *Proposed by C. D. Olds, Purdue University*

Let P_1, P_2, P_3 be any three points on a plane curve C, and let O be any point in the plane of C. Show that if the areas of the three triangles OP_2P_3, OP_3P_1, OP_1P_2 are connected by a relation independent of the coordinates of P_1, P_2, P_3, then C is either a straight line or a conic with center O.

Solution by the Proposer. Let O be the origin of coordinates, and let P_i be the point (x_i, y_i). Then the respective areas a_1, a_2, a_3 are given by

$$\pm\, 2a_1 = y_2x_3 - y_3x_2, \qquad \pm\, 2a_2 = y_3x_1 - y_1x_3, \qquad \pm\, 2a_3 = y_1x_2 - y_2x_1,$$

which expressions are functions of x_1, x_2, x_3. We think of y as a given function of x for points on C.

Since a_1, a_2, a_3 must satisfy a relation independent of x_1, x_2, x_3, the vanishing of the Jacobian $\partial(a_1, a_2, a_3)/\partial(x_1, x_2, x_3)$ leads to the equation

$$y_3'(y_3k + x_3l) + x_3m + y_3l = 0, \tag{1}$$

where

$$k = x_1x_2(y_2' - y_1') - (x_2y_1 - x_1y_2),$$
$$l = x_2y_1y_1' - x_1y_2y_2',$$
$$m = y_1y_2(y_2' - y_1') - y_1'y_2'(x_2y_1 - x_1y_2),$$

and $y_1' = \partial y_1/\partial x_1$, *etc.* Notice that k, l, m are functions of x_1, x_2 only. The following two cases present themselves.

(a) Suppose that $k=l=m=0$. From $k=0$ and $m=0$, eliminate the expression $x_2y_1-x_1y_2$, getting

$$(y_2' - y_1')(y_1y_2 - y_1'y_2'x_1x_2) = 0,$$

so that either $y_2'=y_1'$ or $y_1y_2=y_1'y_2\cdot y_2'y_1$. If $y_2'=y_1'$, then $m=0$ implies that $x_2y_1=x_1y_2$. For a given P_1, the locus of P_2 is the straight line $y_1x=x_1y$. On the other hand, if $y_1y_2=y_1'x_2\cdot y_2'x_1$, then combining this with $l=0$ we get $y_2^2=y_1'^2x_2^2$, or

$$\pm\ x_2 y_1' - y_2 = 0.$$

For any definite constant value of y_1', either of these equations gives a straight line.

(b) If k, l, m are not all identically zero, then (1) gives, when integrated,

$$y_3^2 k + 2y_3 x_3 l + x_3^2 m = f(x_1, x_2).$$

Thus, for arbitrary values of x_1, x_2, the locus of (x_3, y_3) is a conic with its center at O.

Weighed and Found Wanting

E 651 [1945, 42]. *Proposed by E. D. Schell, Arlington, Virginia*

You have eight similar coins and a beam balance. At most one coin is counterfeit and hence underweight. How can you determine whether there is an underweight coin, and if so, which one, using the balance only twice?

Solution by Monte Dernham, San Francisco. Weigh three coins against any other three. If they balance, weigh the seventh coin against the eighth. If the two sets of three do not balance, weigh against each other any two coins of the lighter set. If these two balance, the remaining one of that set is counterfeit.

Also solved by D. W. Alling, Murray Barbour, Ruth Beck, D. H. Browne, W. E. Buker, W. E. Bunyan, H. N. Carleton, Howard Eves, Orrin Frink, Jr., R. E. Greenwood, J. E. Hanson, Robert Hoskins, Irving Kaplansky, V. L. Klee, Jr., H. D. Larsen, A. D. M. Lewis, J. S. Miller, G. H. Neugebauer, John Pack, W. O. Pennell, J. Price, Jacob Schachter, E. P. Starke, R. H. Urbano, Jeanette Van Os, W. R. Van Voorhis, Alan Wayne, T. L. Woolard, and the proposer.

The following generalization was noticed by Kaplansky, Neugebauer, and Pennell. In any set of N coins, where $3^{n-1} \leqq N < 3^n$, not more than n balancings are needed to determine whether there is an underweight coin, and if so, which one. But if there is known to be an underweight counterfeit among N coins, where $3^{n-1} < N \leqq 3^n$, then not more than n balancings are needed to determine the counterfeit.

An Extension of E 569

E 653 [1945, 42]. *Proposed by J. H. Butchart, Grinnell College*

The ends of a chord UV of the circle $r=a$ have the parametric angles ϕ and $k\phi$, where k is a constant greater than 1. Show that the locus of the midpoint of UV is a prolate epitrochoid whose polar equation is

$$r = a \cos \frac{k-1}{k+1}\theta.$$

Show also that the envelope of UV is an epicycloid, the point of contact dividing UV in the ratio $1:k$.

Solution by R. H. Urbano, Albany, N. Y. We have

$$r = a \cos \tfrac{1}{2}(k\phi - \phi), \qquad \theta = \tfrac{1}{2}(k\phi - \phi) + \phi = \tfrac{1}{2}(k\phi + \phi).$$

Therefore

$$r = a \cos \frac{k-1}{k+1}\theta.$$

The line UV, joining $(a \cos \phi, a \sin \phi)$ and $(a \cos k\phi, a \sin k\phi)$, has the equation $F=0$, where

$$F = (x - a \cos \phi)(\sin \phi - \sin k\phi) - (y - a \sin \phi)(\cos \phi - \cos k\phi).$$

The envelope of this line is the simultaneous solution in x and y of the equations $F=0$ and $\partial F/\partial\phi=0$. Since

$$\frac{\partial F}{\partial \phi} = x(\cos \phi - \sin k\phi) + y(\sin \phi - k \sin \phi) + a(k - 1) \cos (k - 1)\phi,$$

the result is

$$x = a(\cos k\phi + k \cos \phi)/(k + 1),$$
$$y = a(\sin k\phi + k \sin \phi)/(k + 1).$$

For each value of ϕ, (x, y) is the point of contact of the line $F=0$ with its envelope, which, being the locus of (x, y), is the well known epicycloid. Finally, (x, y) divides UV in the ratio $1:k$.

Also solved by F. C. Hall.

The Sum of p Consecutive Superfactorials

E 654 [1945, 42]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $n!_1$ denote the coefficient of $x^n/n!$ in the expansion of $e^x/(1-x)$. Show that, for a prime p and any nonnegative integer k,

$$\sum_{n=k}^{k+p-1} n!_1 \equiv -1 \pmod{p}.$$

(*Cf.* E 488 [1942, 478].)

Solution by N. J. Fine, Lukas-Harold Laboratory, Indianapolis. Since $n!_1=1+n(n-1)!_1$, we have

$$(n + p)!_1 = 1 + (n + p)(n + p - 1)!_1 \equiv 1 + n(n + p - 1)!_1 \pmod{p}.$$

Subtracting,

$$(n + p)!_1 - n!_1 \equiv n\{(n + p - 1)!_1 - (n - 1)!_1\}.$$

But $p!_1-0!_1=1+p(p-1)!_1-1\equiv 0$. Therefore $(n+p)!_1\equiv n!_1$ for every n. Consequently

$$\sum_{n=k}^{k+p-1} n!_1 \equiv \sum_{n=0}^{p-1} n!_1 \qquad \pmod{p}.$$

Now, since

$$n!_1 = n! \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{n} \binom{n}{k} k!,$$

$$\sum_{n=0}^{p-1} n!_1 = \sum_{n=0}^{p-1} \sum_{k=0}^{n} \binom{n}{k} k! = \sum_{k=0}^{p-1} k! \sum_{n=0}^{p-1} \binom{n}{k}.$$

The inner sum is $\binom{p}{k+1}$; for it is the coefficient of x^k in the expansion of

$$\sum_{n=0}^{p-1} (1 + x)^n = \frac{(1 + x)^p - 1}{x} = \sum_{k=0}^{p-1} \binom{p}{k+1} x^k.$$

Since $\binom{p}{k+1} \equiv 0 \pmod{p}$ unless $k = p-1$, Wilson's Theorem gives

$$\sum_{n=0}^{p-1} n! \equiv (p - 1)! \equiv -1.$$

Also solved by D. W. Alling, H. W. Becker, J. B. Kelly, John Riordan, E. P. Starke, and the proposer.

Twenty Terms of a Geometrical Progression

E 655 [1945, 42]. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

The first term of a geometrical progression is a three-digit number, and the ratio is 9. Each term is divisible by the sum of its digits. Find the sequence having the largest number of terms. (The one beginning with 100 has thirteen terms.)

Solution by the Proposer. The sum of the digits of a product when 9 is the multiplier is 9 or a multiple of 9. Each term of the progression after the first, therefore, will contain powers of the prime factor 3. For an optimum progression the first term must contain the prime factors 2, 5, 7; *i.e.*, it must be a three-digit multiple of 70, divisible by the sum of its digits. Such numbers are 140, 210, 280, 420, 630, 700, 770, 840, 910. In those cases where the factor 2 occurs to the first power, the progression will break down for a term with digit-sum 36; so these may be omitted from consideration. Only five optimum first terms remain: 140, 280, 420, 700, 840. Of these 140 gives a sequence with 20 terms, the largest number found.

Thus the answer is

$$140, 1260, 11340, \cdots, 18911924047421889 2460.$$

Solved incorrectly by Murray Barbour, D. H. Browne, J. E. Hanson, Robert Hoskins, Irving Kaplansky, and E. D. Schell, who all found the first term 280 or 840, and consequently only 18 terms altogether.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4166. *Proposed by J. H. Butchart, Grinnell College.*

Given the straight lines l, l' and a point F. A variable circle through F and the intersection of l, l' cuts l and l' in A and A' respectively. The circles through F and tangent to l, l' at A, A' respectively meet again on a parabola tangent to l and l' and having F as focus.

4167. *Proposed by R. A. Staal, Student, University of Toronto*

What curves are self-reciprocal with respect to the conic $x^2+y^2=z^2$ (or the circle $x^2+y^2=1$)?

4168. *Proposed by Henry Scheffé, Princeton University*

The matrix $X=(x_{ij})$ is real $m\times n$ with $m\leqq n$. Let the ith row sum be $y_i=\sum_j x_{ij}$. The elements x_{ij} may vary subject to the conditions (1) $XX'=I$, (2) $y_1=y_2=\cdots=y_m$. Write $y_i=y$. Show that the maximum and minimum values of y are $\sqrt{n/m}$ and $-\sqrt{n/m}$ respectively.

4169. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The tangents at the vertices A, B, C of a given triangle to its Feuerbach hyperbola form a triangle whose conjugate circle is tangent to the inscribed circle of ABC at its Feuerbach point.

4170. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The powers of the vertices A, B, C of a given triangle with respect to the circles (ω_1), (ω_2), (ω_3) are respectively (ka^2, kb^2, kc^2), (kb^2, kc^2, ka^2), (kc^2, ka^2, kb^2), where a, b, c are the lengths of the sides of ABC. Find the loci of the centers ω_i as k varies. (2) The circumcenter of ABC is the centroid of triangle $\omega_1\omega_2\omega_3$ which remains similar to itself. (3) The straight line $\omega_2\omega_3$ is perpendicular to the join of the centroid and Lemoine point of ABC.

SOLUTIONS

Self Polar Triangle Inscribed in Another Conic

4063 [1942, 689]. *Proposed by H. S. M. Coxeter, University of Toronto*

In projective geometry the porism of triangles inscribed in one conic and self-polar for another is commonly proved by showing that if one such triangle exists,

we can find another with one vertex at *any* given point on the first conic. This statement is easily seen to be valid in complex geometry. Discuss its possible failure in real geometry.

II. *Solution by R. G. Stanton, University of Toronto.* Let the two conics be represented by homogeneous equations

$$yz + zx + xy = 0, \tag{1}$$

$$ax^2 + by^2 + cz^2 = 0. \tag{2}$$

The reference triangle is inscribed in (1), self-polar re (2). Let (x, y, z) be any point on (1). Its polar re (2) has the tangential coordinates $[ax, by, cz]$. Since the tangential equation for (1) is

$$X^2 + Y^2 + Z^2 - 2YZ - 2ZX - 2XY = 0,$$

this polar meets (1) in two points which coincide if and only if

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0. \tag{3}$$

The four points of intersection of the two conics (1) and (3) give four points on (1) whose polars re (2) touch (1).

If a point P is on (1), let its polar QR, re (2), meet (1) in Q and R to form the self-polar triangle PQR. The problem is to determine for what positions of P this self-polar triangle is real. The transition from real to imaginary positions occurs when QR changes from a secant to an exterior line; thus the transitional stage is one in which QR is tangent to (1), and can occur for only four positions of P, corresponding to the roots of the quartic equation for x/y formed by eliminating z between (1) and (3). Three situations may arise:

(i) the quartic has four real roots;
(ii) the quartic has only two real roots;
(iii) the quartic has no real roots.

In case (i), the four real points will divide the conic (1) into four arcs, on two of which P may lie for real self-polar triangles. In case (ii) there will be only one such arc. In case (iii) every position on the conic will give a real triangle. (One such being already given, all must be real.) Intermediate situations, corresponding to coincident roots, occur when (1) and (2) are in contact.

Examples of these situations are readily provided. If $a = b = 1$, we have two arcs of admissible positions for P, or only one, according as $c < -8$ or $-8 \leqq c < 0$. An instance of case (iii) occurs when (2) is

$$\tfrac{1}{4}x^2 + y^2 - z^2 = 0.$$

Editorial Note. A partial solution is given 1944, 171.

4098 [1945, 49]. An Alternant Type of Determinant

II. *Note by Arnold Dresden, Swarthmore College.* The result obtained in the solution [1945, 49] appears to be practically identical with Theorem 2 in the

article *On the generalized Vandermonde determinant and symmetric functions*, in the Bull. Am. Math. Soc., 1933, vol. 39, no. 6, p. 447.

Hyperboloid and Parallelepiped

4112 [1944, 167]. *Proposed by N. A. Court, University of Oklahoma*

If three rulers, chosen arbitrarily, of the same system of a given hyperboloid are taken for the edges of a parallelepiped, the diagonals of the parallelepiped meet in a fixed point.

Solution by Robert Steinberg, University of Toronto. Let A, B, C, D, A', B', C', D' be the vertices of the parallelepiped so that $ABCD$ and $A'B'C'D'$ are a pair of opposite faces joined by the edges AA', BB', CC', DD', let L, M, N be the points at infinity on AB, AD, AA' respectively, and let O be the point of intersection of the diagonals of the parallelepiped. Suppose that the three given generators are AA', BC, and $C'D'$. Then, three generators of the other system of the hyperboloid are CC', $A'D'$ and AB since each of these three lines meets each of the three given generators. Hence through L we have the two generators ABL and $C'D'L$. Thus the plane LBC' is the polar plane of L. Since AC' and BD' meet at O, this means that O lies in the polar plane of L. Hence, L lies in the polar plane of O. Similarly M and N lie in the polar plane of O, and hence O is the pole of the plane at infinity, *i.e.*, O is the center of the hyperboloid, a fixed point.

Solved also by the proposer, who referred to *Educational Times, Reprints*, vol. 3, 1903, p. 120, Q. 5778, Wilkins-Nanson.

Factorial Coefficients

4118. *Corrected. Proposed by Otto Dunkel, Washington University*

Show that

$$\sum_{t=0}^{n}(-1)^{n+t}\frac{t^{n+4}}{t!(n-t)!}=\frac{(n+4)(n+3)\cdots n}{6!8}\left[15n^3+30n^2+5n-2\right],\quad n\geqq 0,$$

and that each member of this equality is a non-negative integer. If n is a negative integer, the right member is an integer; what meaning may be given to the result in this case?

Solution by J. S. Frame, Michigan State College. We expand the function $f(x)\equiv x^{-n}(1-e^{-x})^n$ in a power series $\sum f_r x^r$ in two distinct ways, and obtain two expressions for the number $N=(n+4)(n+3)(n+2)(n+1)f_4$, which are the left and right hand members respectively of the given identity.

First, if n is a positive integer, we have

$$f(x)=x^{-n}\sum_{t=0}^{n}(-1)^t\binom{n}{t}e^{-tx}=x^{-n}\sum_{t=0}^{n}\sum_{s=0}^{\infty}(-1)^{t+s}\binom{n}{t}\frac{t^s x^s}{s!}.$$

The coefficient f_4 of x^4 is obtained by setting $s=n+4$ in the final summation; from this we get N in the form of the left member:

$$f_4 = \sum_{t=0}^{n} (-1)^{t+n+4} \binom{n}{t} \frac{t^{n+4}}{(n+4)!}; \qquad N = \sum_{t=0}^{n} (-1)^{t+n} \frac{t^{n+4}}{t!(n-t)!}.$$

Secondly, we expand the nth power of the function $g(x) = (1-e^{-x})/x = \sum g_j x^j$ by means of the following lemma.

LEMMA. *If* $\sum_{r=0}^{\infty} f_r x^r = (\sum_{j=0}^{\infty} g_j x^j)^n$, *and* $f_0 = g_0 = 1$, *then*

$$f_m = \sum_{k=1}^{m} \binom{n}{k} \sum_j g_{j_1} g_{j_2} \cdots g_{j_k} \quad \text{where} \quad j_1 + j_2 + \cdots + j_k = m, \ j_s \geqq 1.$$

In this problem we have $g_j = (-1)^j/(j+1)!$,

$$f_4 = \binom{n}{1} g_4 + \binom{n}{2} (g_1 g_3 + g_2 g_2 + g_3 g_1) + \binom{n}{3} (3 g_1 g_1 g_2) + \binom{n}{4} g_1^4,$$

$$= \binom{n}{1} \frac{1}{5!} + \binom{n}{2} \left(\frac{1}{2!4!} + \frac{1}{3!3!} + \frac{1}{4!2!} \right) + \binom{n}{3} \frac{3}{2!2!3!} + \binom{n}{4} \frac{1}{16}.$$

$$N = \binom{n+4}{5} + \binom{n+4}{6} 3\cdot 4\cdot 5\cdot 6 \left(\frac{1}{24} + \frac{1}{36} \right) + \binom{n+4}{7} \frac{4\cdot 5\cdot 6\cdot 7}{8}$$

$$+ \binom{n+4}{8} \frac{5\cdot 6\cdot 7\cdot 8}{16},$$

$$= \binom{n+4}{5} + 25 \binom{n+4}{8} + 105 \binom{n+4}{7} + 105 \binom{n+4}{8}.$$

This expression defines an integer N for all integral values of n. It reduces at once to the right member of the given identity; since

$$N = \frac{(n+4)(n+3)\cdots n}{6!8} [6\cdot 8 + 200(n-1) + 120(n-1)(n-2) + 15(n-1)(n-2)(n-3)].$$

When $n = -m$, $m \geqq 5$, the series for $f(x)$ must be replaced by

$$f(x) = x^m \sum_t (-1)^t \binom{-m}{t} e^{-tx} = x^m \sum_{t=0}^{\infty} \binom{m-1+t}{m-1} e^{-tx},$$

which converges for positive values of x. The number $N = (m-1) \cdots (m-4) f_4$ is

$$N = \binom{m-1}{4} \lim_{x\to 0} \sum_{t=0}^{\infty} \frac{(m-1+t)!}{(m-1)!t!} \frac{d^4}{dx^4} (x^m e^{-tx}), \qquad m = -n.$$

Solved by C. D. Olds, G. Pólya, and G. T. Williams.

Editorial Note. In the notation of 4108 [1945, 281] we have

$$x^n = \sum_{r=1}^{n} {}_nQ_r x^{(r)}, \quad x^{(r)} = x(x-1)\cdots(x-r+1), \quad {}_{n+1}Q_r = {}_nQ_{r-1} + r\,{}_nQ_r, \tag{1}$$
$$ {}_nQ_1 = {}_nQ_n = 1, \quad {}_rQ_r = \Delta^r 0^n/r!.$$

Hence

$$\begin{aligned} {}_{n+r}Q_n &= \Delta^n 0^{n+r}/n! = (U-1)^n 0^{n+r}/n! = \sum_{t=0}^{n} {}_nC_t U^t(-1)^{n-t}0^{n+r}/n! \\ &= \sum_{t=0}^{n} (-1)^{n+t} t^{n+r}/t!(n-t)!. \end{aligned} \tag{2}$$

It was shown also that ${}_{n+r}Q_n = \sigma_r(-n)$, where $\sigma_r(n) = {}_nP_{n-r}$ is the rth elementary symmetry function of $1, 2, \cdots, n-1$. In 3940 [1941, 641] and the above reference it is shown that

$$ {}_nP_{n-r} = \sigma_r(n) = n^{(r+1)} \sum_{t=1}^{r} (-1)^{r+t}\, {}^r a_t (n+t-1)^{(t-1)}; \tag{3}$$

and formulas are given for the computation of a_t^r with the numerical results for $0 \leqq r \leqq 5$. Thus

$$\Delta^n 0^{n+r}/n! = \sum_{t=0}^{n} (-1)^{n+t} t^{n+r}/t!(n-t)! = (n+r)^{(r+1)} \sum_{t=0}^{r} {}^r a_t (n-1)^{(t-1)}. \tag{4}$$

These formulas give the results desired in the problem. It is shown that the polynomial $\sigma_r(n)$ of degree $2r$ in n has the factor $n^2(n-1)^2$ if r is odd $\geqq 3$. Thus

$$\sum_{t=0}^{n} (-1)^{n+t} \frac{t^{n+5}}{t!(n-t)!} = (n+5)^{(6)}(n+1)n(3n^2+7n-2)/2^5 5!3. \tag{5}$$

All the solvers replaced the erroneous $15n$ by $5n$. Olds obtained the results in (1) and (2) in a different notation, and then gave a table of his computations of ${}_{n+r}Q_n$ for $1 \leqq r \leqq 4$ which resulted in the equation of the problem. This was followed by developments of ${}_nP_r$ and ${}_nQ_r$ similar to those given by Weisner in the solution of 4108, but in a different form using in one case the generating function $(e^x-1)^n$. The solutions by Pólya and Williams also used the same generating function.

Gergonne Point

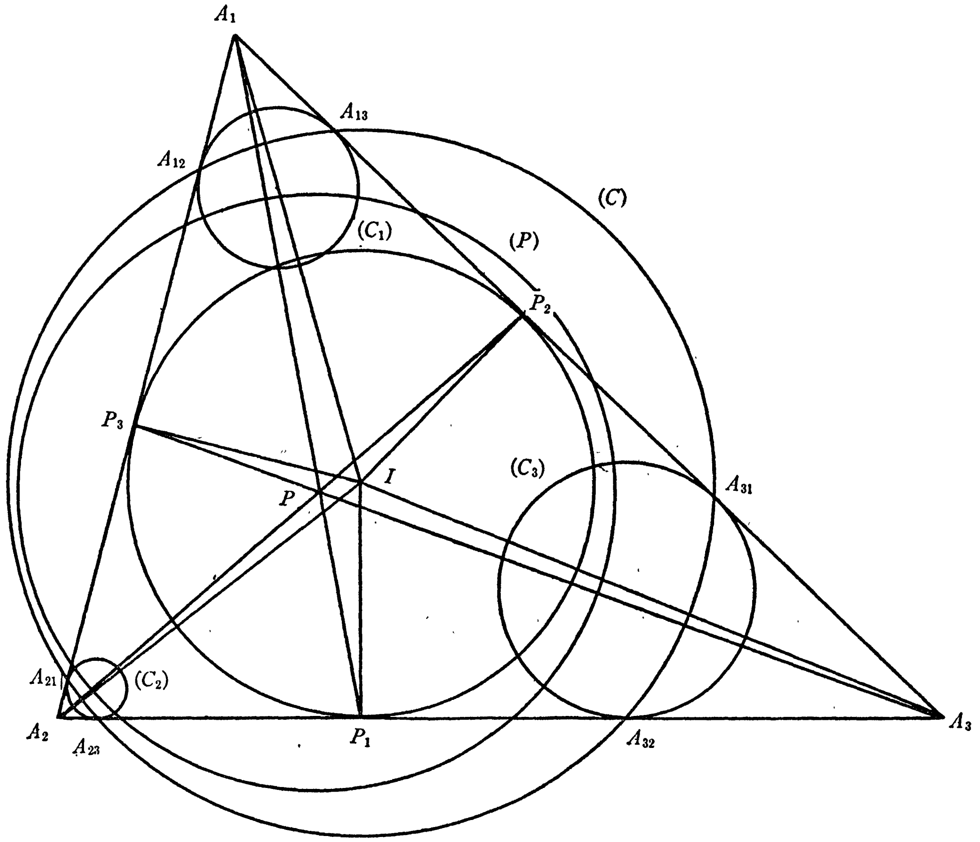
4119 [1944, 234]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The straight lines joining the vertices of a triangle to the points of contact of the inscribed circle with the respective opposite sides meet in a point P. Show that the six points of contact of circles tangent to two sides and orthogonal to a given circle with center P are on a circle concentric with the inscribed circle.

Solution by Howard Eves, Syracuse University. Let $A_1A_2A_3$ be the given triangle, and let P_1, P_2, P_3 be the feet of the perpendiculars from the incenter I on the sides of the triangle, P_i lying opposite A_i. Designate the given circle with center P by (P) and let (C_1) be a circle orthogonal to (P) and touching A_1A_3 at A_{13} and A_1A_2 at A_{12}.

Draw circle (C) concentric with the incircle and passing through A_{13} and A_{12}. Let (C) cut A_1A_3 again in A_{31} and A_1A_2 again in A_{21} and A_2A_3 in A_{23} and A_{32}. Draw (C_2) touching A_2A_1 and A_2A_3 at A_{21} and A_{23}, and (C_3) touching A_3A_1 and A_3A_2 at A_{31} and A_{32}. Since I is the center of (C) it follows that all the segments P_iA_{jk} $(i\neq j\neq k\neq i)$ are equal. Hence P_i and A_i lie on the radical axis of C_j and C_k. Thus P is the radical center of the three circles (C_i), and since (P) is orthogonal to (C_1) it is also orthogonal to (C_2) and (C_3).

This is sufficient to establish the theorem.



Note. In connection with this problem see E 457 (vol. 48, no. 9, p. 636, Nov. 1941).

Editorial Note. The proposer remarked as follows: It is easy to show that the radical center of three circles (C_1), (C_2), (C_3) each tangent to two sides of a triangle, so that the points of contact are six points of a single circle, is the Gergonne point P in whatever manner the three tangent circles are chosen with their centers C_i on the internal bisectors. The theorem of the problem states the converse of this property.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Associate Professor R. S. Burington of Case School of Applied Science has been awarded the Meritorious Civilian Award from the U. S. Navy for his work as director and organizer of a research analysis group in the Research and Development Division of the Bureau of Ordnance.

Professor Garrett Birkhoff of Harvard University and Professor Norman Levinson of the Massachusetts Institute of Technology have been elected fellows of the American Academy of Arts and Sciences.

Dr. Paul Erdös has been awarded a Guggenheim fellowship.

Reverend J. T. O'Callahan of the College of the Holy Cross, Lieutenant Commander in the Navy, has been recommended for the Congressional Medal of Honor for outstanding bravery while in service on the U.S.S. *Franklin*.

Professor Hassler Whitney of Harvard University and Professor E. P. Wigner of Princeton University have been elected members of the National Academy of Sciences.

Associate Professor Harriet W. Allen of Hollins College has accepted a position with the Air Reduction Company of Stamford, Connecticut.

Dr. L. A. Aroian of Hunter College has been granted a leave of absence to serve as research associate in the applied mathematics panel at Berkeley, California.

Associate Professor Edith I. Atkin of Illinois State Normal University has retired.

Associate Professor I. A. Barnett of the University of Cincinnati has been promoted to a professorship.

A. F. Bartholomay of Keuka College has been promoted to an assistant professorship.

Associate Professor E. F. Beckenbach of the University of Texas has been appointed to an associate professorship at the University of California at Los Angeles.

Assistant Professor Z. W. Birnbaum of the University of Washington has been promoted to an associate professorship.

Assistant Professors I. W. Burr and H. F. S. Jonah of Purdue University have been promoted to associate professorships.

Assistant Professor Herbert Busemann of the Illinois Institute of Technology has been appointed to an assistant professorship at Smith College.

Professor Theodosia T. Callaway of Stephens College, Columbia, Missouri, has retired.

Associate Professor R. H. Cameron of the Massachusetts Institute of Technology has been appointed to a professorship at the University of Minnesota.

Professor L. M. Coffin of Coe College, Cedar Rapids, Iowa, has retired with the title professor emeritus.

Professor A. R. Congdon of the University of Nebraska has retired.

Dr. Helen W. Dodson has been appointed to an associate professorship at Goucher College.

Professor Albert Einstein of the Institute for Advanced Study has retired with the title professor emeritus.

Norman Gunderson of Cornell University is now serving in the Allegany Ballistics Laboratory, Cumberland, Maryland.

C. F. Hall of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. Margaret M. Hansman of Colorado College has been promoted to an assistant professorship.

Assistant Professor M. L. Hartung of the University of Chicago has been appointed associate professor of the teaching of mathematics.

Dr. F. E. Hohn of the University of Arizona has been appointed to an assistant professorship at Guilford College, Guilford College, North Carolina.

Assistant Professor H. T. Karner of Louisiana State University has been appointed associate professor of mathematics and dean of men.

Dr. Anne L. Lewis of the University of Chicago has been appointed to an assistant professorship at the Woman's College of the University of North Carolina.

Dr. Murray Mannos of the University of Notre Dame has been promoted to an assistant professorship.

Dr. Helene Reschovsky of Russell Sage College has been promoted to an assistant professorship.

Professor J. B. Reynolds of Lehigh University has been appointed head of the department of mathematics and astronomy.

Professor J. B. Rosenbach of the Carnegie Institute of Technology has been appointed acting head of the department of mathematics.

Professor N. E. Rutt of Louisiana State University has been given leave of absence to teach mathematics in the United States Army Study Center abroad.

Assistant Professor Henry Scheffé of Syracuse University has been granted leave of absence to serve as senior mathematician with the Princeton University station of division 2 of N.D.R.C.

Professor E. R. Smith has retired from the chairmanship of the department of mathematics at Iowa State College. Professor D. L. Holl will succeed him as chairman.

Associate Professor P. A. Smith of Columbia University has been promoted to a professorship. He will serve as executive officer of the department of mathematics for a term of three years.

Dr. R. H. Sorgenfrey of the University of California at Los Angeles has been promoted to an assistant professorship.

Associate Professor Marion E. Stark of Wellesley College has been promoted to a professorship. Assistant Professor Helen G. Russell has been promoted to an associate professorship.

Professor R. P. Stephens of the University of Georgia has retired.

Professor J. L. Synge has returned to the Ohio State University after service overseas.

Professor C. F. Thomas of Case School of Applied Science has retired.

Dr. A. R. Turquette of Cornell University has been appointed visiting assistant professor of philosophy at the University of Illinois.

The following appointments to instructorships have been announced:

East Carolina Teachers College, Greenville, North Carolina: Dr. Ethel Sutherland.

Rutgers University: Edmund Churchill, A. G. Makarov, Dr. L. M. Rauch.

President Emeritus W. A. Bratton of Whitman College died in June 1945. He was a charter member of the Association.

Professor P. J. da Cunha of the University of Lisbon died on February 4, 1945.

Professor K. S. K. Iyengar of the University of Mysore, Bangalore, India, died June 23, 1944.

Professor S. T. Sanders, Jr., of Southwestern Louisiana Institute died on April 10, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE CRISIS IN TRAINING FOR THE SCIENTIFIC PROFESSIONS

In recent months, many scientists have forcefully pointed out that present manpower policies in the United States have seriously endangered the postwar supply of scientists. Especial reference may be made to two articles written by Dr. M. H. Trytten, Director of the Office of Scientific Personnel of the National Research Council; these appeared in *Science* (101: 172–173, 1945) and in *The Scientific Monthly* (LX: 37–47, 1945). On May 28, 1945, the American Council on Education and the Office of Scientific Personnel jointly issued a special bulletin of 32 pages asking for the resumption of training for the scientific professions. Because of the critical nature of the facts contained in this bulletin, a digest of it follows.

War has increased the need of the nation for technical and professionally trained manpower; yet, at the same time, it has decreased and nearly stopped the flow of able-bodied men into these fields so essential to the national health, safety, and interest. War has, also, brought a tremendous increase in our dependence upon specialized knowledge and skill but has sharply curtailed the training of selected individuals for service in these fields.

As America looks to the reconversion period, the increasing demand and greater dependency upon scientific and technological developments will increase. If these demands are to be met, if America is to have essential security in terms of national health, if we are to meet successfully the competition of nations that have preserved their professional personnel, we must begin now to take such steps as may partially make up for the growing deficit caused by the reduced flow of men into these fields. Either by administrative action or by legislation, we must reverse the policy established over the past eighteen months.

A conservative estimate indicates that even if enrollments in medical schools are maintained at present levels, there will result a shortage of some 19,000 doctors available for civilians in the postwar period as compared with the prewar period. Despite this fact, government policies and regulations now in effect will inevitably result in either (1) a sharp reduction in medical school freshmen enrolled in late 1945 and especially in 1946, or (2) the selection of large numbers of poorly qualified applicants. To insure an adequate supply of physicians in the future, about 8,000 students graduating from high school in 1945 should be selected to enter college on a premedical program and deferred from active military service. Selection should be based on high school performance, aptitude in

science as determined by special tests, motivation toward science and medicine, intelligence and emotional stability.

In considering the postwar supply of dental manpower upon which the people of the United States must depend for adequate oral health care, it is desirable to point out that there was a real shortage of dentists in the United States in the years immediately preceding the outbreak of World War II. Yet, under present conditions, the dental schools of the country will be unable to secure a freshman class in 1945. The dental A.S.T.P. has been discontinued; the Navy V-12 program lacks sufficient candidates to provide more than four per cent of the total enrollment capacity of the dental schools; the availability of discharged veterans has proved disappointing; 4 F's and women make up only about eight per cent of the total current enrollment in dental schools. Instead of the maximum possible enrollment of 3,000 freshman students in the thirty-eight dental schools that is imperative in present circumstances, there will be an actual enrollment of not more than 360. The oral health situation which now confronts the people is critical and unless there is prompt action to relieve this condition through pending legislation or a reversal of present policies regarding deferment, the situation of the nation is jeopardized.

The extent to which industry and the life of the country generally are dependent on highly trained personnel is so very great and the attendant change in American industrial and commercial processes has been so rapid that full appreciation of the situation is rare among the people and in government. The absorption of scientists and engineers has been at a very rapid rate as American industry has steadily exploited the fruits of technological advances, and processes have become more exact and more intricate. It is this alertness to scientific advance which has given our industry its enormous productivity power and has made it possible for America to commence with a handicap and yet outstrip the world in equipping its fighting arms. The war has not stopped nor reversed this technological trend. In some directions technological development has been stimulated by the war, in others it has been slowed down. For complete recovery the trend must be recaptured in all fields.

The latest government sponsored survey seeking to measure the demand in industry for highly trained engineers and scientists was made a year ago. The National Roster of Scientific and Specialized Personnel published in May, 1944, a report based on a survey made through the facilities of the U.S.E.S. The National Roster estimated at that time that the accumulated total need on the part of the American industry was likely to be 40,000 scientists and engineers by June, 1944, about the number graduated in eighteen months in American colleges and universities in normal times. This means that industry was already a year and a half behind at that time in recruiting its forces to meet new demands and to recoup losses to Selective Service. It is now one more year behind.

Recently Dean A. A. Potter of Purdue University conducted a poll of something over one hundred companies to determine the present situation with regard to highly trained personnel. His letter of inquiry asked for the number

of engineers needed by these companies to meet critical needs at the present time. The results show that about 48 engineers per company are needed by these firms on the average to handle adequately their present urgent war responsibilities. This is a shortage of nearly 5,000 engineers for this small cross section of American industry alone. These companies estimated that to meet their reconversion needs an additional thirty-six engineers per company would be needed. To appreciate the import of these figures it must be borne in mind that reconversion is largely dependent on highly skilled personnel during the periods of research, development, and plant engineering. Mass employment can not occur until these preparatory steps have been completed.

It must not be overlooked that in the future foreign nations will contribute to the demand for American technological personnel. Technologically trained brains have become a commodity of supreme importance to military and economic well-being. Foreign countries have not failed to appreciate this.

In spite of this great potential demand for engineers, the Society for the Promotion of Engineering Education reports enrollments in engineering in American colleges of something over 110,000 in 1940–41, the last "normal" year, about 50,000 in 1943–44 and about 38,000 in 1944–45; of this latter group, however, 17,000 were freshmen, most of whom are awaiting induction.

The field of physics is of especial interest because of its particular and strategic importance in this war and in all research for military purposes, as well as for its importance in the research work underlying recovery and reconversion. At all levels of training, the reduction in the training of physicists has been drastic. The number of persons achieving the doctorate level has dropped each year since 1941, the drop each year being of the order of twenty-five per cent of the 1940 number of graduates. Since this number includes many who have been able to complete their training in spite of being heavily occupied with war research, the picture fails to tell the whole story. It is also clear that the flow of persons trained to the doctorate level can not be resumed in the future until the lower levels have regained their rate of flow. Thus, not only will there be a shortage of physical scientists at all levels but the deficit must be projected into the future to perceive the cumulative effect on our scientific competence.

In the case of chemistry, the figures reported by the American Chemical Society show enrollments of male students following the same pattern. Enrollment is made up almost entirely of women and a small per cent of 4 F's. Since women, on the average, do not remain long in the professions and less frequently go on to achieve higher levels of training, the contribution to the scientific competence of the nation is not proportionate to the numbers enrolled. Dr. Charles Parsons, of the American Chemical Society, states that at the present time the need for highly trained chemists is many times the available supply and is growing progressively even more serious.

In the case of mathematics, exact data are not available but every indication is that the situation is probably worse than in other fields on account of more adverse Selective Service policies.

Scientists trained to the doctoral level are the future leaders of American science and are the staff and line officers of the scientific campaigns of the future, be they in civilian or military researches. It is amazing that in the very period when the worth of such personnel is so brilliantly demonstrated, the record shows we have canceled our production of reserves. There is no great possibility that the flow of trained personnel can appreciably pick up before the end of resistance in the Pacific. Selective Service quotas do not show signs of being reduced and if they are, the reduction will only affect the induction of older persons. It seems realistic to look for no large scale resumption of training in the technological branches in American schools before 1947. It will thus be about 1950 before industry can expect to begin large scale recruiting again and then only at the baccalaureate level. Full scale resumption of training at the doctorate level must occur much later.

It must be emphasized that the stoppage of training is much more serious than figures indicate. Figures can not give a measure of the effects produced which will be difficult to counteract, such as dislocation and dispersal of faculty personnel, lack of trained faculty replacement and accelerated attrition in the ranks of scientists.

What information is available on Russia indicates that training in engineering and the sciences has been vastly stimulated during the war. Edgar Snow has reported in books and articles that much emphasis has been placed on education at all levels and in the technological branches particularly. Russian scientists in the United States report that the training facilities of the Soviet Union are in full utilization. In England there has been throughout the war a very intelligent attitude toward training in the technological branches. In these branches a student who is in good standing and who is deemed by his faculty as likely to finish his training in the required time is held at the university to finish his course. At each university there is an office known as a Joint Recruiting Board which represents the university and the services. If the faculty recommends to this board the reservation of the student so he may finish his course, the board reserves him. They follow his progress and on graduation advise with him so as to indicate to him where he can best be used in the national interest either in the services or in industry. As a result of this policy England has had a steady flow of students into the critical categories, at least equal to the prewar norm, and probably about fifty per cent above it in physics and some branches of engineering. In the case of Canada a substantially similar course of action has been taken; approximately the same method of control as in England has been adopted and stimulation of training in the technological branches has resulted. A news release in the *New York Times* (February 28, 1945) entitled "Facts and Figures of Australia at War" reports that at the University of Sydney in 1943 medical enrollments were up eleven per cent, sciences forty-three per cent and engineering fifty per cent. It is thus clear that among our allies the training of students in the scientific branches is at least equal to that of pre-war years and in some branches definitely above pre-war levels.

Recognizing the seriousness of the situation in the scientific professions, conferences have been held with both Selective Service and the Inter-Agency Committee of the War Manpower Commission. While both of these agencies frankly recognized the serious situation that would inevitably result in stopping the flow of men into professions essential to the national interest, they stated that it was inadvisable to reestablish student deferment by regulation. They pointed out that the quota demands of the Army and Navy were such that not only could there be no deferment for 18 year old men but that increasing pressure would be brought to bear upon men up through age 33. The present policy of Selective Service is clearly stated in a letter dated April 27, 1945, from which the following paragraph is quoted:

"The urgent manpower requirements of the armed forces for young men who are physically and mentally qualified to engage in active combat service necessitate an increasingly strict examination of registrants, ages 18 through 29, for purposes of occupational classification. To the extent that registrants in this age group are deferred on occupational grounds local boards find it necessary to induct registrants over 30 years of age in order to meet the calls of the armed forces. The manpower requirements of the armed forces plus the manpower needs in war production do not permit deferment of students who wish to undergo training in specialized fields except as provision has been made for students in medicine, dentistry, veterinary medicine, or osteopathy."

Even this brief summary of the increasing need and the decreasing supply to meet the need in the scientific professions indicates that action is imperative. There are two possible courses of action—modification of existing administrative policies or legislation. The Selective Service and Training Act of 1940 and supplementary regulations which have the full effect of law are adequate to authorize the Selective Service System to defer such numbers as are necessary to meet the imperative need in essential fields. However, if this need is to be met through administrative action it will entail a complete reversal of the policy which began with the curtailment of student deferment in 1942. That Selective Service does not now contemplate such a reversal policy is indicated in the quotation just above. If such reversal of policy is not made through existing administrative agencies, it entails the necessity of immediate legislative action. Bills providing for student deferment on a quota basis and in terms only of demonstrated needs have already been introduced into the 79th Congress.

An editorial in the May 1 issue of *Industrial and Engineering Chemistry* entitled "Shall We Sell America Short Scientifically?" concludes with the following:

"With reckless abandon, our Government is selling America technologically short. Unless an aroused public demands an immediate investigation and appropriate action, our country is headed for a second- and third-rate role in the postwar period. God help us if we become engaged in a third World War in the next 25 years."

Local Board Memorandum 115 of the Selective Service System, as amended June 22, 1945, indicates no change in policy in the induction of young men ages 18 through 29. In fact, Section 4 of this Memorandum states that "certifying agencies (The National Roster has been the certifying agency for mathematics since March 21, 1945) are being required to reduce the total number of outstanding certifications" for deferment from military service.

In view of such trends in Selective Service policy, the Executive Committee of the A.A.A.S. at a meeting held on April 22 voted to endorse and urge the passage by Congress of Bill H.R. 2827. This Bill, known as the McDonough Bill, was introduced in the House of Representatives on April 2, 1945, and is quoted below.

"Be it enacted by the Senate and House of Representatives of the United States of America in Congress assembled, That (1) in order to make possible the training, education, and availability of such numbers of persons as are necessary for the health, safety, and welfare of the Nation in the sciences of fundamental importance to the conclusion of the war, the safe reconversion of the national life to the ways of peace, and the Nation's potency in the world's future economy, the President shall, under such rules and regulations as he may prescribe, provide for the deferment from military service of

(a) not to exceed 20,000 young men annually for training to meet essential needs in the physical sciences and in their application to technology and engineering, and of teachers to conduct said training program; and

(b) 15,000 trained scientists and engineers now employed in research or by industry in work essential to the health, safety, and welfare of the Nation.

Section 2. "The discharge or assignment to essential civilian pursuits of

(a) not to exceed 20,000 technically-trained enlisted men, especially chemists, chemical engineers, physicists, and mathematicians, not utilizing their highest skills in the practice of their professions in the Army or Navy, to industries and educational institutions urgently in need of such men; and

(b) not to exceed 15,000 enlisted men partially trained in those branches of science and engineering in which shortages exist, but whose collegiate training was interrupted by military service and who had shown promise of completing with distinction their preparation for professional work; providing they undertake to immediately resume and continue their collegiate training to graduation.

Section 3. "No provision of law in force on the date of enactment of this Act shall be construed to authorize any action inconsistent with the provisions and purposes of this Act."

Imaginaries. The danger, however, is not less real because it is imaginary; imagination acts upon man as really as does gravitation and may kill him as certainly as does a dose of prussic acid.—Frazer, *Golden Bough*, p. 223 of the condensed edition, published by Macmillan.—*Arnold Dresden.*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-fifth regular meeting of the Southern California Section of the Mathematical Association of America was held at the George Pepperdine College in Los Angeles on Saturday, March 10, 1945. Professor P. G. Hoel, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was sixty-four, including the following thirty-nine members of the Association: O. W. Albert, Clifford Bell, E. T. Bell, L. T. Black, W. D. Cairns, Myrtie Collier, P. H. Daus, D. C. Duncan, W. H. Glenn, Jr., P. C. Hammer, E. J. Hills, P. G. Hoel, D. H. Hyers, C. G. Jaeger, Glenn James, E. M. Justin, G. R. Kaelin, L. C. Lay, Margaret B. Lehman, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, W. B. Orange, W. T. Puckett, Jr., H. R. Pyle, V. V. Quilliam, W. C. Randals, E. C. Rex, J. M. Robb, G. E. F. Sherwood, R. H. Sorgenfrey, D. V. Steed, V. C. Throckmorton, S. E. Urner, F. A. Valentine, Morgan Ward, R. L. White, Euphemia R. Worthington, Max Zorn.

At the business meeting the following officers were elected for the next year: Chairman, R. P. Dilworth, California Institute of Technology; Vice-Chairman, H. J. Hamilton, Pomona College; Program Committee, P. C. Hammer, E. J. Hills, D. H. Hyers, and P. H. Daus, Secretary. It was decided to hold the next meeting at the California Institute of Technology in Pasadena on March 9, 1946.

The following papers were presented.

1. *Mathematics needed for the training of nurses*, by Dr. E. J. Hills, Los Angeles City College.

It was stated that the mathematics needed by nurses includes decimal fractions, percentage, and proportions. Special devices for the solution of typical problems were explained. Illustrative problems pertaining to drugs and solutions were cited. The discussion was based upon the speaker's recent book entitled *Arithmetic of Drugs and Solutions.*

2. *Some fundamentals of freshman mathematics*, by Professor O. G. Harrold, Pomona College, introduced by Professor C. G. Jaeger.

This address consisted of a brief survey of some unusual efforts to re-examine matters of rigor in an introduction to mathematics at the college and pre-college level. The speaker described a course now being developed in which the emphasis is on form rather than content.

3. *The foundations of the theory of probability*, by Professor Hans Reichenbach, University of California at Los Angeles.

The speaker approached the foundations of the theory of probability from an axiomatic point of view, and discussed the philosophic implications when it is applied to physical situations.

4. *Heron's problem*, by Professor E. T. Bell, California Institute of Technology.

As generalized by Tannery, Heron's problem is the problem of solving the simultaneous system

$$a(x + y) = u + v, \qquad xy = buv$$

in which a and b are known integers, for integral values of x, y, u, and v. Several individuals have written papers which purported to present the complete solution of this problem, but in each case the work was incomplete, inexplicit, or unproved. The speaker presented a complete solution with its proof. By suitable specialization of the parameters in the complete solution, the special cases $a=b$ and $a=1$ considered by Heron, Planude, and others can be solved.

5. *The use of conformal transformations to solve certain practical problems in aerodynamics*, by Dr. G. H. Peebles, Douglas Aircraft Company, introduced by Professor P. G. Hoel.

This paper dealt with the problem of designing airfoils, ducted airfoils, intakes, turbine blades, and slotted flaps to give prescribed flow patterns.

6. *Applications of mathematics—analysis by means of mass spectrometer*, by Miss Sibyl N. Rock, Consolidated Engineering Corporation, introduced by Professor G. E. F. Sherwood.

Miss Rock employed lantern slides to explain the mass spectrometer and its use in making certain chemical analyses. She also described an electric analogue computer used to solve a system of simultaneous linear equations arising in connection with the work. The time required for solution by means of this machine and by the usual calculator methods was stated. The question of errors was also considered.

7. *Educational problems in statistical quality control*, by Dr. P. C. Hammer, Lockheed Aircraft Corporation.

The speaker discussed the problem of presenting the quality control course to men with diverse educational backgrounds. He stated that at the Lockheed Aircraft Corporation the problem has been partially solved by presenting statistical concepts by means of experiments. Some of the materials used in these experiments were washers, rivets, playing cards, and ball bearings. The success of this type of instruction in statistics indicates that similar methods could be profitably employed in mathematics courses.

P. H. DAUS, *Secretary*

THE ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at the Polytechnic Institute of Brooklyn, Brooklyn, New York, on Saturday, April 21, 1945. Professor Jewell Hughes Bushey, Chairman of the Section, presided.

The attendance was one hundred and eight, including the following forty-seven members of the Association: Claire F. Adler, R. G. Archibald, Aaron Bakst, Brother Bernard Alfred (Welch), Frank Boehm, C. B. Boyer, Benjamin Braverman, A. B. Brown, Jewell Hughes Bushey, H. R. Cooley, T. F. Cope, W. H. H. Cowles, W. H. Fagerstrom, J. M. Feld, R. M. Foster, Marion C. Gray, Mary W. Gray, George Grossman, C. C. Grove, C. E. Heilman, L. S. Hill, Joseph Jablonower, Herman Karnow, Edna E. Kramer-Lassar, Helen L. Kutman, Nathan Lazar, C. H. Lehmann, Joseph Milkman, F. H. Miller, P. B. Norman, L. F. Ollmann, Max Peters, L. M. Reagan, Moses Richardson, John Riordan, S. G. Roth, Charles Salkind, Harry Schor, Aaron Shapiro, James Singer, F. E. Smith, E. R. Stabler, H. E. Wahlert, Etta A. Waite, Alan Wayne, D. E. Whitford, John Williamson.

At the business meeting the following officers were elected for the coming year: Chairman, F. H. Miller, Cooper Union; Vice-Chairman, H. E. Wahlert, New York University; Secretary, C. B. Boyer, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School.

The following program was presented:

1. *Demonstrative algebra*, by Professor E. R. Stabler, Hofstra College.

The speaker outlined a simple logical unit of algebraic postulates and theorems, essentially the elementary theory of number fields, for possible use in courses in high school or college algebra. Some advantages which might occur from the teaching of such a unit were cited.

2. *Triangular permutations*, by John Riordan, Bell Telephone Laboratories.

Permutations are called triangular when they are subject to a set of conditions on positions forbidden to elements which appear as a triangle in the square array formed with elements as columns and positions as rows. The properties and the problem of the enumeration of such permutations were discussed.

3. *Hadamard's determinant theorem*, by Professor John Williamson, Queens College.

Professor Williamson defined a Hadamard or H-matrix to be a square matrix of order $n>1$, each element of which is ± 1, and whose determinant has the maximum possible value $n^{n/2}$. He outlined the proof by which Paley showed that, if $n\equiv 0 \pmod 4$ and $n\leqq 200$, an H-matrix of order n exists except possibly for $n=92, 116, 156, 172, 184, 188$. He stated that it could be shown by other methods that an H-matrix of order 172 does exist, and that by generalization of Paley's methods the existence of H-matrices of certain other orders can be established. He also considered the maximum values Δ_n of the determinants of such matrices for certain small values of $n\not\equiv 0 \pmod 4$. These values of n were 3, 5, 6, and the corresponding values of Δ_n were 2^2, $3\cdot 2^4$, and $5\cdot 2^5$, respectively. It was remarked that $9\cdot 2^6\leqq\Delta_7\leqq 10\cdot 2^6$, and it seems probable that $\Delta_7=9\cdot 2^6$.

4. *Changing objectives in the teaching of algebra and trigonometry in the senior high schools*, by Benjamin Braverman, Seward Park High School.

In algebra and trigonometry as taught, say in 1915, the content was arranged topically and developed logically. Each topic was developed in all its theoretical ramifications, and highly involved techniques were taught. Now the content is organized in a psychological way, and selected because of its direct connection with the life situations of the student. The emphasis is on the understanding of concepts; skills and techniques are taught only as needed, and in problem situations that have meaning to the pupil.

5. *Changing objectives in the teaching of geometry in the senior high schools*, by Samuel Welkowitz, Franklin K. Lane High School, introduced by Dr. Nathan Lazar.

Mr. Welkowitz advocated that geometry be exhibited as an illustration of the scientific method, that the role of deduction and the nature of proof be emphasized, that we lift the pupil from the second to the third dimension, and that we provide practice on more useful constructions without restriction to the straightedge and compasses. He also recommended a modification of the order of topics to permit a gradual approach to formal deduction, the selection of the easier topics for the first semester, and the introduction of a greater amount of algebraic and arithmetic drill.

H. E. WAHLERT, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

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NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the SECRETARY-TREASURER, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 10th of the month in which the change becomes effective.

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WHAT IS THE LAPLACE TRANSFORM?

D. V. WIDDER, Harvard University

1. Introduction. A *Laplace integral* is an integral of the form

$$(1) \qquad f(s) = \int_0^\infty e^{-st}\phi(t)dt,$$

where $\phi(t)$ is any function which, for some value of s, gives the integral meaning. The integral then exists for a whole interval of values of s, so that a function $f(s)$ is defined. Since equation (1) may be thought of as *transforming* $\phi(t)$ into $f(s)$, it is frequently called the *Laplace transform.*

If, for example, $\phi(t)=1$, then

$$(2) \qquad f(s) = \lim_{R\to\infty} \int_0^R e^{-st}dt = \lim_{R\to\infty} (1 - e^{-sR})/s \qquad s \neq 0.$$

When $s>0$ this limit is $1/s$; when $s<0$ there is no limit. More generally, if s is complex, $s=\sigma+i\tau$, the limit (2) exists or fails to exist according as σ is positive or negative. The general integral (1) behaves similarly: it *converges* in a right half-plane, $\sigma>\sigma_c$, and *diverges* for $\sigma<\sigma_c$. The number σ_c, which may be $+\infty$ or $-\infty$, is called the *abscissa of convergence.*

The name for the integral (1) was chosen because Laplace* used it extensively in his theory of probability. The modern revival of interest in the transform probably was caused by Riemann's discovery that the distribution of prime numbers depends upon the position of the zeros of the Zeta-function,

$$(3) \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\cdot$$

His famous conjecture, still unverified, that any zero of this function with positive real part must have the real part 1/2 has had a tremendous influence on the development of mathematics. Now $\zeta(s)/s$ is a Laplace integral. Accordingly it was very natural that in looking for specific properties of the special function (3) the general properties of the integral (1) should have been discovered.

In this brief note no attempt will be made to give references for the various results described. They may be found in the treatises listed in the bibliography.

2. Relation to power series. A natural way of generalizing a power series

$$(4) \qquad F(z) = \sum_{n=0}^{\infty} a_n z^n$$

is to replace the integral exponent n by an arbitrary real number λ_n. But for complex z the function z^{λ_n} usually has many values. One convenient way of specifying which value is intended is to set $z=e^{-s}$,

* See, for example, pages 111 to 180 in volume VII of the collected works of Laplace.

$$F(e^{-s}) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}. \tag{5}$$

If λ_n tends to $+\infty$, this is a *Dirichlet series*. The Zeta-function is seen to be a special case of the series (5) by taking $\lambda_n = \log n$.

It is now natural to generalize a step further by changing λ_n to a continuous variable t. The summation becomes an integral, the sequence a_n becomes a function $\phi(t)$, and the Dirichlet series becomes the Laplace integral. Accordingly it is not surprising that many of the properties of the integral can be correctly conjectured from the corresponding ones for power series. For example, the region of convergence of a power series is $|z| < \rho$. Hence we might expect that the region of convergence of a Laplace integral is $|e^{-s}| = e^{-\sigma} < \rho$ or $\sigma > -\log \rho$, a right half-plane. We have stated this fact and verified it in a particular example.

Let us list several properties that do carry over from power series:

	Power series	Laplace integrals
1. Convergence	$\|z\| < \rho$	$\sigma > \sigma_c$
2. Differentiation	$F'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$	$f'(s) = -\int_0^{\infty} e^{-st} t\phi(t)dt$
3. Analyticity	$\|z\| < \rho$	$\sigma > \sigma_c$
4. Uniqueness	$F \equiv 0$ implies $a_n \equiv 0$	$f \equiv 0$ implies $\phi(t) \equiv 0$
5. Inversion	$a_n = \frac{1}{2\pi i}\int_{\|z\|=k} F(z) z^{-n-1} dz$	$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{st}ds$
6. Products	$FG = \sum_{n=0}^{\infty} c_n z^n$	$fg = \int_0^{\infty} e^{-st}\omega(t)dt$
	$c_n = \sum_{k=0}^{n} a_k b_{n-k}$	$\omega(t) = \int_0^{t} \phi(x)\psi(t-x)dx$

In property 4. it must be understood that $\phi(t)$ is *essentially* zero. It may be different from zero at some points, but at any rate its integral over any interval must vanish. In 5. $k < \rho$ and $c > \sigma_c$. Here the analogy is not quite complete since the substitution $z = e^{-s}$ carries the circle $|z| = k$ into only a piece of the vertical line $\sigma = c = -\log k$. In 6. the power series expansion of $G(z)$ has coefficients b_n and $g(s)$ is the Laplace transform of $\psi(t)$.

It is also important to observe certain fundamental differences between the two theories. Whereas a power series converges absolutely inside its region of convergence the same is not true for Laplace integrals. Indeed for the latter there are abscissas of conditional, absolute and uniform convergence, generally all different. Series (4) converges out to the singularity of $F(z)$ nearest the origin; $f(s)$ need have no singularity on the line $\sigma = \sigma_c$. Again, every analytic function has a power series expansion; the function $f(s) = s$ is entire but is not the Laplace transform of any function. Finally, there is invariably a difference in the meth-

ods of proof in the two theories. Often the gap is bridged by an integration by parts, for this process replaces a conditionally convergent Laplace integral by one which converges absolutely.

3. The bilateral transform. If the lower limit of integration in the integral (1) is replaced by $-\infty$, the new equation defines the *bilateral Laplace transform*. It is analogous to a Laurent series, and its region of convergence is a vertical strip. An important example is the Gamma-function,

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx = \int_{-\infty}^\infty e^{-st}e^{-e^{-t}}dt.$$

The region of convergence is $\sigma > 0$; the vertical strip has been enlarged into a half-plane. The inversion formula 5. applies as well to the bilateral transform.

4. Relation to the Fourier transform. Set $s = iy$ in a bilateral transform,

$$g(y) = f(iy) = \int_{-\infty}^\infty e^{-iyt}\phi(t)dt.$$

This is the Fourier transform of $\phi(t)$ into $g(y)$. It is thus a bilateral Laplace transform considered along a single vertical line. However, it becomes most useful when the strip of convergence reduces to a line. The Laplace theory is then fairly vacuous. By setting $c = 0$ and $s = iy$ the inversion formula 5. becomes

$$\phi(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(s)e^{st}ds = \frac{1}{2\pi}\int_{-\infty}^\infty g(y)e^{iyt}dy.$$

This is the usual inversion of the Fourier transform; but of course it is not usually a corollary of formula 5. since for its validity the line of integration must lie *inside* a strip of convergence.

5. A table of transforms. It is useful to have a table of Laplace transforms to be used like any table of integrals. Here is a highly abridged one:

	$f(s)$	$\phi(t)$	σ_c
A.	$\frac{1}{s}$	1	0
B.	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	0
C.	$\frac{1}{s-a}$	e^{at}	α
D.	$\frac{1}{s^2+1}$	$\sin t$	0
E.	$\frac{s}{s^2+1}$	$\cos t$	0

The pair B. may be derived from A. by differentiation; C. comes from the same source when s is replaced by $s-a$. Here α is the real part of a. Both D. and E. come from C. when $\sin t$ and $\cos t$ are expressed in terms of e^{it} and e^{-it}.

6. Differential equations. As an illustration of the many applications of the general theory let us solve a differential equation. It will be clear from the example that the method is very general. The transform (1) applied to an ordinary linear differential equation with constant coefficients reduces it to an algebraic one. The solution of the latter is then retransformed by inversion, or by use of a table. More generally, if the original equation is partial in any number of independent variables, one application of the Laplace transform reduces the number of variables by one.

Let it be required to find a function $y(t)$ such that $y(0)=1$, $y'(0)=2$ and

$$y'' + y = 2e^t. \tag{6}$$

Denote the Laplace transform of $y(t)$ by $Y(s)$. Integration by parts gives

$$\int_0^\infty e^{-st}y''(t)dt = -y'(0) - y(0)s + s^2\int_0^\infty e^{-st}y(t)dt$$

on the assumption that the integrated part vanishes at $+\infty$, at least for large values of s. Using the pair C. of §5 we see that the transformed equation is

$$-2 - s + s^2Y(s) + Y(s) = \frac{2}{s-1}$$

$$Y(s) = \frac{s^2+s}{(s-1)(s^2+1)} = \frac{1}{s-1} + \frac{1}{s^2+1}.$$

A second reference to the table and an appeal to the uniqueness property 4. shows that $y(t)=e^t+\sin t$. This function has the required properties, so that it is unnecessary to check the assumption made about its behavior at $+\infty$.

The procedure is elementary and could be used in an introductory study of differential equations. The proof that the method always leads to the solution has hitherto depended on contour integration in the complex plane. In a forthcoming text on advanced calculus by the author a simpler proof depending only on real variable theory will appear.

7. Real inversion. The most familiar determination of the coefficients of the power series (4) is $a_n=F^{(n)}(0)/n!$. An analogous inversion of the Laplace transform (1) has recently been discovered. It is

$$\phi(t) = \lim_{k\to\infty}\frac{(-1)^k}{k!}f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1} \tag{7}$$

Observe that $\phi(t)$ is determined by the values of the successive derivatives of $f(s)$ near $s=+\infty$. Since $z=0$ corresponds to $s=+\infty$ when $z=e^{-s}$ formula (7)

is indeed similar to the Taylor "inversion" of a power series. For the pair $f(s)=s^{-2}$, $\phi(t)=t$ equation (7) becomes

$$t = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)t.$$

For the pair C. it is

$$e^{at} = \lim_{k\to\infty}\left(1-\frac{at}{k}\right)^{-k-1}$$

8. Relation to the moment problem. The *moment problem* of Hausdorff is the determination of a non-decreasing function $\beta(x)$ such that

$$\mu_n = \int_0^1 x^n d\beta(x) \qquad n = 0, 1, 2, \cdots, \tag{8}$$

where the integral is a Stieltjes integral. Hausdorff showed that the problem has a solution if and only if, for all n,

$$\mu_n \geqq 0,\ \Delta\mu_n = \mu_{n+1} - \mu_n \leqq 0,\ \Delta^2\mu_n = \mu_{n+2} - 2\mu_{n+1} + \mu_n \geqq 0, \cdots .$$

Such a sequence is given by $\mu_n = 1/(n+1)$ or by $(1/2)^n$, and these arise from equation (8) by taking first $\beta(x)=x$ and then $\beta(x)=0$ or 1 according as $x<1/2$ or $x>1/2$. Both functions are non-decreasing.

If in equation (8) we replace n by a continuous variable s and set $x=e^{-t}$ we obtain

$$\mu(s) = \int_0^\infty e^{-st}d[-\beta(e^{-t})] \qquad 0 \leqq s < \infty.$$

But this is a *Laplace-Stieltjes transform*

$$f(s) = \int_0^\infty e^{-st}d\alpha(t). \tag{9}$$

By analogy with the Hausdorff result we might expect that a non-decreasing function $\alpha(t)$ would exist satisfying equation (9) if and only if

$$f(s) \geqq 0,\ f'(s) \leqq 0,\ f''(s) \geqq 0, \cdots \qquad 0 < s < \infty.$$

Such a function is called *completely monotonic*. Examples are $f(s)=1/(s+1)$ and $f(s)=e^{-s}$, and these arise from equation (9) by use of the non-decreasing functions $\phi(t)=e^{-t}$ and $\phi(t)=0$ or 1 according as $t<1$ or $t>1$. The conjectured result is true and is known as Bernstein's theorem. The result is particularly remarkable in view of the fact that the signs of the derivatives of a function on the real axis should determine not only its analyticity in a half-plane but its representation in the form (9).

At first sight it may not be easy to see why the Stieltjes integral has been introduced at this stage. It is done largely to produce elegant results like those

of Hausdorff and Bernstein given above. The function e^{-s} though completely monotonic, could never have a representation (1). The same is true of any convergent Dirichlet series with positive coefficients. It is only when we introduce the Stieltjes integral, thus combining the class of such Dirichlet series with the functions defined by equation (1) with $\phi(t) \geqq 0$, that we can obtain a neat characterization of completely monotonic functions.

9. The Stieltjes transform. If the transform (1) is applied to itself there results, after a change in the order of integration,

$$f(s) = \int_0^\infty e^{-sx}\, dx \int_0^\infty e^{-xt}\, \phi(t) dt = \int_0^\infty \frac{\phi(t)}{s+t}\, dt. \tag{10}$$

This is called the *Stieltjes transform* because it was put to effective use by Stieltjes in his theory of continued fractions. A sample pair is $f(s) = 1/(\pi\sqrt{s})$, $\phi(t) = 1/\sqrt{t}$. This may be verified by formula 482 of B. O. Peirce's table of integrals. From the origin of the transform as an *iterated* Laplace transform many of its properties can be conjectured. An inversion formula related to property 5. of §3 is

$$\phi(t) = \lim_{\epsilon \to 0} \frac{f(-t - i\epsilon) - f(-t + i\epsilon)}{2\pi i}\,. \tag{11}$$

Another, matching the one given in §7, is

$$\phi(t) = \lim_{k \to \infty} \frac{(-t)^{k-1}}{k!(k-2)!} \left[t^k f(t)\right]^{(2k-1)}. \tag{12}$$

It is interesting to test this equation with the pair of transforms given above.

A result analogous to Bernstein's is that a non-negative number P and a non-decreasing function $\alpha(t)$ such that

$$f(s) = P + \int_0^\infty \frac{d\alpha(t)}{s+t}$$

exist if and only if

$$f(s) \geqq 0, \quad (-1)^{k-1}[s^k f(s)]^{(2k-1)} \geqq 0 \qquad 0 < s < \infty,\ k = 1, 2, \cdots.$$

The function $f(s) = 1/(\pi\sqrt{s})$ has this property, and the corresponding function $\alpha(t) = 2\sqrt{t}$ is clearly non-decreasing.

10. Operational considerations. One of the most fascinating aspects of the subject is its relation to the calculus of differential operators. Such considerations enable one to unify existing knowledge and to predict new results. Space is lacking for an adequate description of the method. Suffices it to say, for example, that the inversion operator for the Stieltjes transform (10) may be considered to be $-\pi^{-1} \sin(\pi s D)$, where D stands for the operation of differentiation with respect to s. If one uses the familiar exponential expression for the sine before applying the operator to $f(s)$ one is led to formula (11). On the other hand,

the infinite product expansion of the sine yields equation (12). One could use operational methods to predict inversion formulas for the higher iterates of the Laplace transform. These iterates have not yet been studied in great detail.

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CIRCLE-TO-LINE TRANSFORMATIONS [1]

JOHN DE CICCO, Illinois Institute of Technology

1. Differential equations of the cubic type. Any ordinary differential equation of the second order which may be written in the form

$$y'' = A(x, y) + B(x, y)y' + C(x, y)y'^2 + D(x, y)y'^3, \tag{1}$$

is termed a differential equation of the cubic type. This class of differential equations has been studied extensively by Lie, R. Liouville, Tresse, Kasner, and Wilczynski. Among the important systems of curves defined as integral curves of differential equations of this type are the geodesics on any surface [2], the velocity systems of a positional field of force [3], natural families, isogonal families, Γ and Γ_0 families [4].

The class of differential equations of the cubic type (1) is transformed into itself under the group of arbitrary point transformations in the plane. In his study of the geometry of differential elements of the second order under the group of arbitrary point transformations [5], Kasner classified according to rank all differential equations of the second order which are algebraic in y' and y''. The differential equations of the first rank are exactly those of the cubic type (1).

A differential equation of the cubic type (of the first rank) may be characterized by the property that the locus of the centers of curvature of the ∞^1 integral curves passing through any point P of the plane is a general cubic curve with an isolated singularity at P, the tangent lines being the minimal lines through P. This cubical locus degenerates into a straight line only in the case of velocity systems.

2. Summary of results. In the present paper, we shall submit what is believed to be a first published proof of a theorem of Kasner which states that *the only systems of ∞^2 circles defined by differential equations of the cubic type* (1) *are the linear systems of circles* [6].

By this theorem and Beltrami's theorem, it can be proved that the only surfaces which can be mapped in a point-to-point fashion on a plane such that the geodesics may be represented by circles are those of constant curvature [7].

From this theorem of Kasner, we are able to obtain the complete set of point-to-point transformations T from the (x, y)-plane to the transformed (X, Y)-plane such that any straight line in the (X, Y)-plane corresponds to a circle in the (x, y)-plane. This set consists of a rational set of transformations of eleven parameters.

Some properties of these circle-to-line transformations T are obtained. Any such transformation T is rational but its inverse T^{-1} is in general irrational. The complete correspondence between the (x, y)-plane and the (X, Y) plane is two-to-one. The inverse T^{-1} is rational if and only if T is the product of a Moebius transformation by a collineation. (A Moebius transformation is a point-to-point correspondence by which every circle is converted into a circle; whereas a collineation is a point-to-point map carrying every straight line into a straight line.) In that event, the inverse T^{-1} of T is the product of a collineation by a Moebius transformation, and can be expressed as a fractional linear polygenic function in $u=x+iy$ and $v=x-iy$, with constant coefficients. Finally the only groups contained in our eleven-parameter set are the group of ∞^8 collineations and the mixed group of $2\infty^6$ Moebius transformations.

3. The statement and proof of Kasner's theorem concerning systems of ∞^2 circles of the cubic type. In this section, we shall prove the following theorem which may be considered to be an extension of the result stated in Section 2.

THEOREM 1. *A differential equation of the second order of the cubic type* (1) *can possess at most* $6\infty^1$ *circles as integral solutions. If it contains more than* $6\infty^1$ *circles as integral solutions, then every solution is a circle and the family of* ∞^2 *circles is linear.*

For those integral solutions of (1) which are circles, the differential equation of all ∞^3 circles: $(1+y'^2)y'''-3y'y''^2=0$, must be satisfied. Hence substituting (1) into this differential condition for circles, we obtain the differential equation of the first order and sixth degree in y' [8]

$$
\begin{aligned}
(A_x + AB) + (A_y &+ B_x + B^2 + 2AC - 3A^2)y' \\
&+ (B_y + C_x + A_x + 3BC + 3AD - 5AB)y'^2 \\
&+ (C_y + D_x + A_y + B_x - 2B^2 - 4AC + 4BD + 2C^2)y'^3 \\
&+ (D_y + B_y + C_x - 3BC - 3AD + 5CD)y'^4 \\
&+ (C_y + D_x - C^2 - 2BD + 3D^2)y'^5 + (D_y - CD)y'^6 = 0.
\end{aligned} \tag{2}
$$

This equation shows that a differential equation of the cubic type (1) can possess at most $6\infty^1$ circles as integral solutions. If it has as integral solutions more than $6\infty^1$ circles, then all integral solutions are circles, and the preceding equation is an identity in y'.

From the resulting seven relations, we find that all the partial derivatives may be eliminated yielding the equations

$$(A - C)(B - D) = 0, \qquad (A - C)^2 = (B - D)^2. \tag{3}$$

From these we derive the result that $C=A$ and $D=B$. This shows that if the integral solutions of the cubic type (1) are all circles, then the differential equation (1) must represent a velocity system.

Using the conditions $C=A$, $D=B$, it is found that our seven relations reduce to the three relations

$$A_x = -AB, \qquad B_y = AB, \qquad A_y + B_x = A^2 - B^2. \tag{4}$$

By these equations, we find that there exists a function $\phi(x, y)$ such that

$$A = \phi_y, \qquad B = -\phi_x. \tag{5}$$

Replacing the A and B in (4) by (5), it follows that the function ϕ must satisfy the pair of simultaneous partial differential equations of the second order

$$\phi_{xy} = \phi_x\phi_y, \qquad \phi_{xx} - \phi_{yy} = \phi_x^2 - \phi_y^2. \tag{6}$$

First if $\phi=$const., then $A=B=C=D=0$, and the circles are the ∞^2 straight lines of the plane. Henceforth we may exclude the case where ϕ_x and ϕ_y are both zero.

It is observed that (6) may be written in the form

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{1}{\phi_x + i\phi_y} + \frac{x - iy}{2}\right) = 0. \tag{7}$$

Introducing the conjugate complex variables $u=x+iy$, $v=x-iy$, it evidently follows that the second parenthesis in (7) is an analytic function of u. That is

$$\phi_x + i\phi_y = \frac{2}{f(u) - v}, \tag{8}$$

where f is an analytic function of u. By taking the conjugate of the above equation and solving the result for ϕ_x and ϕ_y, we obtain

$$\begin{aligned} \phi_x &= \frac{1}{f(u) - v} + \frac{1}{g(v) - u}, \\ \phi_y &= \frac{-i}{f(u) - v} + \frac{i}{g(v) - u}, \end{aligned} \tag{9}$$

where $g(v)$ is the conjugate of the function $f(u)$.

Imposing the compatibility conditions on the function $\phi(x, y)$ whose partial derivatives are given by (9), it is seen that $f(u)$ (and hence $g(v)$) satisfies the condition [9]

$$(10) \qquad \frac{f_u}{(f-v)^2} = \frac{g_v}{(g-u)^2}.$$

Consider now the case where f is constant, say $f = x_0 - iy_0$. Then (10) is satisfied and (9) becomes

$$(11) \qquad \phi_x = \frac{-2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}, \qquad \phi_y = \frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}.$$

By (5), it follows that the differential equation of the cubic type (1) is in this case

$$(12) \qquad y'' = -\frac{2(1+y'^2)[(y-y_0)-y'(x-x_0)]}{(x-x_0)^2+(y-y_0)^2}.$$

This yields all the ∞^2 circles through the fixed point (x_0, y_0), which is a linear system. Henceforth we may assume that f is not constant.

Since f is not constant, we can take the reciprocal of (10) and operate on both sides with $\partial^4/\partial u^2 \partial v^2$. The result is

$$(13) \qquad \frac{\partial^2}{\partial u^2}\frac{1}{f_u} = \frac{\partial^2}{\partial v^2}\frac{1}{g_v}.$$

From this, we deduce that

$$(14) \qquad f_u = \frac{1}{au^2+bu+c},$$

where a is real and (b, c) are complex.

Next let us consider the case where a and b are both zero. Then f must be of the form $ku+l$ where $k \neq 0$, and l are complex constants. The substitution of this into (10) shows that $k = -e^{-2i\alpha}$ and $l = 2pe^{-i\alpha}$ where α and p are real numbers. Therefore by (9), we find [10]

$$(15) \qquad \begin{aligned} \phi_x &= \frac{-\cos\alpha}{x\cos\alpha + y\sin\alpha - p}, \\ \phi_y &= \frac{-\sin\alpha}{x\cos\alpha + y\sin\alpha - p}. \end{aligned}$$

By (5), it is observed that the differential equation of the cubic type (1) is in this case

$$(16) \qquad y'' = \frac{(1+y'^2)(-\sin\alpha + y'\cos\alpha)}{x\cos\alpha + y\sin\alpha - p}.$$

The complete solution of this yields the totality of ∞^2 circles whose centers all lie on the straight line $x\cos\alpha + y\sin\alpha - p = 0$. These form a linear system.

Henceforth we consider the case where a and b of (14) are not both zero. Substituting (14) into (10) we find

$$(f - v)^2(au^2 + bu + c) = (g - u)^2(av^2 + \beta v + \gamma), \tag{17}$$

where β is the conjugate of b, and γ is the conjugate of c. Using the operator $\partial^2/\partial u\partial v$ on this equation and simplifying, we find

$$(2au + b)f + \beta u = (2av + \beta)g + bv. \tag{18}$$

Hence there exists a real number d such that

$$f = \frac{d - \beta u}{2au + b}. \tag{19}$$

Substituting (19) into (14) and simplifying, we find

$$(2au + b)^2 + (b\beta + 2ad)(au^2 + bu + c) = 0. \tag{20}$$

If $a = 0$, we find from the preceding identity in u that $b = 0$. This case has been considered already. Hence $a \neq 0$, and the identity (20) in u, yields the conditions

$$b\beta + 2ad + 4a = 0, \qquad b^2 - 4ac = 0. \tag{21}$$

Hence we find that f must be of the form

$$f = -\frac{2a\beta u + b\beta + 4a}{2a(2au + b)}. \tag{22}$$

The last of equations (21) shows that the derivative of f with respect to u is $f_u = 4a/(2au+b)^2$. Thus the function f as given by (22) satisfies the equation (10) identically.

Substituting the value of f as given by (22) into the equations (9), simplifying and then replacing $(b+\beta)$ by $-4ax_0$, $(b-\beta)$ by $-4ay_0i$, and $1/a$ by $k \neq 0$, we find

$$\phi_x = \frac{-2(x - x_0)}{(x - x_0)^2 + (y - y_0)^2 + k}, \quad \phi_y = \frac{-2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2 + k}, \tag{23}$$

where (x_0, y_0, k) are all real numbers. The differential equation of the cubic type (1) is then

$$y'' = -\frac{2(1 + y'^2)[(y - y_0) - y'(x - x_0)]}{(x - x_0)^2 + (y - y_0)^2 + k}. \tag{24}$$

The integral curves are the ∞^2 circles orthogonal to the real or imaginary circle $(x-x_0)^2+(y-y_0)^2+k=0$. This system is also linear.

Thus in all possible cases the only systems of ∞^2 circles which may be defined by differential equations of the cubic type (1) are those orthogonal to a fixed circle (proper with real or imaginary radius, rectilinear, or null). These form a linear system of circles. Any such system may be obtained as a linear combination of the three equations $a_j(x^2+y^2)+b_jx+c_jy+d_j=0$ for $j=1, 2, 3$, of any three non-co-axial circles. This completes the proof of Theorem 1.

4. The eleven-parameter set S_{11} of point transformations T whereby straight lines correspond to circles. We shall discuss the following result.

THEOREM 2. *If a point transformation T carries more than $6\infty^1$ circles into straight lines, then every circle of a certain linear congruence of circles is converted by T into straight lines, and T must belong to the eleven-parameter set S_{11} of transformations*

$$X = \frac{a_2(x^2+y^2)+b_2x+c_2y+d_2}{a_1(x^2+y^2)+b_1x+c_1y+d_1},\qquad Y = \frac{a_3(x^2+y^2)+b_3x+c_3y+d_3}{a_1(x^2+y^2)+b_1x+c_1y+d_1}, \tag{25}$$

where (a_j, b_j, c_j, d_j) are real numbers for $j=1, 2, 3$ whose matrix is of rank three [11].

It is noted that any transformation which does not belong to the eleven-parameter set S_{11} defined by the equations (25) can carry at most $6\infty^1$ circles into straight lines.

Let T be any transformation defined by the equations

$$X = X(x, y),\qquad Y = Y(x, y), \tag{26}$$

where the functions (X, Y) are single valued differentiable functions of (x, y) with non-vanishing jacobian $J=\partial(X, Y)/\partial(x, y)$ in a suitable region of the (x, y)-plane.

By extending the preceding transformation twice, it is seen that the ∞^2 straight lines of the (X, Y)-plane correspond to a system of ∞^2 curves in the (x, y)-plane, which are defined by a differential equation of the cubic type (1). By Theorem 1, it follows that if T carries more than $6\infty^1$ circles of the (x, y)-plane into more than $6\infty^1$ of the totality of ∞^2 straight lines of the (X, Y)-plane, then every circle of a certain linear congruence of circles in the (x, y)-plane is converted by T into a straight line.

It may be that the linear-congruence of circles consists of the totality of ∞^2 straight lines in the (x, y)-plane. In that case T is a collineation which is included in the set (25).

Henceforth we can assume that the linear congruence of circles in the (x, y)-plane does not consist entirely of straight lines. In that event, these ∞^2 circles must be orthogonal to a fixed circle C which may be null or rectilinear or proper with a real or imaginary radius.

First of all, let us consider the case where the fixed circle C is a null-circle, that is, C is a point A. Then all the circles in the (x, y)-plane which correspond to straight lines in the (X, Y)-plane must be those passing through A.

The transformation T must be a one-to-one correspondence except for the point A. For assume the contrary and let us suppose that two distinct points p_1 and p_2, neither of which is A, in the (x, y)-plane, correspond to a single point P in the (X, Y)-plane. Then the unique circle through (p_1, p_2, A) must correspond to every straight line through P in the (X, Y)-plane. Thence every point in the

(X, Y)-plane corresponds to a point in the (x, y)-plane on the circle determined by (p_1, p_2, A). This is impossible because the jacobian J is assumed to be not identically zero. Therefore the transformation T is a one-to-one correspondence except for the point A.

If M represents an inversion with center at A in the (x, y)-plane, then the product MT^{-1} is a one-to-one correspondence converting straight lines into straight lines, and hence is a collineation P. Therefore $T=PM$, that is, T is the product of an inversion by a collineation. Any such transformation T belongs to the eleven-parameter set S_{11} defined by the equations (25).

Finally we consider the case where the fixed circle C is not a point. Then any two points in the (x, y)-plane which are inverse with respect to the circle C must correspond to a single point in the (X, Y)-plane, since the system of circles orthogonal to C is transformed into itself under an inversion with respect to C. Thus the transformation T is at least two-to-one.

We shall show that T is two-to-one. Suppose the contrary so that three distinct points (p_1, p_2, p_3) in the (x, y)-plane, where (p_1, p_2) are inverse with respect to the fixed circle C, correspond to a single point P in the (X, Y)-plane. Then every straight line in the (X, Y)-plane through the point P will correspond to the circle in the (x, y)-plane determined by the three points (p_1, p_2, p_3), which is in the linear congruence of circles since p_1 and p_2 are inverse with respect to C. Hence every point in the (X, Y)-plane corresponds to a point in the (x, y)-plane on the unique circle determined by (p_1, p_2, p_3). This is impossible since the jacobian J was assumed to be not identically zero in a suitable region of the (x, y)-plane. Therefore our transformation T is two-to-one.

This transformation T can be considered to be one-to-one by introducing the convention that two points in the (x, y)-plane shall be considered as one if and only if they are inverse with respect to the fixed circle C.

Now if S is any other transformation by which straight lines in the (X, Y)-plane correspond to circles orthogonal to the fixed circle C, then ST^{-1} is a one-to-one correspondence carrying straight lines into straight lines and hence must be a collineation P. Thus T is of the form PS, that is, T is the product of the transformation S followed by a collineation P.

Let three non-coaxial circles orthogonal to the fixed circle C be given by $a_j(x^2+y^2)+b_jx+c_jy+d_j=0$, for $j=1, 2, 3$. The matrix of the real numbers (a_j, b_j, c_j, d_j) is of rank three. Clearly one such transformation S by which all straight lines in the (X, Y)-plane correspond to circles in the (x, y)-plane must be given by equations of the type (25). Any other such transformation T is the correspondence S followed by a collineation P, which product is of the same form (25).

Thus we have shown that any transformation T by which straight lines in the (X, Y)-plane correspond to circles in the (x, y)-plane must be given by a pair of equations of the forms (25). Since any such transformation carries a certain linear congruence of circles into straight lines, our Theorem 2 is completely proved.

5. Concluding remarks. By Theorem 2, it is seen that any transformation T of our eleven-parameter set is rational. If the fixed circle C is not null, then the inverse T^{-1} is algebraic and irrational. Hence the inverse T^{-1} is rational if and only if T is either a collineation, or a Moebius transformation, or the product of a Moebius transformation by a collineation. The inverses T^{-1} of all such transformations T are given by the fractional linear polygenic functions in $u=x+iy$ $v=x-iy$ with constant complex coefficients. That is, any such rational inverse T^{-1} is given by an equation of the form

$$U = \frac{\alpha_2 u + \beta_2 v + \gamma_2}{\alpha_1 u + \beta_1 v + \gamma_1}, \tag{27}$$

where the matrix of the complex numbers $(\alpha_j, \beta_j, \gamma_j)$ for $j=1, 2$, is of rank two. All such transformations T with rational inverse form a ten-parameter subset S_{10} of S_{11}.

If one inquires as to the groups contained in the eleven-parameter set S_{11}, it is seen that if T is any transformation of S_{11} and belonging to a group G contained in S_{11}, then its inverse T^{-1} must belong to G and hence to S_{11} and therefore is of the form (27). But any transformation T^{-1} of the form (27) can belong to the subset S_{11} if and only if it is a collineation or a Moebius transformation. Hence the only groups contained in our eleven-parameter set S_{11} are the eight-parameter group of collineations and the mixed six-parameter group of Moebius circular transformations.

From the above remarks is deduced the well-known result that the group of all point transformations carrying circles into circles is the Moebius group. Also these are the only conformal transformations in our set [12].

Finally it may be stated that the jacobian J of any transformation T of the set S_{11} defined by the equations (25), vanishes only along the points of the fixed circle C defined in Theorem 2. This means that in the derivative plane, the Kasner circles corresponding to the points of the fixed circle C all pass through the origin [13].

The results of this paper are valid also in the complex cartesian plane, where any point is defined as an ordered pair of independent complex numbers.

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8. As an aid to deriving the equation (2), we use the equations

$$\begin{aligned} &y''^2 = A^2 + 2ABy' + (B^2 + 2AC)y'^2 + 2(AD + BC)y'^3 + (C^2 + 2BD)y'^4 + 2CDy'^5 + D^2y'^6,\\ (2')\quad &y''' = (AB + A_x) + (B^2 + 2AC + A_y + B_x)y' + (3BC + 3AD + B_y + C_x)y'^2\\ &\qquad + (2C^2 + 4BD + C_y + D_x)y'^3 + (5CD + D_y)y'^4 + 3D^2y'^5, \end{aligned}$$

which are obtained from (1) by squaring and differentiating.

9. In deriving the equation (10), we make use of the following operational relations

$$(10')\qquad \begin{aligned} &\frac{\partial}{\partial u} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), && \frac{\partial}{\partial v} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),\\ &\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, && \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right). \end{aligned}$$

The operators $\partial/\partial u$ and $\partial/\partial v$ are important in the theory of polygenic functions. See the last reference at end of paper.

10. As on aid to deriving the equations (15), it is noted

$$(15')\qquad \begin{aligned} f(u) - v &= -e^{-2i\alpha}u + 2pe^{-i\alpha} - v = -x(1 + e^{-2i\alpha}) + iy(1 - e^{-2i\alpha}) + 2pe^{-i\alpha}\\ &= -2e^{-i\alpha}(x\cos\alpha + y\sin\alpha - p). \end{aligned}$$

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Mathematics and humanism. You know how to measure the circle, you can compute the distances between the stars . . . . But if you are a real master of your profession, measure me the mind of man. Tell me how great it is or how puny.—Seneca, quoted by Cajori in *Mathematics in Liberal Education.*

. . . or on how Crane's poetry can only be defined, reviewed, and generally exposited in terms of mathematical formulae—ahem! ahem, now!—

$$\frac{\sqrt{an + pxt}}{237} = \frac{n - F_3(B^{18} + 11)}{2}.$$

—Thomas Wolfe, *You Can't Go Home Again*, p. 485.—*E. D. Schell.*

AN ELECTROMAGNETIC ANALOGY IN MECHANICS*

A. WEINSTEIN AND J. R. POUNDER, University of Toronto

1. Introduction. We propose to make a few observations on two elementary problems often treated in textbooks on classical mechanics and electromagnetic theory. We hope that our remarks will lead to a simplified and unified treatment of the subject. The problems considered are:

I. The motion of a point-charge under the influence of static uniform electric and magnetic fields.

II. The motion of a heavy particle on a rotating earth.

In the usual treatments [1], the differential equations of motion in problem I are obtained and integrated without making any approximations. The equations of motion in problem II are approximate, inasmuch as terms involving the square of ω, the earth's angular velocity, are neglected; moreover, they are usually solved only approximately, in power series. The final presentations of the solution in the two problems are not such as would suggest any connection between them. However, by writing the equations of motion in the two cases in vector form, and resolving in different directions, we shall recognize them to be identical, except for notation. We intend to give first a unified direct treatment, and then establish the link with the usual presentation.

The formulae found for the motion of the particle in problem II agree with those obtained in a paper by Denizot, who, however, did not notice the connection with the electromagnetic problem. Denizot's results, published in book form [2], were adversely criticized in the *Fortschritte der Mathematik* at the time of their appearance, and have now been completely forgotten.

2. The equations of motion. We begin with the equations of motion of the two problems:

$$m\ddot{\mathbf{r}} = e\mathbf{E} + e\dot{\mathbf{r}} \times \mathbf{H} \tag{1}$$

$$m\ddot{\mathbf{r}} = -m\mathbf{g} + m\dot{\mathbf{r}} \times 2\boldsymbol{\omega}. \tag{2}$$

Equation (1) is a statement of the classical Lorentz law of force, together with Newton's law of motion. The notation is as follows (vectors being denoted throughout by letters in heavy type):

m is the mass of the point-charge
e is its total electric charge
$\mathbf{r}$ is its position vector
$\mathbf{E}$ is the constant electric field
$\mathbf{H}$ is the constant magnetic field

Equation (2) is a statement of Newton's law of motion in a frame of reference moving with the observer. The notation is as follows:

m is the mass of the particle

* Presented to the American Mathematical Society on October 28, 1944, under the title "On Two Elementary Problems of Classical Mechanics and Electromagnetic Theory."

r is its position vector
g is the vector acceleration due to the combined gravitational and (static) centrifugal fields
ω is the vector angular velocity of the earth (its magnitude ω is .0000729 sec.$^{-1}$)

The dot denotes differentiation with respect to the time t.

These equations are of the same form, the gravitational force corresponding to the electric force and the Coriolis force to the magnetic force. In fact, we can convert (1) into (2) by replacing e/m by 1, **E** by $-$**g**, and **H** by 2**ω**. That this obvious fact has not been recognized up to now is probably owing to the premature resolution of the vector equation (2) into vertical and horizontal components. Our procedure shows once more the advantage of using vectors.

We shall denote the angle btween **H** and **E** by $\pi/2-\mu$, and the angle between **ω** and $-$**g** by $\pi/2-\lambda$, λ being the latitude.

3. The solution of the electromagnetic problem. For reasons which will become apparent later, we now proceed to solve problem I in a more explicit way than is usually presented. For the sake of definiteness we shall consider only solutions in which the particle starts from rest. We resolve equation (1) in three mutually perpendicular directions as follows (see figure 1):

1st, parallel to **H** (z-direction)
2nd, perpendicular to both **H** and **E** (y-direction)
3rd, perpendicular to both the preceding, *i.e.*, in the plane of **H** and **E** (x-direction).

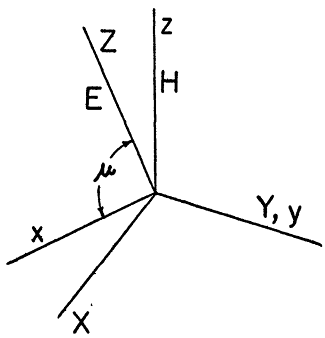

FIG. 1.

We get the following system of equations:

$$(3)\qquad \begin{cases} \ddot{x} = \dfrac{e}{m}(E\cos\mu + \dot{y}H) \\[2ex] \ddot{y} = -\dfrac{e}{m}\dot{x}H \\[2ex] \ddot{z} = \dfrac{e}{m}E\sin\mu \end{cases}$$

with the initial conditions

$$\left.\begin{aligned} x &= \dot{x} = 0 \\ y &= \dot{y} = 0 \\ z &= \dot{z} = 0 \end{aligned}\right\} \quad \text{for} \quad t = 0.$$

The solution, obtained in the usual elementary way, is:

$$(4) \qquad \begin{cases} x = \dfrac{mE\cos\mu}{eH^2}\left(1 - \cos\left(\dfrac{eH}{m}t\right)\right) \\ y = -\dfrac{E\cos\mu}{H}\left(t - \dfrac{m}{eH}\sin\left(\dfrac{eH}{m}t\right)\right) \\ z = \dfrac{eE}{m}\sin\mu\,\dfrac{t^2}{2} \end{cases}$$

We see, then, that the point-charge moves on a cycloid lying in an x, y-plane which moves parallel to itself with constant acceleration in the z-direction.

4. The solution of the mechanical problem. At this point we pass on to the solution of problem II. We resolve equation (2) in three directions corresponding to those in problem I (see figure 2):

1st, parallel to ω (z-direction).
2nd, perpendicular to both ω and $\mathbf{g}$ (y-direction).
3rd, perpendicular to both the preceding, *i.e.*, in the plane of ω and $\mathbf{g}$ (x-direction).

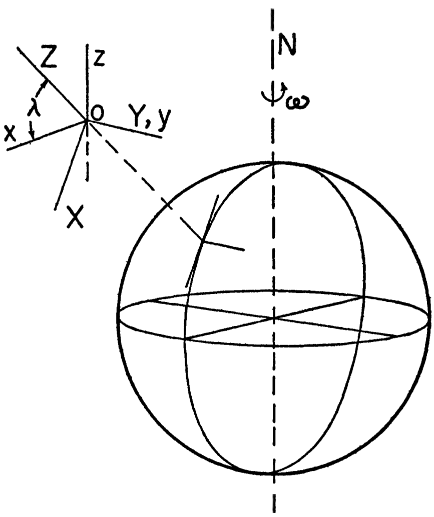

FIG. 2.

We can obtain the solution here from equations (4) by replacing H by 2ω, E by $-g$, e/m by 1, and μ by λ. We get

$$(5)\qquad \begin{cases} x = -\dfrac{g\cos\lambda}{4}(1-\cos(2\omega t)) \\ y = \dfrac{g\cos\lambda}{2}\left(t - \dfrac{1}{2\omega}\sin(2\omega t)\right) \\ z = -g\sin\lambda\,\dfrac{t^2}{2} \end{cases}$$

5. Motion in a weak magnetic field. We now wish to investigate our solution of problem I when H is small. This is of interest because of the analogy with problem II (where ω *is* small), and because the solution of I is usually given in the following form (see for instance Appell, Synge and Griffith, *l.c.* (1)):

$$(6)\qquad \begin{cases} x = A + B\cos\left(\dfrac{eH}{m}t\right) \\ y = A' - B\sin\left(\dfrac{eH}{m}t\right) - \dfrac{E\cos\mu}{H}t \\ z = C + Dt + \dfrac{eE\sin\mu}{m}\,\dfrac{t^2}{2} \end{cases}$$

where A, A', B, C, D, are constants of integration, not expressed in terms of initial conditions. A superficial inspection of equations (6) would lead to the absurd conclusion that, as H tends to zero, the velocity tends to infinity.

In order to obtain the asymptotic expressions of the solution for small values of H, we expand equations (4) as power series in $(eH/m)t$:

$$x = \frac{mE\cos\mu}{eH^2}\left(\frac{(eH)^2t^2}{m^2 2} - \frac{(eH)^4t^4}{m^4 24} + \cdots\right)$$

$$\cong \frac{e}{m}E\cos\mu\,\frac{t^2}{2} - \frac{e^3}{m^3}H^2E\cos\mu\,\frac{t^4}{24}$$

$$y = -\frac{E\cos\mu}{H}\left(\frac{(eH)^2t^3}{m^2 6} - \cdots\right)$$

$$\cong -\frac{e^2}{m^2}HE\cos\mu\,\frac{t^3}{6}$$

$$z = \frac{e}{m}E\sin\mu\,\frac{t^2}{2}\cdot$$

We now define three directions as follows:

Z-direction: parallel to $\mathbf{E}$

Y-direction: same as y-direction

X-direction: perpendicular to Y- and Z-directions,

i.e., we rotate the original axes about Oy through an angle $(\pi/2-\mu)$. Then

$$X \cong -\frac{e^3}{m^3} H^2 E \sin\mu \cos\mu \frac{t^4}{24}$$

$$Y \cong -\frac{e^2}{m^2} HE \cos\mu \frac{t^3}{6}$$

$$Z \cong \frac{e}{m} E \frac{t^2}{2} - \frac{e^3}{m^3} H^2 E \cos^2\mu \frac{t^4}{24}$$

so that, as H tends to zero, X and Y tend to zero, while Z tends to $(eE/m)t^2/2$, as would be expected.

6. Deviation from the vertical of a falling particle. Translating these results into the language of problem II, where the X, Y, Z, directions point south, east, and vertically upwards, we obtain the usual formulae:

$$\left\{\begin{aligned} X &\cong \omega^2 g \sin\lambda \cos\lambda \frac{t^4}{6} \\ Y &\cong \omega g \cos\lambda \frac{t^3}{3} \\ Z &\cong -\frac{gt^2}{2} + \omega^2 g \cos^2\lambda \frac{t^4}{6} \end{aligned}\right. \tag{7}$$

7. Note. It should be noted that the formulas for the deviation of a falling particle obtained by Denizot are somewhat different from our formulae (7). This is owing to the fact that Denizot takes into account the variable part of the centrifugal force and considers instead of (2) the equation

$$\ddot{\mathbf{r}} = -\mathbf{A} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{A}$ is the vector representing the acceleration due to gravitation. This system of three linear equations can also be solved rigorously in an elementary way, provided $\mathbf{A}$ is considered to be constant. But this is equivalent to the usual assumption that the length of the trajectory is small compared with the radius of the earth, and in that case we are also justified in neglecting the variable part of the centrifugal force. The only difference between Denizot's formulae and ours (equations (7)) is that the numerical factor in the terms proportional to t^4 is 1/8 instead of 1/6.

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J. L. Synge and B. A. Griffith, Principles of Mechanics, McGraw-Hill, New York, 1942, section 13.4 (esp. pp. 376–379) and section 13.5 (esp. pp. 391–395).

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DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

MINIMAL TANGENTS

IRVING KAPLANSKY, New York City

A popular problem in elementary calculus texts is to find the shortest tangent to a curve cut off between the axes, or the tangent that cuts off the triangle of smallest area, *etc.* The algebra is usually found to be fairly heavy, unlike the gratifyingly simple answers. It is perhaps interesting to solve once for all a general problem of this kind.

At the point (x, y) of a curve let the tangent be drawn; its x- and y-intercepts, say u and v, are given by

$$u = (xy' - y)/y', \qquad v = y - xy'.$$

We have $u' = yy''/(y')^2$, $v' = -xy''$. Suppose $z = f(u, v)$ is to be minimized. We set

$$z' = f_u u' + f_v v' = y''[f_u y/(y')^2 - f_v x]$$

equal to zero. The root $y'' = 0$ corresponds to a point of inflection which yields an uninteresting extremum. The desired solution is therefore given by

$$(y')^2 f_v/f_u = y/x. \tag{1}$$

Examples. 1. If $z = u^n + v^n$, then $f_v/f_u = v^{n-1}/u^{n-1} = (-y')^{n-1}$ and (1) becomes

$$(-y')^{n+1} = y/x. \tag{2}$$

The case $n = 2$ corresponds to minimizing the length of the tangent, the case $n = 1$ to minimizing the sum of its intercepts.

2. If $z = uv$, then $f_v/f_u = u/v = -1/y'$, yielding $-y' = y/x$. This is the problem of minimizing the area of the triangle cut off by the tangent, and fits in as the case $n = 0$ of (2). The answer has the geometrical interpretation that the triangle formed by the origin, the point of tangency, and the x-intercept is isosceles.

3. A further variant is to seek a curve for which z is constant. This is given by solving (1) as a differential equation. The solution of (2) is

$$xy = c \qquad (n = 0),$$

$$x^{n/(n+1)} + y^{n/(n+1)} = a^{n/(n+1)} \qquad (n > 0).$$

We thus establish well known properties of the rectangular hyperbola, the parabola ($n = 1$), and the 4-cusped hypocycloid ($n = 2$).

ON POLYNOMIALS WITH MULTIPLE ROOTS

W. J. STERNBERG, Cornell University

Let $f(x)=0$ be an algebraic equation of degree n with multiple roots. We collect roots of the same multiplicity. Let

$$\begin{array}{l} \alpha_{11},\ \alpha_{12},\ \cdots,\ \alpha_{1r_1} \text{ be different roots of multiplicity } n_1, \\ \alpha_{21},\ \alpha_{22},\ \cdots,\ \alpha_{2r_2} \text{ be different roots of multiplicity } n_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{k1},\ \alpha_{k2},\ \cdots,\ \alpha_{kr_k} \text{ be different roots of multiplicity } n_k, \end{array}$$

where $0<n_1<n_2<\cdots<n_k$ and $r_1n_1+r_2n_2+\cdots+r_kn_k=n$. Put

$$(x-\alpha_{i1})(x-\alpha_{i2})\cdots(x-\alpha_{ir_i})=g_i(x), \qquad (i=1,2,\cdots,k).$$

Then, assuming that the coefficient of the highest power of x is one,

$$f(x)=[g_1(x)]^{n_1}[g_2(x)]^{n_2}\cdots[g_k(x)]^{n_k}.$$

We state that the coefficients of the polynomials $g_1(x),\cdots,g_k(x)$ depend rationally on those of $f(x)$, or, to put it briefly, are "rational."

To prove this, let

$$q(x)=g_1^{n_1-1}g_2^{n_2-1}\cdots g_k^{n_k-1}$$

be the greatest common divisor of $f(x)$ and its derivative $f'(x)$. Let

$$f(x)/q(x)=h(x)=g_1g_2\cdots g_k,$$
$$f(x)/[h(x)]^{n_1}=f_1(x)=g_2^{n_2-n_1}g_3^{n_3-n_1}\cdots g_k^{n_k-n_1}.$$

The coefficients of $q(x)$, $h(x)$, $f_1(x)$ are, of course, "rational." Putting $n_2-n_1=\nu_2$, $n_3-n_1=\nu_3,\cdots,n_k-n_1=\nu_k$, we get

$$f_1(x)=g_2^{\nu_2}g_3^{\nu_3}\cdots g_k^{\nu_k}, \qquad 0<\nu_2<\nu_3<\cdots<\nu_k.$$

The polynomial $f_1(x)$ is thus of the same type as $f(x)$, but includes only $k-1$ of the polynomials $g_i(x)$. By applying to $f_1(x)$ the analogous method, we obtain

$$q_1(x)=g_2^{\nu_2-1}\cdots g_k^{\nu_k-1}, \qquad h_1(x)=g_2\cdots g_k, \qquad f_2(x)=g_3^{\nu_3-\nu_2}\cdots g_k^{\nu_k-\nu_2}.$$

The same procedure can be continued, and we conclude that the coefficients of $g_2(x),\cdots,g_k(x)$ are "rational." Moreover, $h(x)/h_1(x)=g_1(x)$. The coefficients of $g_1(x)$ are thus "rational." This completes the proof.

We mention a special case. If a root, say α_{i1} has a multiplicity n_i which is different from those of all other roots, then the corresponding polynomial $g_i(x)$ is linear, *i.e.* $g_i(x)=x-\alpha_{i1}$, and α_{i1} itself is "rational." All roots of this kind are thus "rational."

A NEW MATCH-GAME

H. D. GROSSMAN and DAVID KRAMER, New York City

Both Nim and the simple match-game: two players alternately take up to a certain number from a group of matches, the object being to take or not to take the last match, are well known. A periodical now out of print, *Games Digest*, April, 1938, suggested without solution a subtle variation of the latter, which we broaden as follows: two players alternately take up to a certain number from a group of matches, the object being to make one's opponent finish with an *odd* or an *even* total.* Analysis and generalization are simplified by focusing attention on the opponent's total (rather than one's own) and (as in Nim) on the number of matches left on the table rather than on the number taken. We shall call the players A and B. The winning moves are defined below; any other move will be called a losing one; the proof follows. A's or B's parity is the oddness or evenness of the number of matches he has at the moment.

The following solution can be worked out:

If the maximum number n one may take is	and if one's ultimate objective is to *change* his opponent's present parity, the winning move is to leave on the table a number of matches congruent to	and if one's ultimate objective is to *keep unchanged* his opponent's present parity, the winning move is to leave on the table a number of matches congruent to
I. *even*;	1 (mod $n+2$).	0 or $n+1$ (mod $n+2$).
II. *odd*;	1 or $n+1$ (mod $2n+2$).	0 or $n+2$ (mod $2n+2$).

If the move or moves indicated are impossible, you have a theoretically lost game, though you will probably win it by the subsequent failure of your opponent to make winning moves. Usually just one move is possible, except in the upper right-hand corner of the table where both moves are often possible (the latter probably preferable because it shortens the game more).

Proof. The proof will be divided into three parts:

1. A winning move can never be followed by a winning move.
2. A losing move can always be followed by a winning move.
3. The last possible winning move wins the game.

The symbols k, k' $(=k-1)$, and n represent integers. At the end of the game with an even number of matches, zero, on the table, both parities or neither will be as A (or as B) wishes. Then when the number on the table is even, both parities or neither are as A (or as B) wishes, but when the number on the table is odd, one of the parities is as A (or as B) wishes and the other is not. Naturally whatever A wishes, B always wishes the opposite.

* A special case of this game is also given in H. E. Dudeney, Amusements in Mathematics, pages 117 and 240.

I, 1. (n *even*)

A's winning move leaving	cannot be followed by B's winning move leaving	
$k(n+2)$ with B's and A's parities both as A wishes	$k'(n+2)+1$,	since B must take from 1 to n matches.
$k(n+2)+1$ with not B's but A's parity as A wishes	$k(n+2)+1$,	
$k(n+2)+n+1$ with B's but not A's parity as A wishes	$k(n+2)$ or $k(n+2)+n+1$,	

I, 2. (n *even*)

A's losing move leaving	can be followed by B's winning move leaving
$k(n+2)+2, k(n+2)+3, \cdots, k(n+2)+n$	$k(n+2)$ or $k(n+2)+1$, B being able to choose whichever the situation demands.
$k(n+2)$ with neither B's nor A's parity as A wishes	$k'(n+2)+n+1$.
$k(n+2)+1$ with B's but not A's parity as A wishes	$k(n+2)$ or $k'(n+2)+n+1$.
$k(n+2)+n+1$ with not B's but A's parity as A wishes	$k(n+2)+1$.

The proof of I, 3. is at once obvious. Since we start with a finite number of matches, we reach the last match in a finite number of steps. Leaving B one match changes his parity; leaving him none does not.

II, 1. (n *odd*)

A's winning move leaving	cannot be followed by B's winning move leaving	
$k(2n+2)$ with B's and A's parities both as A wishes	$k(2n+2)+1$ or $k'(2n+2)+n+1$,	since B must take from 1 to n matches
$k(2n+2)+1$ with not B's but A's parity as A wishes		
$k(2n+2)+n+1$ with neither B's nor A's parity as A wishes	$k(2n+2)$ or $k'(2n+2)+n+2$,	
$k(2n+2)+n+2$ with B's but not A's parity as A wishes		

II, 2. (n *odd*)

<table>
<tr><th>A's losing move leaving</th><th colspan="2">can be followed by B's winning move leaving</th></tr>
<tr><td>$k(2n+2)+2,\ k(2n+2)+3,\ \cdots,\ k(2n+2)+n$</td><td>$k(2n+2)$ or
$k(2n+2)+1$,</td><td rowspan="2">B being able to choose whichever the situation demands.</td></tr>
<tr><td>$k(2n+2)+n+3,\ k(2n+2)+n+4,\ \cdots,$
$k(2n+2)+2n+1$</td><td>$k(2n+2)+n+1$ or
$k(2n+2)+n+2$,</td></tr>
<tr><td>$k(2n+2)$ with neither B's nor A's parity as A wishes</td><td colspan="2">$k'(2n+2)+n+2$.</td></tr>
<tr><td>$k(2n+2)+1$ with B's but not A's parity as A wishes</td><td colspan="2">$k(2n+2)$.</td></tr>
<tr><td>$k(2n+2)+n+1$ with B's and A's parities both as A wishes</td><td colspan="2">$k(2n+2)+1$.</td></tr>
<tr><td>$k(2n+2)+n+2$ with not B's but A's parity as A wishes</td><td colspan="2">$k(2n+2)+n+1$.</td></tr>
</table>

The proof of II, 3 is the same as I, 3.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

NATIONAL OFFICERS OF KAPPA MU EPSILON FOR 1945-46

President: E. R. Sleight, Albion College, Albion, Michigan
Vice-President: Fred W. Sparks, Texas Technological College, Lubbock, Texas
Secretary: E. Marie Hove, University of New Mexico, Alburquerque, New Mexico
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Historian: Sister Helen Sullivan, O.S.B., Mount St. Scholastica College, Atchison, Kansas

CANTOR CENTENNIAL

Mathematics Clubs may wish to take cognizance of the fact that this year is the centennial of the birth at St. Petersburg on March 3, 1845, of the German mathematician, Georg Cantor, famous as the founder of the theory of transfinite numbers.

CLUB REPORTS 1944-45

Mathematics Club, Brown University

In each of the three semesters the Mathematics Club announced in advance its program of meetings, which were held in Wilson 307 at 2000 o'clock. Refreshments followed the formal part of each meeting. The Club picture was taken at the first meeting after Christmas. Picnics were planned for September and May. Papers for the summer semester 1944 were published in the last report. Those for the fall semester were as follows:

The design of a computing machine, illustrated by machines and lantern slides, by Professor W. Prager, Brown University, L. N. Pease presiding.

The Fibonacci series, by Evelyn Lindsay, and

The origin of logarithms, by Henry Epstein, Frances Jenckes presiding.

Ptolemy of Alexandria and his geometrical theorem, by Julianne Heller, and

The fourth dimension, by Sumner Levine, R. P. Breeding presiding.

Papers for the spring semester were as follows:

Some problems of geometry, by Professor Will Feller, Brown University, Shirley Gallup presiding.

The game of Nim, by R. B. Abel, and

One-sided surfaces, by Miriam Rose, E. N. Clarke presiding.

Magic squares, by D. T. Cross, and

Rational triangles and quadrilaterals, by P. R. Garabedian, Marjorie Hackett presiding.

A Mathematics Club Prize Competition was held during the spring semester. The prize of five dollars was awarded for the best undergraduate solution or discussion of a set of three problems in mathematics. Competitors were permitted to make full use of the Mathematical Library.

The club was directed by a Committee on Program and Arrangements, consisting of a Faculty Representative: Professor R. C. Archibald; a Student Chairman: Shirley Marilyn Gallup; and the following committee members: R. P. Breeding (fall '44), D. T. Cross (spring '45), S. L. Ehrlich, C. V. Harding, Jr., Frances Jenckes, Evelyn Lindsay, Lynn Pease.

Rho Theta, St. Louis University

Three meetings of the *Rho Theta* Mathematics Club were held in the first half of the academic year, 1944–45. A paper was read at one of them, entitled

Laplace transforms in applied mathematics, by Professor A. E. Ross.

The other two meetings were devoted to the discussion of problems arising in connection with the forthcoming affiliation with the National Honor Mathematics Fraternity, *Pi Mu Epsilon.*

The officers for 1944–45 were: President, Helen Jackson; Vice-President, Carl Kisslinger; Secretary-Treasurer, Mary F. Nawrocki.

Pi Mu Epsilon, St. Louis University

The installation of the Missouri Gamma Chapter of *Pi Mu Epsilon* at St. Louis University took place on Thursday, February 22, 1945. The program commenced with an academic procession participated in by the Arts and Science faculties. Professor P. R. Rider of Washington University in St. Louis presented the charter and delivered the principal address as the installation officer of *Pi Mu Epsilon.* The twenty-five active members of the *Rho Theta* Club became charter members of the Missouri *Gamma* Chapter and in addition forty-seven new members were initiated. The response for St. Louis University was made by the Reverend T. M. Smith, S.J., followed by a lecture on

Pathological functions, by Professor T. H. Hildebrandt of the University of Michigan.

At the banquet which culminated the day's activities, the new chapter was welcomed by the President of St. Louis University, the Reverend Patrick J. Holloran, S.J. The response for the National Organization was made by Professor Rider and for the *Rho Theta* Mathematical Fraternity, the forerunner of the new Chapter, by Miss Helen Jackson, President. The banquet speech was entitled:

The development of the mathematical sciences at St. Louis University, presented by the Reverend J. B. Macelwane, S.J., Dean of the Institute of Geophysical Technology.

At the first meeting of the Chapter, held on April 9, 1945, a paper was read, entitled:

The vector concept in elementary analytic geometry, by Brother Thomas Matthews, F.S.C.

The officers for 1944–45 are: Director, Helen Jackson; Secretary-Treasurer, Mary Nawrocki. Professor F. Regan is the Faculty Adviser and the Corresponding Secretary.

Junior Mathematics Club, Iowa State College

One meeting was held each quarter during the year, attended by an average of thirty-five persons. The following talks were given:

Navigation, by Professor D. L. Holl

The calendar, by Allen Miller and Owen Sauerlender, subsequently published under the title *The drifting years*, in *The Iowa Engineer* (vol. 45, Mar. 1945, pp. 133–135)

Matrices, by Carl Langenhop and Russell Carr.

A special invitation to the last meeting was extended to *Pi Mu Epsilon* members. The *Pi Mu Epsilon* prize in mathematics for outstanding achievements in mathematics in the freshman and sophomore years was awarded to Miss Mary Ann Williams.

The members of the committee for the year were as follows: Jean Grosser, Leslie Smith, Amanda Christensen, Adelaide Madsen, Aliene Barrett; Faculty Adviser, Professor Fred Robertson.

Mathematics Society, Massachusetts Institute of Technology

The following talks were presented during the past year:
Origins of the calculus, by Professor D. J. Struik
Continued fractions, by W. S. Loud
Fourier division, by L. R. Norwood
Cute curves, by D. L. Thomsen
Elementary topology, by O. G. Selfridge
Trisecting the angle, by R. G. Selfridge
Solution of integral equations, by R. E. Graves
The Riemann integral, by F. E. Browder
Topics in the theory of Bessel functions, by D. Mintzer
Metrics of n-dimensional space, by R. Kraichnan
Functions of a complex variable, a series of four papers, by L. R. Norwood, S. H. Crandall, R. E. Graves, and D. Mintzer
Epsilontics, by W. S. Loud
Elementary matrix theory, by I. Stempnitzky
Elementary tensor theory, by L. B. Wadel
The continuum of real numbers, by J. W. Shearer
The transcendence of e, by R. G. Selfridge
Imaginaries in geometry, by L. A. Zadeh
Diffusion, by Professor H. B. Phillips
Operational methods, by S. T. Epstein
Non-linear differential equations, by W. S. Loud
Finite geometry, by R. P. Abelson
Fourier transforms, by R. E. Graves
Probability theory, by O. G. Selfridge
Elements of Sienckiewicz theory with applications, by W. Pitts
Introductory statistics, by E. J. Gehrig
Incremental and progressional calculus, by L. B. Wadel.

In addition to the presentation of the above talks, the Society sponsored mathematics contests for Freshmen and Sophomores last fall. First places were taken by F. E. Browder and P. D. Jones, respectively. Officers of the Society were as follows. For July–October, 1944: President, L. R. Norwood; Vice-President, O. G. Selfridge; Secretary-Treasurer, S. H. Crandall. For November–June, 1944–45: President, R. E. Graves; Vice-President, L. B. Wadel; Secretary-Treasurer, R. G. Selfridge. For July–October, 1945: President, L. B. Wadel; Vice-President, F. E. Browder; Secretary-Treasurer, R. B. Davis.

Kappa Mu Epsilon, Texas Technological College

The following papers were presented during the year to the Texas Alpha Chapter of *Kappa Mu Epsilon*:

Many valued logic, by Associate Professor D. L. Webb

Life and work of Pythagoras, by Edward Turrentine

Scotopic and photopic vision, by Beverly Price

Some of the mechanical processes in making steel for guns and the construction of their barrels, by Jim Wanner

The Fourier series, its background and some original investigations, by Virginia Bowman

The fall term initiation banquet and Christmas party were combined and held in the Chimayo Room of the Hilton Hotel on December 14, 1944. Thirteen new members were initiated during the 1944–1945 session. Officers for the 1945–1946 session are: President, Jim Wanner; Vice-President, Maisie Carter; Secretary, Sarah Scroggins; Treasurer, Ben Logan; Corresponding Secretary, Lida B. May; Faculty Sponsor, Dr. D. L. Webb.

Kappa Mu Epsilon, Mount St. Scholastica College

The Kansas Gamma Chapter numbered thirty-six active members and held meetings semi-monthly. The topic for special research and study during the academic year was:

Opportunities for women trained in mathematics.

Vital information on this subject was obtained by sending form letters to fifty of the nation's leading industrial and business concerns. A symposium based on the results was presented at an all-college assembly. The participants were: Patricia Warwick, Ann Hughes, Mary Growney, Mary Davis, Virginia Harrison, Katherine Zeller, and Jane Hajovsky.

Outstanding features of the year were the following: *Mathematics and philosophy*, a lecture by the Reverend Malachy Sullivan, O.S.B.; the publication of a departmental newspaper, *The Exponent*, which was distributed to interested collegians as well as to former chapter members, and the purchase of War Bonds to promote the war effort.

At the May meeting fourteen new members were initiated, representing the states of Kansas (4), Illinois (3), Iowa (2), Alabama, Missouri, Nebraska, Pennsylvania, and Texas. Following the initiation a formal buffet supper was served. At this time the announcement was made that Katherine Zeller had merited the annual award for having contributed the most to the fraternity during the past year.

Officers elected for the year 1945–1946 are: President, Mary Lou Maloney; Vice-President, Katherine Zeller; Secretary, Mary Jane Fox; Treasurer, Mary Davis; Chapter Publicity, Victoria Fritton; Chapter Musician, Elizabeth Gulde; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Statistics and Applications. By J. G. Smith and A. J. Duncan. New York, McGraw-Hill Book Co., Inc., 1944. 10+720 pages. $4.00.

This book is a companion volume to the same authors' *Sampling Statistics and Applications*, forming a set on *Fundamentals of the Theory of Statistics*. It is designed for a beginning course and evolved over a period of ten years from mimeographed notes to its present form. Calculus is used sparingly. The contents include principles of gathering and presenting statistics, frequency distribution analysis, probability theory, normal curve, elementary sampling procedures, correlation, time series analysis, and forecasting. The more advanced points of theory are generally relegated to the companion volume. Parts of the manuscript benefited from criticism and suggestions by members of the Bureau of Labor Statistics and the Princeton mathematics department.

The book is of considerable dimensions, the introduction alone occupying 157 pages. Here are outlined applications to many fields: economics, business, politics, education, psychology, and the physical sciences. There is a description of methods of gathering statistics including discussion of types of questionnaires, illustrations being drawn from forms used by various government agencies. There is even a replica of instructions issued to census enumerators. A valuable chapter on sources goes as far back as the Domesday Book and is particularly complete as regards U. S. government publications of a statistical nature. A chapter on presentation is illustrated with six types of tables and nineteen types of charts and cartograms. The introduction ends with a discussion of certain elementary symbolic conventions and a description of graphical procedures: time series, ratio and log-log charts, discrete and continuous frequency distributions, and bivariate series.

A second division deals with the analysis of frequency distributions. Here occur descriptions of the various measures of central tendency and dispersion and their interrelations. Higher moments receive an unusually complete and lucid treatment. Change of class interval, arrangement of work sheets, and short cuts in computation are discussed in great detail and in a thoroughly practical way.

Part III, on the normal distribution, follows the classical sequence, probability—binomial distribution—normal curve. Noteworthy is an excellent critique of the Laplace and von Mises concepts of probability, leading to an acceptance of the "intuitive-axiomatic" approach in the form exposed by J. Neyman. The approach of the binomial distribution to normality is illustrated diagrammati-

cally and made acceptable with some simple analytic geometry. There is also a summary of conditions tending in practice to produce normality. Fitting a normal curve to a given histogram with subsequent χ^2 test is described. A laudable point here is the carrying of a single set of data through various aspects of a statistical analysis. Thus, data on heights of Princeton freshmen, used earlier to illustrate class interval and moments, is here fitted with a normal curve and tested. A chapter on sampling, statistical inference, and confidence limits includes a tabulation of the standard errors of the more important sampling statistics. Diagrams and clear explanations take the place of rigorous derivation.

There are 176 pages devoted to multivariate analysis, starting with simple regression and correlation and running the gamut of multiple and partial correlation into a discussion of the normal frequency surface. No pains are spared to clarify this rather confusing field. For example, detailed work sheets and diagrams and a precise attention to notation lead one through a two and a three variate analysis up to a final tour de force in four variates. Analysis of variance enters in a purely formal way in connection with multiple regression. The description of normal frequency surface includes excellent perspective figures and a detailed evaluation of the undetermined coefficients in terms of the fundamental parameters.

Part V is the study of dynamic variability from index numbers to business cycles. Notable is a paragraph on adjustment of indexes to bench marks with presentation of a method used by the Bureau of Labor Statistics. Time series are analyzed successively for trend, seasonal variation, and cyclical fluctuation. For use in higher order trends orthogonal polynomials are given a very practical discussion. Of course, orthogonality is here defined in terms of equally spaced ordinates and not over a continuous interval. Calculation of index of seasonal variation and determination of major and minor cycles after prior removal of primary factors from the raw series are described with tables and graphs. A final relatively short division deals decriptively with forecasts, a subject which the authors judiciously term an art rather than a science. Historical analogy and cross-cut analysis are the main ingredients.

There is both a subject and author index and an appendix with eight mathematical tables. Throughout the book there are 159 figures, 101 tables, and footnote references to about 300 books and papers. Very few misprints or misstatements were noted, none of them serious. There are no exercises.

The book is neither written by mathematicians nor designed primarily for mathematicians so one must overlook a certain lack of mathematical elegance sometimes apparent. The book's most striking positive characteristics are its heroic comprehensiveness in content and references, its clear elementary exposition, and its richly detailed illustrative material drawn from real life.

J. L. VANDERSLICE

College Algebra and Trigonometry. By F. H. Miller. New York, John Wiley and Sons, 1945. 12+324 pages. $3.00.

The sub-title of this text is *A Basic Integrated Course.* The titles are indicative of the premises on which the book is based; namely, that certain portions of the conventional college courses in algebra and trigonometry give the student the essential background for subsequent mathematical study, and that these fundamental parts can advantageously be integrated into a single course to serve as a foundation for the study of analytic geometry and calculus.

The choice of the material included in the book is influenced by the viewpoint of the author, that the analytical aspects of the constituent subjects are deserving of emphasis. Throughout the presentation the general concepts and the analytical techniques are the goals. Unusual deftness is shown in the treatment of a number of topics in this manner.

Chapters I and II contain a discussion of number systems, a postulational approach to algebra, and a brief treatment of functions and graphs. Chapter III is concerned with the fundamentals of trigonometry, with stress being put on the general definitions of the trigonometric functions and on radian measure. The integration of algebra and trigonometry is begun in Chapter IV, on identities and conditional equations.

Some noteworthy features of the rest of the book are the following. Emphasis is put on analytical trigonometry; only one section of three pages is devoted to the logarithmic solution of triangles. The author avails himself of the particular order and manner in which he interweaves the algebra and trigonometry to use the trigonometric functions as illustrations of transcendental functions, to relate the methods of solving algebraic and trigonometric equations, to prove selected trigonometric relations by mathematical induction, to give an unusually clear and complete treatment of complex numbers, to apply the principles and operations of algebraic inequalities to trigonometric inequalities, to study inverse trigonometric functions along with inverse algebraic functions, to give the variation of algebraic functions and the variation of trigonometric functions a common setting, and to show how certain transcendental equations of high degree may be solved with the use of methods of solving rational integral equations. Chapter XIV, on exponential and logarithmic functions, contains some well chosen material on the number e, on exponential equations, and on logarithmic equations.

The impression of the reviewer is that in *College Algebra and Trigonometry* the general concepts and analytical techniques of algebra and trigonometry necessary for the study of analytic geometry and calculus are aptly chosen and worthily treated.

T. L. WADE

Engineering Mathematics. By Harry Sohon. New York, D. Van Nostrand Company, Inc., 1944. 6+278 pages. $3.50.

Professor Sohon is Assistant Professor of Electrical Engineering at the Moore School of Electrical Engineering of the University of Pennsylvania. He has attempted to collect in this volume such mathematics as "will strengthen the student in algebra and provide him with certain mathematical tools which depend on the calculus" and has written it for the advanced undergraduate in engineering, for graduate students in engineering, and for the practicing engineer.

The Chapters in order are: Interpolation Formulas, Determinants, Dimensional Analysis, Complex Numbers and Hyperbolic Functions, Algebraic Equations, Approximate Solutions of Algebraic Equations, Fourier Series, Differential Equations, Gamma Functions and Bessel Functions, Vector Algebra, Vector Calculus, Stretched String and Round Diaphragm, and Skin Effect Problems.

It seems to this reviewer that the author has attempted to embody a large amount of elementary and advanced mathematical material in a minimum amount of space. As a result some sections in the text are almost in the character of a handbook and, in a few sections, the brevity leads to lack of coherence and continuity. Such shortness of treatment in parts of the text accounts for the absence of those helpful suggestions that make for understanding and learning—as, for example, the remark that the coefficients in Newton's interpolation formula are binomial coefficients.

There are, perhaps, three schools of thought with regard to the teaching of advanced undergraduate mathematics to engineers. One school, to characterize it in the extreme, believes in instruction by the mathematician in mathematics and by the engineer in matters engineering and a very careful cleavage in that instruction. A second school of thought, again to describe it in the extreme, believes in a single classroom with combined instruction in both advanced mathematics and engineering. The third school believes in separate instruction by the mathematician and the engineer but with frequent utilization of engineering examples in mathematics and vice versa. The present text can be classified as belonging to the first school of thought for each chapter has an opening paragraph suggesting some of the engineering applications of the material contained in that chapter. Thereafter the reading material and problems for assignment are usually mathematical in statement and content.

The chapter on Fourier Series contains an interesting analysis to show that the customary use of the trapezoidal rule in approximate analysis is to be preferred on the basis of expected accuracy to the application of Simpson's rule. The chapter on Dimensional Analysis is well written and contains several illustrations to show the underlying theory and to illustrate Buckingham's Pi Theorem. The discussion of Graeffe's method is excellent and the author illustrates with simple examples the basic ideas and restricts his theory to the cases of real roots, repeated real roots, and simple complex roots, which is sufficient for engi-

neering application. If the entire text were about one-third longer because of elaboration of material and were written in the style of these three chapters, the reviewer would find this text much more meritorious.

Misprints, rather to be expected in this period of emergency, are not too numerous and do not vitiate the content. One error of commission is made throughout the chapter on Hyperbolic Functions when the $\pm$ sign is not prefixed to the square-roots and no mention is made of the range of validity of these formulas. This same mistake sometimes is found in elementary texts on trigonometry and calculus.

The choice of content for the text includes most of the topics from advanced undergraduate mathematics that the engineering student is likely to need in specializing senior courses and in many first-year graduate courses in engineering.

J. W. CELL

Alignment Charts. By Maurice Kraitchik. New York, D. Van Nostrand Company, Inc., 1944. 6+94 pages. $2.50.

The author of this little volume was for many years in close contact with Maurice d'Ocagne, who was a pioneer in developing the science of nomography. He has here presented that science with simplicity and in a readable manner. Furthermore, the mathematical presentation is adequate and generally satisfying.

As is proper, the treatment rests on the three-rowed determinant. There is an introductory chapter on determinants as a preparation for what is to come. This material and some following chapters on the construction of scales cover twenty-five pages.

On page 26 the alignment chart is introduced in its generality. Then follow chapters on the most important special types of charts; such as those with three parallel scales and those with two parallel scales. There are chapters on special charts and other matters.

There are nearly fifty charts of various kinds. These are excellently constructed (with the single exception of Figure 19, page 36) and they would well repay study on the part of the would-be nomographer. The graduations on the scales and their numbering are models of good practice. The charts cover a variety of fields with, however, a rather strong predilection for investment problems.

The book's most noticeable defect is the unorthodox use of the word "function." We read everywhere of variables "satisfying a function" and repeatedly the author uses "function" when he means "equation."

The reviewer would have liked a somewhat fuller treatment of projective transformations. However, one should not expect too much in ninety-odd pages.

The book is generally clear and careful in its explanations and it is easy to read.

L. R. FORD

Theory of Functions. By Konrad Knopp. Translated by Frederick Bagemihl, New York, Dover Publications, 1945. 8+146 pages. $1.25.

The two parts of Knopp's Funktionentheorie have been recognized for a long time as excellent summaries of the principal results of the theory of analytic functions. The topics are well chosen and systematically arranged. While the author is careful to be accurate, he is occasionally willing to make informal supplementary remarks when these help to clarify the situation for the reader. Throughout the style is as clear and readable as is consistent with the extreme brevity.

The book under review is the translation into English of Part I, Foundations (fifth German edition). This includes the main properties of single valued analytic functions, starting with the concept of analytic function and a recapitulation of elementary properties. This is followed by a section on integral theorems, one on power series and analytic continuation, and a final section on Laurent series and singularities. The translator has done his work well, and has even made some minor corrections and improvements.

The book assumes some familiarity with the theory of real numbers, including sequences, limits and series and also with the definition and differentiation of elementary functions for complex values, and with infinite series of constant complex terms. There are some, but not very many, exercises. However, any reader with the required background can gain much from this little book. The translation should be particularly valuable to the American student for purposes of supplementary reading, self-study, or review.

PHILIP FRANKLIN

Fermagoric Triangles. By Pedro Pizá. Publication No. 1 of the Polytechnic Institute of Puerto Rico, San Germán, P.R. New York, G. E. Stechert and Co. (U. S. Distributor), 1945. 155 pages. $3.00.

A triangle with sides a, b, c, is called fermagoric of degree n in case $a^n+b^n=c^n$. For $n=2, 3, 4, 5$, the sides of such a triangle may be expressed as functions of two real parameters (say) r, s, involving at most two square root extractions. Thus for these values of n, and for rational values of r, s, the corresponding fermagoric triangles are constructible with ruler and compasses. Ingenious constructions of such triangles are a feature of the book. The author contends that F. Klein's dictum concerning the impossibility of parameterization of the Fermat equation should be revised in the light of his results. Klein's statement referred of course to expression of a, b, c as *rational functions* of parameters, reducing the Fermat equation to an *identity* in these parameters.

The latter part of the book is concerned with geometric studies of the n-fermat, that is, the algebraic plane curve of degree $4n$, traced by a point always forming a fermagoric triangle of degree n with two other fixed points as vertices of the base c.

There are many excellent figures and a wealth of numerical examples in the text.

The author, while confessing himself an amateur, is nevertheless remarkably well informed on the history of the Fermat problem, and writes with an enthusiasm all too rare in these days. One happy phrase among many lingers in the memory. In reviewing existing proofs of Fermat's theorem for many special primes, Pizá chides: "But since Euclid so gracefully demonstrated in the *Elements* that the series of prime numbers is infinite, many, no matter how many, can never be all."

C. J. EVERETT

Statistics for Sociologists. By M. J. Hagood. New York, Reynal and Hitchcock Co., 1941. 934 pages. $4.00.

As the title of the book indicates, it has been written primarily for sociologists who want to learn how to use statistical methods in their field of research and be able to evaluate critically the results of statistical analysis made by others. Since most of the sociologists have little mathematical background, the treatment of the subject, as given in the book, is elementary and non-mathematical. The mathematical tools used do not go beyond freshman college algebra. Consequently, proofs and derivations of formulas are not given, but the meaning of statistical procedures and interpretation of the results are stressed throughout the book.

The discussion of statistical methods and procedures is subdivided into two logically distinct parts, descriptive statistics and inductive statistics. In descriptive statistics, as distinguished from inductive statistics, no attempt is made to draw inference as to the parameters of the population from which the sample has been taken. The purpose of descriptive statistics is merely to condense the information contained in the sample into a few "summarizing measures" or "statistics." Under the heading "descriptive statistics" a number of summarizing measures, such as proportions, rates, frequency tables, various kinds of means, several measures of dispersion, *etc.*, are discussed. The problem of statistical inference is taken up in the part on inductive statistics. The understanding of descriptive statistics requires, of course, less intellectual effort than that of inductive statistics and, therefore, such a subdivision may have some advantages from the pedagogic point of view. On the other hand, it seems to the reviewer that the treatment of descriptive statistics as an independent logical unit has some drawbacks also, since the various "summarizing measures" can be justified, in general, only on the basis of their usefulness for purposes of statistical inference. Under the heading "statistics of relationship" contingency tables, analysis of variance and covariance, partial and multiple regressions and correlations are discussed.

Throughout the book the author takes great pains to explain the meaning of the various statistical procedures and to state the conditions under which they are applicable. While as a whole the author was successful in this effort, especially in explaining the meaning of tests of significance and estimation by confidence intervals, the discussion is perhaps not everywhere entirely adequate.

In the chapter on regressions there is no method given for setting up confidence limits for partial regression coefficients, although exact procedures for this purpose are available. The reviewer also noticed a few statements which are not entirely correct. For example, the definition of an unbiased estimate as given on page 388 is not correct. The statement that the sample standard deviation, using the number of degrees of freedom in the denominator, is an unbiased estimate of the population standard deviation is also incorrect. One can only say that the sample variance is an unbiased estimate of the population variance. In spite of these shortcomings, the book as a whole is certainly interesting and useful reading for sociologists who want to learn about statistical methods. The reader will also welcome the numerous illustrations and examples taken from fields of applications of interest to sociologists.

ABRAHAM WALD

Applied Mathematics for Technical Students. By M. S. Corrington. (Rochester Technical Series.) New York and London, Harper and Brothers. 1943. 3+226 pages. $2.20.

"This textbook was written as part of a program of the Rochester Athenaeum and Mechanics Institute for developing teaching materials of a practical nature which are closely related to the actual requirements of various jobs in industry. . . . This book . . . was written for trade schools, factory training courses, or pre-engineering studies; any mechanic or engineer can use it for home study and reference" (from the preface). The table of contents reads: I. Arithmetic with Applications, II. Fundamentals of Algebra, III. Logarithms, IV. Quadratic Equations, V. Simultaneous Equations, VI. Trigonometry with Applications. The book consists of standard material treated in a standard manner. The last three chapters are good on the whole, but the first half gives the impression of being hurriedly or carelessly written. The techniques are set forth in compactly stated rules, which are usually clear, but are sometimes awkwardly put, and even ungrammatical on occasion. There are many practical shop problems with a good worldly touch. Many of the problems are presented as excellent mechanical drawings. The author lays strong emphasis on checking.

There are quite a number of annoying details, mostly in the first half of the book. Multiplication is indicated by juxtaposition from the beginning, but this is not explained until page 58. Many problems on arithmetic in Chapter I involve mensuration formulae which are only given later. The problems on page 8 involve multiplication of fractions, which is explained on page 11. Decimals are used in problems before being explained in the text. Significant figures are mentioned on page 2, but are never defined or discussed. Some of the problems worked out in the book would set bad examples for the student in the matter of significant figures retained in the answers. The definitions of *direct* and *inverse proportion* are incorrect; two quantities are said to "vary directly" if one is a monotonically increasing function of the other. Letters are used to stand for numbers long before this procedure is explained; the author's meaning might be

guessed, however, by the student from the illustrative examples. The explanation of rational and irrational numbers on pages 51 and 52 is unclear and possibly misleading. According to the definitions on page 53 neither $5+x/2$ nor x^2 is a polynomial in x. The discussion on pages 69 and 70 gives the impression that ax^2+bx+c can be factored into linear factors with integral coefficients whenever a, b, and c are integers. The discussion of complex numbers and related difficulties in working with fractional exponents is quite confusing. It would probably have been better to omit this altogether or else to make the discussion more detailed. The reference on page 115 to Hermite's theorem on the transcendentality of e is irrelevant and would be meaningless to any reader of the book. On page 123 the notion of a function is used without any previous discussion. Plotting of curves is introduced on page 124 without any discussion of coordinates; later on pages 191 and 192 the same fault is repeated in connection with the trigonometric functions of a general angle. The student could probably understand the concrete applications of coordinates by studying the examples worked out in the book. The separate discussion of "incomplete quadratic equations" on page 127 is unnecessary. The fact that the inverse trigonometric equations are many valued is not mentioned.

There are many good warnings against common errors throughout the book. It would be better for use as a refresher under the guidance of a teacher rather than for self study. The format of the book and the layout of the pages are quite attractive.

P. C. ROSENBLOOM

NEW BOOKS RECEIVED

Sampling Statistics and Applications. By J. G. Smith and A. J. Duncan. New York, McGraw-Hill Book Co., 1945. 12+498 pages. $4.00.

Spherical Trigonometry After the Cesaro Method. By J. D. H. Donnay. New York, Interscience Publishers, Inc., 1945. 11+83 pages. $1.75.

The deflation spiral. I don't know just what a spiral is. I heard when in school that the involute of an Archimedean spiral has a circular asymptote of finite diameter. It is a beautiful way to put it, but I didn't know just what it meant.—W. H. Davis, Chairman of National War Labor Board, *Chester Wright Labor Letter*, November 13, 1943.—*E. D. Schell.*

Ideal and Real. The mathematician starts a line from an imaginary point that he informs us exists theoretically without occupying any space, which is a contradiction of terms according to his human acceptation of knowledge derived from scientific experiment, if science is based on verified facts. He assumes straight lines exist, which is a necessity for his calculating; but such a line he has never made . . . . He begins his study in the unknown, it ends in the unknowable.—John Uri Lloyd, *Etidorhpa*, p. 271.

The category of reality belongs not to science but to religion. It arises not as an aid to intellectual analysis, but as a means of escape or deliverance from the perplexities and confusions of deceitful appearances in a disorderly world.—M. R. Cohen, *A Preface to Logic*, p. 16.—*E. D. Schell.*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 686. *Proposed by E. D. Schell, Arlington, Virginia*

The professor writes out a series of positive terms, $\sum a_n$, on the blackboard. Because of his carelessly written addition signs, his students consider instead the product $\prod a_n$. Unfortunately the misunderstanding cannot be discovered even though he calls out the partial sum as he writes each term. Given that a_1 and a_2 are integers, find the first five terms.

E 687. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement of water. Prove that the ball is just completely submerged.

E 688. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Consider the right triangle ABC with sides $a=12$, $b=5$, $c=13$. ($a^2+b^2=c^2$.) Keeping the base BC fixed, displace the vertex A without altering the perimeter (*i.e.*, along an ellipse with foci B and C) until the value of $\cos A$ is reduced from 5/13 to 7/38. Show that the sides of the new triangle satisfy the relation

$$a^3 = b^3 + c^3.$$

E 689. *Proposed by Morgan Ward, California Institute of Technology*

Let A, B, C, D be four collinear points in the order written, and let P be any other point in space. Prove that the inequality

$$PA + PD \geqq PB + PC$$

holds for all positions of P if and only if $AB=CD$.

E 690. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is a prime number greater than 3, prove that the polynomial

$$(x+y)^p - x^p - y^p$$

is divisible by $x^3+2x^2y+2xy^2+y^3$. (Cf. E 614 [1944, 591].)

SOLUTIONS

Three-Line Latin Rectangles

E 650 [1944, 586–587]. *Proposed by Lloyd Dulmage, University of Manitoba*

If we first arrange n letters in a row, in a definite order, and then arrange below these letters

(*a*) a second row containing the same n letters so that no letter is repeated in any column, the number of possible arrays is ${}_2K_n$;

(*b*) a second row containing p of the letters of the first row together with $n-p$ other letters, so that no letter is repeated in any column, then the number of arrays is ${}_2K_{n,p}$;

(*c*) a second row containing a definite p of the letters of the first row (and $n-p$ empty spaces) so that exactly some q of these p letters do not appear below any of the p chosen letters (but the remaining $p-q$ letters appear below some $p-q$ of the p chosen letters), no letter being repeated in any column, then the number of arrays is ${}_2K_{n,p,q}$;

(*d*) two rows (second and third) each containing the same n letters, so that no letter is repeated in any column, then the number of arrays is ${}_3K_n$.

Show that the functions so defined satisfy the following relations:

$$(1)\qquad {}_2K_{n,p} = \sum_{r=0}^{p} (-1)^r \binom{p}{r}(n-r)!,$$

$$(2)\qquad {}_2K_{n,p,q} = \binom{p}{q}\binom{n-p}{q}\, {}_2K_{p,p-q},$$

$$(3)\qquad {}_2K_n = \sum_{q=0}^{\min(p,n-p)} {}_2K_{n,p,q}\cdot {}_2K_{n-p,n-p-q}$$

(for each value of p from 0 to n),

$$(4)\qquad {}_3K_n = \sum_{p=0}^{n} (-1)^p \binom{n}{p} T_p,$$

where

$$T_p = \sum_{q=0}^{\min(p,n-p)} {}_2K_{n,p,q}({}_2K_{n-p,n-p-q})^2.$$

Editorial Note. The proposer's solution will be published next year in the *Proceedings of the Canadian Mathematical Congress.* It is a matter of personal taste, whether one prefers this explicit (though complicated) formula for ${}_3K_n$, or the symbolic one given recently by Riordan [1944, 450–452]. Both far excel the attempts by Jacob and Kerawala (cited at the end of Riordan's article).

A Square of the Form *aabc*

E 652 [1945, 42]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which scales of notation can a four-digit number *aabc* be the square of a two-digit number *mn*, if $c=b+1$ and $n=m+1$?

Solution by E. P. Starke, Rutgers University. If r is the radix, we have by hypothesis

$$ar^3 + ar^2 + br + c = (mr + n)^2, \qquad a \neq 0, \tag{1}$$

which, with $c=b+1$, $n=m+1$, reduces easily to $ar^2+b=m^2(r+1)+2m$, or

$$a(r - 1) = m^2 + (2m - a - b)/(r + 1). \tag{2}$$

Hence

$$(2m - a - b)/(r + 1) = I, \tag{3}$$

an integer. Since no digit exceeds r, the possible values of I are $0, 1, -1$. From (2) and (3) we have

$$a(r - 1) = m^2 + I, \qquad 2m = a + b + I(r + 1).$$

If $I=1$, elimination of r gives $(m-a)^2+ab+2a+1=0$, which is impossible for positive digits.

If $I=-1$, we have the two equations

$$a(r - 1) = m^2 - 1, \qquad 2m = a + b - r - 1.$$

Since $a \leqq r-1$, the former equation gives $a^2 \leqq m^2-1$, or $a<m$, but then the latter becomes $r-b=a-2m-1$, in which the left member must be positive, the right negative. Hence this case is also impossible.

Thus we must have $I=0$, $a(r-1)=m^2$, $2m=a+b$. Since $a \leqq r-1$, the integer $r-1$ must contain a square factor; otherwise $a=r-1=m=b$, whence $c=r$, which is impossible. Hence

$$r = xy^2 + 1,$$

where $y>1$. We may take $a=x$, $m=xy$, $b=2xy-x$, $c=2xy-x+1$, $n=xy+1$. These satisfy (1) identically. Evidently all are positive; moreover, a, m, n, b, c are all less than r, the last because $0<x(y-1)^2$. Thus the necessary and sufficient condition on r is that it can be put in the form $r=xy^2+1$, $y>1$.

The first few values are as follows:

$$\begin{aligned} r &= 5, \quad 9, \quad 10, \quad 10, \quad 13, \cdots \\ m &= 2, \quad 4, \quad 3, \quad 6, \quad 6, \cdots \\ a &= 1, \quad 2, \quad 1, \quad 4, \quad 3, \cdots \\ c &= 4, \quad 7, \quad 6, \quad 9, \quad 10, \cdots \end{aligned}$$

Also solved by Irving Kaplansky, H. L. Lee, Helen K. Milleson, W. J. Robinson, E. D. Schell, and the proposer.

A Speculation in War Bonds

E 656 [1945, 95]. *Proposed by S. H. Gould, National Research Council, Ottawa*

Each member of a very large office-staff agrees to buy a fifty-dollar war bond. One of the members suggests that each bond, instead of being given to its purchaser, be disposed of by lot. He wishes to wager one dollar that he himself will suffer loss through his own suggestion. How much should be wagered against him?

Solution by Monte Dernham, San Francisco. If there are n office workers, it is evident that in $n-1$ out of n different contingencies, all equally probable, the first bond will not go to the particular member who made the suggestion; similarly, in $(n-1)^n$ contingencies out of a total of n^n he will not draw any of the n war bonds, and will win his wager. The odds in favor of his losing the wager are therefore $n^n-(n-1)^n$ to $(n-1)^n$, or W to 1 where

$$W = \left(\frac{n}{n-1}\right)^n - 1 = \left(1 - \frac{1}{n}\right)^{-n} - 1.$$

As n increases, the decreasing function W approaches $e-1$. Thus the odds invariably exceed 1.718 to 1, no matter how large the personnel. With the aid of a sufficiently accurate table of logarithms, we find from the relation

$$n\{\log n - \log (n-1)\} = \log (W+1)$$

that $W=1.725008 \cdots$ for $n=203$, but $W=1.724973 \cdots$ for $n=204$. Hence, for a staff of 204 or more persons, the correct wager, to the nearest cent, is \$1.72.

Also solved by D. W. Alling, D. H. Browne, N. J. Fine, Irving Kaplansky, H. D. Larsen, Bart Park, E. D. Schell, Harry Schor, E. P. Starke, and the proposer.

Three Terms of a Geometrical Progression

E 658 [1945, 95]. *Proposed by Norman Anning, University of Michigan*

Find three three-digit numbers in geometrical progression which can be derived from one another by cyclic permutation of digits.

Solution by E. P. Starke, Rutgers University. Let the three numbers be

$$x = 100a + 10b + c, \qquad y = 100b + 10c + a, \qquad z = 100c + 10a + b.$$

If y is the geometric mean, we have $y^2 = xz$ which, upon combining terms and dividing through by the factor 999, becomes

$$10(b^2 - ac) = (a^2 - bc)$$

or

$$c = (10b^2 - a^2)/(10a - b). \tag{1}$$

The trivial case $a=b=c$ is neglected. Further, if a, b have a common divisor d, then $a'=a/d$, $b'=b/d$, $c'=c/d$ give a solution whenever a, b, c do, and conversely. Hence we may take a, b relatively prime and put (1) in the form

$$c = 999a^2/(10a - b) - 10b - 100a.$$

Now $10a-b$ is also relatively prime to a, so that $10a-b$ must be a factor of 999. Thus the possible values are

$$10a - b = 1, 3, 9, 27, 37,$$

giving $(a, b)=(1, 9)$, $(1, 7)$, $(1, 1)$, $(3, 3)$, $(4, 3)$. For the first two $c>10$, the next two are trivial, and $a=4$, $b=3$ gives $c=2$, a solution. The desired numbers are 432, 324, 243; also 864, 648, 486.

Also solved by D. W. Alling, Murray Barbour, D. H. Browne, Monte Dernham, S. E. Field, F. C. Hall, Robert Hoskins, V. L. Klee, Jr., H. L. Lee, Walter Penney, E. D. Schell, J. A. Tierney, and Hazel Schoonmaker Wilson.

An Extension of the Vandermonde Determinant

E 660 [1945, 95]. *Proposed by C. D. Olds, Purdue University*

Let $z_0, z_1, \cdots, z_k$ be $k+1$ different complex numbers, all contained in the circle $|z| \leqq r$. Let

$$B_{kp} = \begin{vmatrix} 1 & z_0 & z_0^2 \cdots & z_0^{k-1} & z_0^{k+p} \\ 1 & z_1 & z_1^2 \cdots & z_1^{k-1} & z_1^{k+p} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ 1 & z_k & z_k^2 & z_k^{k-1} & z_k^{k+p} \end{vmatrix}.$$

Prove that

$$\left|\frac{B_{kp}}{B_{k0}}\right| \leqq \binom{k+p}{p} r^p.$$

I. *Solution by N. J. Fine, Indianapolis.* B_{k0} is the well known Vandermonde determinant, whose value is $\prod_{i>j}(z_i - z_j)$. Expand B_{kp} by elements of the last column to get

$$B_{kp} = (-1)^k \left\{ z_0^{k+p} \prod_{i>j,\, i,j\neq 0} (z_i - z_j) - z_1^{k+p} \prod_{i>j,\, i,j\neq 1} (z_i - z_j) + \cdots \right\}.$$

Thus

$$\frac{B_{kp}}{B_{k0}} = \frac{z_0^{k+p}}{\prod_{i\neq 0}(z_0 - z_i)} + \frac{z_1^{k+p}}{\prod_{i\neq 1}(z_1 - z_i)} + \cdots + \frac{z_k^{k+p}}{\prod_{i\neq k}(z_k - z_i)}.$$

Let $F(z)=\prod_{i=0}^{k}(z-z_i)$. Then

$$\frac{B_{kp}}{B_{k0}}=\sum_{i=0}^{k}\frac{z_i^{k+p}}{F'(z_i)}=\sum_{i=0}^{k}\left(\text{residue at } z=z_i \text{ of } \frac{z^{k+p}}{F(z)}\right)$$

$$=\frac{1}{2\pi i}\int_{|z|=R>r}\frac{z^{k+p}}{F(z)}dz=\frac{1}{2\pi i}\int z^{p-1}dz\Big/\prod_{i=0}^{k}\left(1-\frac{z_i}{z}\right)$$

$$=\frac{1}{2\pi i}\int z^{p-1}\prod_{i=0}^{k}\left(1+\frac{z_i}{z}+\frac{z_i^2}{z^2}+\cdots\right)dz$$

$$=\sum z_0^{\alpha_0}z_1^{\alpha_1}\cdots z_k^{\alpha_k},$$

where the sum is taken over all sets of non-negative integers α_i such that $\sum\alpha_i=p$. The desired result now follows, since $\binom{k+p}{p}$ is the coefficient of z^p in the expansion of $(1-z)^{-(k+1)}$.

II. *Solution by Irving Kaplansky, Columbia University.* It is well known that $B_{kp}=B_{k0}H_p$, where H_p is the complete homogeneous polynomial of the pth degree in $z_0, \cdots, z_k$. The number of terms in H_p is the number of ways of selecting p objects from $k+1$ with repetititions allowed, which is $\binom{k+p}{p}$. Hence

$$\left|\frac{B_{kp}}{B_{k0}}\right|\leqq\binom{k+p}{p}r^p.$$

Also solved (by induction) by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4171. *Proposed by Tibor Radó, Ohio State University*

Let $\mathbf{x}_n$, $n=0, 1, 2, \cdots$ be a sequence of vectors in euclidean three-space such that $|\mathbf{x}_n|>\delta$ for all n, where δ is a fixed positive constant, and absolute value signs designate the length of the vector involved. Prove that the relation

$$|\mathbf{x}_n|+|\mathbf{x}_0|-|\mathbf{x}_n+\mathbf{x}_0|\to 0 \quad \text{as} \quad n\to\infty$$

holds if and only if there exists a sequence of positive scalars a_n such that $|a_n\mathbf{x}_n - \mathbf{x}_0| \to 0$ as $n \to \infty$.

4172. *Proposed by R. P. Agnew, Cornell University*

Prove or disprove the following statement involving cosine series. If $a_1, a_2, a_3, \cdots$ is a sequence of real constants, and if L_n is the length of the part of the graph of the function

$$f_n(x) = a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$

lying in the interval $-\pi \leqq x \leqq \pi$ then $L_1 \leqq L_2 \leqq L_3 \leqq \cdots$.

4173. *Proposed by Herbert Robbins, U. S. Naval Academy, Annapolis*

Let $a < b$ be given numbers and let $f(t)$ be defined, continuous, non-negative, and strictly increasing for $a \leqq t \leqq b$. By the law of the mean for integrals, for every $p > 0$ there will exist a unique number $a \leqq x_p \leqq b$ such that

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt.$$

Find $\lim_{p\to\infty} x_p$.

4174. *Proposed by Irving Kaplansky, Harvard University*

Stone has called a ring "Boolean" if all its elements satisfy the equation $x^2 = x$. Show that a ring in which $x^2 = \pm x$ is either Boolean or the direct sum of a Boolean ring and the Galois field of three elements.

4175. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The twelve point sphere of any tetrahedron is the locus of the points such that the sum of the squares of their distances to the vertices diminished by their powers with respect to the circumsphere, is equal to a third of the sum of the squares of the edges.

SOLUTIONS

Factorial Coefficients

4122 [1944, 290]. *Proposed by Otto Dunkel, Washington University*

Show that $\Delta^{n-r}0^n/(n-r)!$, where n and r are non-negative integers, $n \geqq r$ is a polynomial $f_r(n)$ in n of degree $2r$. The polynomial $f_r(x)$, $r \geqq 1$, has a positive integral value for all integral values of x, positive or negative, except for $x = 0, 1, 2, \cdots, r-1, r$ for which values it vanishes. If r is an odd integer $\geqq 3$, $r-1$ and r are double roots.

Solution by the Proposer. From 4108 [1945, 281] we have

$$\Delta^{n-r}0^n/(n-r)! = {}_nQ_{n-r} = \sigma_r(-n+r) = \sum_{t=1}^{r} a_t^r n^{(r+t)} = f_r(n),$$

which says that $f_r(n)$ is a polynomial of degree $2r$ in n which has the factor

$n^{(r+1)}=n(n-1)\cdots(n-r)$. If r is odd $\geqq 3$ it has the factor $n^{(r+1)}(n-r)(n-r+1)$, since it was shown in 3940 [1941, 641] that $\sigma_r(n)=n^{(r+1)}P_r(n)$ where $P_r(n)$ has the factor $n(n-1)$ if r is odd $\geqq 3$. From the equality $f_r(n)={}_nQ_{n-r}$ it follows that $f_r(n)$ is a positive integer for $n\geqq r+1$; from the equality $f_r(-n)=\sigma_r(n+r)$, $f_r(-n)$ is a positive integer for $n\geqq 1$, since $\sigma_r(n+r)$ is the rth elementary symmetric function of $1, 2, \cdots, (n+r-1)$. This completes the proof.

Number Theory

4121 [1944, 290]. *Proposed by Alfred Brauer, University of North Carolina*

In generalization of results of Sylvester and Mirimanoff the following theorem was proved, H. F. Baker, *Proc. London Math. Soc.*, (2) vol. 4, 1906, pp. 131–135.

Let d and m be relatively prime positive integers and $mm'\equiv 1$ mod d. Denote by $(m'x)$ the least positive residue of $m'x$ mod d. Then

$$\frac{d^{\phi(m)}-1}{m}\equiv\sum_x\frac{(m'x)}{m-x}\bmod m,$$

where x runs over the positive integers less than m and relatively prime to m.

Recently, P. Kesava Menon proved the following theorems, *Proc. Indian Academy of Science*, Sect. A, vol. 17, 1943, pp. 107–113.

Let d, m, and n be integers such that I. $dn=m-1$; II. $dn=m+1$. Then we have

$$\text{I.}\quad \frac{d^{\phi(m)}-1}{m}\equiv\frac{1}{d}\sum_x\frac{\{x/n\}}{x}\bmod m;$$

$$\text{II.}\quad \frac{d^{\phi(m)}-1}{m}\equiv\frac{1}{d}\sum_x\frac{[x/n]}{x}\bmod m;$$

respectively, where x runs over the positive integers less than m and relatively prime to m, and where $\{a\}$ denotes the smallest integer greater than or equal to a, and $[a]$ denotes the greatest integer less than or equal to a.

Prove that these results are special cases of Baker's theorem.

Solution by the Proposer. I. We have

$$A\equiv\frac{1}{d}\sum_x\frac{\left\{\frac{x}{n}\right\}}{x}\equiv\sum_x\frac{\left\{\frac{x}{n}\right\}}{dx}\pmod m,\qquad 0<x<m,\ (x,m)=1.$$

We denote the smallest negative residue of dx (mod m) by y and set

$$(1)\qquad dx=km-y.$$

Since $m\equiv 1$ (mod d), we have $(d, m)=1$, hence $(dx, m)=(y, m)=1$. If x runs over a reduced system of residues (mod m), then dx and y also run over such a system. It follows that

$$A \equiv \sum_x \frac{\left\{\frac{x}{n}\right\}}{m-y} \pmod{m}. \tag{2}$$

On the other hand, set

$$B \equiv \sum_y \frac{(m'y)}{m-y} \pmod{m}, \qquad 0 < y < m,\ (y, m) = 1.$$

Because $m \equiv 1 \pmod{d}$, we have $m' \equiv 1 \pmod{d}$, hence $(m'y) = (y)$ and

$$B \equiv \sum_y \frac{(y)}{m-y} \pmod{m}. \tag{3}$$

In order to prove that $A \equiv B \pmod{m}$, it is sufficient to show that

$$\left\{\frac{x}{n}\right\} = (y)$$

for every given x less than m and relatively prime to m, where y is given by (1). We set

$$\left\{\frac{x}{n}\right\} = \left\{\frac{dx}{dn}\right\} = \left\{\frac{dx}{m-1}\right\} = l.$$

Then we have

$$l(m-1) \geqq dx > (l-1)(m-1). \tag{4}$$

Since $m-1 \geqq x$, we obtain from (4)

$$d(m-1) \geqq dx > (l-1)(m-1),$$
$$d > l - 1,$$
$$d \geqq l. \tag{5}$$

On the other hand, it follows from (1) that

$$km > dx > (k-1)m. \tag{6}$$

By (4) and (6) we have

$$l(m-1) \geqq dx > (k-1)m,$$
$$(l-k+1)m > l > 0,$$
$$l > k-1,$$
$$l \geqq k. \tag{7}$$

Let us now assume that

$$l \geqq k+1. \tag{8}$$

It follows from (4) that

$$(l-1)(m-1) < dx,$$
$$(l-1)nd < dx,$$

hence by (8), (6), and (5)

$$knd < dx < km = k(nd+1) = knd + k < knd + l \leqq d(kn+1).$$

This is impossible since a multiple of d cannot lie between two consecutive multiples of d. Hence (8) is not possible, and we have $l=k$ by (7).

It follows from (1) that

$$(9) \qquad (y) \equiv y \equiv km \equiv k \pmod{d}$$

since $m \equiv 1 \pmod{d}$. Moreover $(y) \leqq d$, and $k = l \leqq d$ by (5), hence by (9)

$$(y) = k = l = \left\{\frac{x}{n}\right\}.$$

II. We use again (1). Here, instead of (2) and (3), we obtain

$$A \equiv \sum_x \frac{\left[\dfrac{x}{n}\right]}{m-y} \pmod{m}, \qquad 0 < x < m,\ (x, m) = 1;$$

$$B \equiv \sum_y \frac{(m'y)}{m-y} \equiv \sum_y \frac{(dy-y)}{m-y} \pmod{m}, \qquad 0 < y < m,\ (y, m) = 1.$$

If $(a, m) = 1$, then

$$\frac{1}{a} + \frac{1}{m-a} \equiv \frac{m-a+a}{a\,(m-a)} \equiv 0 \pmod{m},$$

hence

$$\sum_y \frac{1}{m-y} \equiv 0 \pmod{m},$$

where y runs over the positive integers less than m and relatively prime to m. Therefore

$$A \equiv \sum_x \frac{\left[\dfrac{x}{n}\right]+1}{m-y} \pmod{m},$$

and it is sufficient to prove that for every x with $0 < x < m$, and $(x, m) = 1$,

$$(dy-y) = \left[\frac{x}{n}\right] + 1 = \left[\frac{dx}{dn}\right] + 1 = \left[\frac{dx}{m+1}\right] + 1.$$

We set $[dx/(m+1)]+1 = l$. We then have

$$(10) \qquad l(m+1) > dx \geqq (l-1)(m+1)$$

and

$$km > dx > (k-1)m. \tag{11}$$

Hence

$$km > (l-1)(m+1) = (l-1)m + l - 1,$$
$$k > l - 1,$$
$$k \geqq l. \tag{12}$$

Let us now assume that $k \geqq l+1$. Since $m+1 > x$, it follows from (10) that $d > l-1$, hence $d \geqq l$. Therefore, by (10) and (11),

$$lnd > dx > (k-1)(nd-1) \geqq l(nd-1) \geqq lnd - d = (ln-1)d.$$

This is also impossible, and we obtain $k = l$ by (12).

On the other hand, it follows from (1) that

$$y \equiv km \equiv -k \pmod{d}$$

since $m \equiv -1 \pmod{d}$. Hence

$$dy - y \equiv -y \equiv k \pmod{d},$$
$$(dy - y) = k$$

since $k \leqq d$. Therefore

$$(dy - y) = k = l = \left[\frac{dx}{m+1}\right] + 1.$$

A review of Kesava Menon's paper appeared in the *Mathematical Reviews*, vol. 5, 1944, p. 34.

Extremal Traces for Matrices

4124 [1944, 352]. *Proposed by T. W. Anderson, Jr., Princeton University*

Consider the set of n by n matrices whose entries are positive integers or zero. Let the sum of the entries of the ith row be r_i, $i = 1, 2, \cdots, n$, and the sum of the entries of the jth column be c_j, $j = 1, 2, \cdots, n$. For specified r_i and c_j, positive or zero integers, with

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j$$

what are the minimum and maximum sums of entries in the main diagonal, *i.e.*, the minimum and maximum traces?

Solution by H. F. Tuan and P. Halmos, Princeton University. For arbitrarily given r_i and c_j (positive or zero integers, with $\sum_{c=1}^{n} r_i = \sum_{j=1}^{n} c_j$), the set of matrices satisfying the given conditions is non-empty. This can be seen easily by applying mathematical induction on the total number of rows and columns. In fact, this process would give the maximum trace at once, for we can construct a matrix with the diagonal elements equal to min (r_i, c_i).

The transformation of adding 1 to the (h, i) entry and 1 to the (j, k) entry and subtracting 1 from the (h, k) entry and 1 from the (j, i) entry leaves the sum of each row and column invariant. This transformation is "permissible" whenever no negative entry is introduced. Using this transformation let us note some properties of a matrix (a_{ij}) with the minimum trace.

First we wish to show that there can be at most one positive entry in the main diagonal. Suppose a_{ii} and a_{jj} $(i \neq j)$ were both non-zero. Then adding 1 to (i, j) and 1 to (j, i) and subtracting 1 from (i, i) and 1 from (j, j) would be permissible. This would reduce the trace by 2 leaving the row and column totals unchanged. Since then (a_{ij}) would not have had the minimum trace, the transformation was not permissible and, hence, only one entry in the main diagonal can be non-zero. If all the entries in the main diagonal are zero, then the minimum trace is zero. Suppose a_{ii} is positive; then a_{jk} $(j, k \neq i)$ is zero. For if a_{jk} were positive, we could subtract 1 from (i, i) and 1 from (j, k) add 1 to (i, k) and 1 to (j, i), thus reducing the trace. However; the trace was minimum, hence $a_{jk}=0$, $a_{ik}=c_k$ $(k \neq j)$, $a_{ji}=r_j$ $(j \neq i)$, and $a_{ii}=r_i-\sum_{j\neq 1}c_j=c_i-\sum_{j\neq 1}r_j$. Hence the minimum trace is the maximum of zero and the n numbers $c_i-\sum_{j\neq i}r_j=r_i-\sum_{j\neq i}c_j$ $(i=1, 2, \cdots, n)$. It is evident that among the n numbers $c_i-\sum_{i\neq j}r_j$ $(i=1, 2, \cdots, n)$ there can be at most one which is positive, and this one, if actually positive, is the minimum trace.

Now let us consider the matrix (a_{ij}) with maximum trace. Consider an element a_{ii}. If there is another non-zero element a_{ik} in the same row and another non-zero element a_{ji} in the same column then we can add 1 to (i, i) and 1 to (j, k), subtract 1 from (i, k) and 1 from (j, i) thus increasing (i, i) and the trace. But the matrix has maximum trace so this is impossible. Hence either the ith row consists entirely of zeros except a_{ii} or the ith column consists entirely of zeros except a_{ii}, and a_{ii} is the minimum of r_i and c_i. The maximum trace, therefore, is $\sum_{i=1}$ min (r_i, c_i).

Solved also by J. B. Kelly.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor C. J. Rees of the University of Delaware has received a citation for his work in a civilian capacity with the 14th Air Force Headquarters.

Dr. H. H. Alden has been appointed to an associate professorship at the University of Wyoming.

Dr. E. E. Blanche has been appointed to the teaching staff of the Army University organized by the War Department for American veterans at Florence, Italy.

Professor L. L. Dines of the Carnegie Institute of Technology has retired with the title professor emeritus.

Assistant Professor H. W. Eves of Syracuse University has been appointed assistant professor and head of the department of mathematics at the College of Puget Sound, Tacoma, Washington.

Assistant Professor C. H. Fischer of the University of Michigan has been promoted to an associate professorship.

Professor H. M. Gehman of the University of Buffalo has been granted leave of absence to teach in the U. S. Army Study Center at Shrivenham, England. Associate Professor Harriet Montague is acting chairman of the department in his absence.

Professor Maria D. Graham of the East Carolina Teachers College, Greenville, North Carolina, has retired.

Assistant Professor Beatrice L. Hagen of Pennsylvania State College has been promoted to an associate professorship.

Dr. Margaret P. Martin of Columbia University has been appointed to an assistant professorship in biostatistics at the University of Minnesota.

Professor L. E. Mehlenbacher of Arizona State Teachers College has been appointed to an associate professorship at the University of Detroit.

Dr. W. K. Morrill of John Hopkins University has been promoted to an assistant professorship.

Professor C. C. Morris of Ohio State University has retired.

Associate Professor C. W. Munshower of Colgate University has been promoted to a professorship.

Assistant Professor E. P. Northrop of the University of Chicago has been promoted to an associate professorship.

Dr. Helen B. Owens of Pennsylvania State College has been promoted to an assistant professorship.

Associate Professor H. H. Pixley of Wayne University has been appointed assistant dean of the College of Liberal Arts.

Assistant Professor K. C. Schraut of the University of Dayton has been promoted to an associate professorship.

The following appointment to an instructorship has been announced:

University of Michigan: Dr. George Piranian.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

MATHEMATICS IN ENGINEERING GRADUATE STUDY

The Society for the Promotion of Engineering Education has a Committee on Graduate Study under the chairmanship of Dean L. E. Grinter of the Illinois Institute of Technology. This Committee was created in 1942, and was asked to study trends in graduate education in engineering. After three years of study, the Committee recently made its report in the form of a manual which analyzes objectives and procedures in engineering graduate study and makes recommendations concerning good practices and suggestions for improvements. The statement contained in this report about the study of mathematics as a part of the graduate curriculum in engineering is quoted.

"Graduate study in engineering emphasizes the mathematical or scientific approach to the solution of technical problems. Analytical studies are far more important in graduate work than informational courses since the art of engineering is to be learned mainly in practice rather than in graduate classes. Mathematics is the primary tool of engineering but undergraduate curricula seldom include required courses in mathematics beyond the integral calculus.

"An ability to use mathematics beyond undergraduate courses in differential and integral calculus is essential for engineering graduate study. A knowledge of differential equations has a particular value in mechanics, the functions of a complex variable are much used in electrical engineering, while an understanding of statistics as applied to the interpretation of engineering data adds to the effectiveness of the investigator. In addition, the rigorous logic of the mathematical approach (premises clearly stated; conditions of sufficiency exactly defined) is excellent training for research workers in all fields.

"It is recommended, therefore, that the master's program in engineering include one or more courses in mathematics beyond integral calculus and that the doctorate in engineering should commonly be strengthened by requiring the student to complete the equivalent of a minor study in mathematics. In some institutions these courses can be taken within the framework of the college of engineering. To meet the mathematics requirement, such courses in advanced mathematics should be taught by competent mathematicians who may also be engineers. If the courses are primarily courses in mechanics, electricity, or thermodynamics, they may be of the greatest value and still not be sufficiently rigorous mathematically to meet the objective discussed. Evidently, courses in engineering analysis should be devoted to the problem of expressing physical conditions through mathematical equations. The solution of the equations is taught in mathematics courses. Either is incomplete without the other."

PREDOCTORAL FELLOWSHIPS IN THE NATURAL SCIENCES

The National Research Council announces that it is now ready to receive nominations and applications for the predoctoral fellowships in the natural (i.e. mathematical, physical, and biological) sciences which it is administering under a grant from the Rockefeller Foundation. These fellowships are intended to assist young men and women, whose graduate study has been prevented or interrupted by the war, to complete their work for the doctorate. It is hoped that these fellowships will do much to accelerate the recovery of the scientific vigor and competence of the country which is so seriously threatened by the loss of almost two graduate school generations of scientifically trained men and women.

This program will be administered by a Committee on Predoctoral Fellowships of the National Research Council whose members are Henry A. Barton, Charles W. Bray, Detlev W. Bronk, Luther P. Eisenhart, Ross G. Harrison (Chairman—National Research Council, ex officio), W. A. Noyes, Jr., and John T. Tate, chairman: Enid Hannaford, secretary.

The annual stipend will be $1200 for single persons and $1800 for married men. In general it is expected that each recipient will spend at least eleven months per year on academic work. An additional allowance up to $500 per year will be made for tuition fees. Fellowships granted to individuals who are eligible for educational support from the "G. I. Bill of Rights" will be at such stipends as to bring the total income from these two sources to that which would be received at the above rates.

Each fellow, before entering on his graduate studies, will submit for review by the Committee on Predoctoral Fellowships a schedule, approved by the dean of his graduate school, for the completion of his work for the doctorate. This schedule, as approved by the committee, will constitute an informal agreement upon the basis of which stipend payments will be made. At the discretion of the university concerned the fellowship stipend may be supplemented by university grants. All such supplementary sources of income should be made a matter of record with the committee. The progress of the fellows will be subject to periodic review by the committee which reserves the right to cancel fellowships when in their judgment satisfactory progress is not being maintained.

Prospective candidates for these fellowships are urged to apply at once even though they may be unable to undertake their graduate study in the immediate future. Information concerning these fellowships and Nomination-Application blanks are being mailed out widely to graduate schools and wartime research laboratories. They may also be obtained by writing directly to the Secretary, Committee on Predoctoral Fellowships, National Research Council, 2101 Constitution Avenue N. W., Washington 25, D. C.

NEW NROTC UNITS

The Navy has authorized the establishment of 25 new NROTC units in addition to the 27 units already located at colleges and universities. Some V-12 students with technical interests were transferred to institutions having the new units before July 1, and additional trainees will be transferred before November 1. Essentially the traditional ROTC program will be inaugurated in all the newly selected institutions about November 1, 1945.

The new units will be at Dartmouth College, Columbia University, University of Rochester, Villanova College, Princeton University, Cornell University, Pennsylvania State College, Case School of Applied Science, Miami University, Illinois Institute of Technology, Iowa State College of Agriculture and Mechanic Arts, University of Mississippi, University of Wisconsin, University of Kansas, University of Nebraska, Alabama Polytechnic Institute, Vanderbilt University, University of Idaho, Oregon State College, Purdue University, University of Illinois, University of Missouri, University of Louisville, University of Utah, Stanford University.

POSTWAR EDUCATIONAL SERVICES FOR VETERANS AND SERVICE PERSONNEL

After signing the armistice in 1918, the Army was unprepared to carry out an educational and recreational program for its personnel before embarkation for home and during the occupation of the Rhineland. Finally, during January, 1919, a plan for an educational program was worked out by military and civilian authorities. This program was designed to be given in post schools, divisional education centers, civilian universities (for qualified graduate students), and in the AEF Educational Center located at Beaune, France. Included in the organization of the Center was the well known AEF University.

The AEF University was created to serve the needs of military personnel who desired undergraduate work of college grade and to give study opportunities to graduates for whom there were inadequate accommodations at civilian universities. It was opened March 17, 1919, and continued for only two and a half months. During that time the total registration was about 13,000 men. The faculty of the AEF University comprised military personnel with previous experience as educators and civilian instructors who were recruited into the Army Educational Corps. Students were required to have the equivalent of four years of high school before they could enter the AEF University; otherwise their needs were to be served by the post and division schools.

Keeping in mind the lessons of the last war, the Army worked for over two years on educational plans for its personnel stationed in inactive theaters of operation. These plans were completed well before VE day, and an extensive educational program is now under way, especially in the European and Mediterranean theaters. Qualified professional personnel in the Army and some civilians make up the staff for the program. Two large Army university study centers have been opened, one in England near Oxford and another in France near Paris; a number of civilian staff members for these institutions were recruited from colleges and universities in this country shortly after VE day. Educational

opportunities now generally available to military personnel do not differ greatly from those to be found in large American communities. The institutional program includes literacy training, elementary, high school, technical school, and college courses, with considerable emphasis on vocational training. Advisory and vocational guidance services are provided. Included in the plans are furloughs and field trips for educational purposes to places of cultural and historical interest. Equipment of the Army technical services and foreign civilian educational facilities are being utilized. Large quantities of classroom supplies were shipped abroad months before the end of the war.

Four types of schools have been created for participants in the educational program, namely, the unit school, the technical school, civilian universities, and Army university study centers. The unit schools have been established whereever there are separate units of 1,000 men or less. Students in these centers receive general education including literacy training, elementary, high school and junior college courses. Where equipment and instructors are available, vocational courses are offered. Technical schools have been organized where it is possible to use Army Technical Service facilities and other special equipment. These schools offer highly specialized vocational courses of a kind not generally available in the unit schools. Students are selected for training in the technical schools on a quota basis from the units taking part in the program. Civilian university centers abroad are being used and the two Army university study centers have been established to train personnel whose needs and interests go beyond the level of the unit school and the technical school. Courses are being provided in the arts and sciences as well as in the professional fields.

In addition to the institutional program, the Armed Forces Institute continues to serve specialized and individual needs. Courses available in mathematics under the auspices of the USAFI were listed in the MONTHLY for May, 1945. Since the compilation of that list, the USAFI has announced the availability to service personnel of the text, *EM* 327, *An Introduction to Statistical Analysis*, by C. H. Richardson.

In anticipation of demobilization, the Army has established classification and counseling sections in hospitals and separation centers throughout the United States. Here the returning service men and women are given an opportunity to discuss their problems of readjustment to civilian life. Qualified military personnel have been carefully selected and thoroughly trained in counseling, and have access to a large amount of information gathered through cooperative arrangements with governmental agencies, business, industry, labor organizations, *etc.* In order that the counselors may be well informed as to special offerings in the field of higher education, colleges and universities are invited to send to separation centers and hospitals *any printed publications* regarding special offerings for service men and women. Information concerning the number of copies required for distribution to service centers and hospitals should be obtained by contacting each Service Command Headquarters. Requests for information should be addressed to the Commanding General, Attn: Officer in

Charge Separation Classification and Counseling, of the nearest Service Command, addresses of which are as follows: Headquarters, First Service Command, Army Base, Boston, Massachusetts; Second Service Command, Governors Island, New York; Third Service Command, U. S. Post Office Building, Baltimore, Maryland; Fourth Service Command, U. S. Post Office Building, Atlanta, Georgia; Fifth Service Command, Fort Hayes, Columbus, Ohio; Sixth Service Command, Civic Opera Building, 20 North Wacker Drive, Chicago, Illinois; Seventh Service Command, Federal Building, Omaha, Nebraska; Eighth Service Command, Santa Fe Building, Dallas, Texas; and the Ninth Service Command, Fort Douglas, Utah.

Veterans' Guidance Centers have been established in 50 educational institutions throughout the country, under agreements entered into by the institutions and the Veterans' Administration. Additional agreements are rapidly being made with other institutions, and it is expected that the total number of centers will reach several hundred within a few months. The purpose of these centers is to give advice and guidance to veterans who have suffered service-connected disabilities that are pensionable and constitute vocational handicaps. Disabled veterans who are to receive vocational rehabilitation under Public Law 16, 78th Congress, are sent to these centers for advice in selecting the courses they will undertake. While there, teachers, vocational experts, psychologists, and doctors interview the veterans and give them tests to determine the type of activity they should undertake in the hope of achieving complete rehabilitation. Veterans who undertake educational courses under the G. I. Bill are not required to accept guidance or direction in selecting their courses. The services of the experts in these centers are, however, also available to veterans undertaking education under this act. The Veterans' Administration urges that those planning to return to school under the G. I. Bill take advantage of this opportunity, so that they may be assured of getting the greatest benefit from their education.

In the U. S. Navy, a program of academic, vocational and orientation education and training is well established under the Bureau of Naval Personnel. This program, known as Educational Services, has two major functions, namely, to disseminate information on the background, progress, and outcomes of the war, and to provide opportunities to continue education while in the service. This program, reaching a large percentage of Naval personnel, has grown steadily and consistently during the two and one-half years of its existence.

During the present post-war period, Educational Services provides information aimed particularly at helping the serviceman to adapt himself to his present assignment and to the post-war world. Moreover, various types of study have been made available to Navy personnel. Hundreds of thousands of men have already enrolled in classes. Subjects being taught range all the way from instruction in the three R's for non-readers and non-writers to college subjects. Use is being made of many of the Navy shops for instruction in shop and prevocational subjects. Use of USAFI correspondence courses and texts for individ-

ual study continues for personnel who are on small ships or stations or who are pursuing subjects of rather special interest.

The Demobilization Division of the Bureau of Naval Personnel is serving as coordinator for all programs that relate to the demobilization and civil readjustment of Naval personnel. The Civil Readjustment Section of the Demobilization Division has the responsibility of coordinating all programs that are concerned with the readjustment of Naval personnel to civilian life. In addition to conducting the final interview before demobilization, the Civil Readjustmen Program, through its District Civil Readjustment Officers, maintains a follow-up service to all veterans of the Navy.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE MAY MEETING OF THE WISCONSIN SECTION

The thirteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the State Teachers College, Milwaukee, Wisconsin, on Saturday, May 5, 1945. Sessions were held in the morning and in the afternoon. Sister Mary Felice, the Chairman of the Program Committee, presided at the morning session. The afternoon session was a joint meeting with the Milwaukee Mathematics Club and the Mathematics Section of the Wisconsin Educational Association. Mr. Lester Garbe, President of the Milwaukee Mathematics Club, presided at the afternoon session.

There were forty-five in attendance, including the following twenty-two members of the Association: R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, Sister M. Mirabella Boehmer, F. A. Butter, Jr., K. L. Clark, H. P. Evans, E. G. Harrell, W. W. Hart, R. C. Huffer, J. F. Kenney, Lionel London, C. C. MacDuffee, Sister Mary Felice, R. E. Norris, Elli Otteson, H. P. Pettit, P. L. Trump, B. R. Ullsvik, J. I. Vass, Louise A. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, Sister Mary Felice, Mount Mary College; Program Committee, R. C. Huffer, Beloit College, H. P. Pettit, Marquette University, B. R. Ullsvik, State Teachers College, Eau Claire. It was voted that the officers of the Wisconsin Section be authorized to make whatever arrangements seem advisable with other state groups interested in the advancement of mathematical education. It was voted that the next meeting be held in May 1946, at Mount Mary College in Milwaukee, the exact date to be set by that institution.

The following papers were presented on the morning program:

1. *A construction aid for conic sections*, by Professor H. P. Pettit, Marquette University.

It was pointed out that the construction of points on a conic by means of an auxiliary line through the intersection of the axis and the directrix, and making an angle arc sin (e) with the axis, is well known. This auxiliary line is tangent

to the conic at an end of the latus rectum. The speaker explained that for a central conic the foci can be constructed easily when the standard form is known. He then took point R on the minor axis such that $OR = OF = ae$, O being the center, and determined point S on OR, with $OS = OA = a$. He explained that the line SK parallel to RA meets OA in K on the directrix, and that the line SK is the auxiliary line. Thus the tangent at the vertex, the auxiliary line, and the tangent at the end of a minor diameter (or an asymptote in case of the hyperbola) were seen to furnish guide lines for a reasonably accurate sketch of the conic.

2. *On Kiepert's configuration*, by Dr. E. G. Harrell, State Teachers College, Platteville.

The speaker employed trilinear coördinates to develop various properties of Kiepert's configuration. Certain envelopes and other loci were also discussed.

3. *A differential notation for logarithms*, by R. S. Hoar, introduced by Professor Marden.

The logarithm of a to base b was denoted by $\mathrm{l}a/\mathrm{l}b$. The symbols $\mathrm{l}a$ and $\mathrm{l}b$ were called "logarentials." It was suggested that the new notation affords a useful mnemonic for converting logarithms from one base to another. It was remarked that various alternative meanings might be given to a detached logarential, one of which serves as a mnemonic for differentiating logarithms. A suggestion was made on the possible usefulness of the corresponding integral notation for anti-logarithms.

4. *Irrational numbers*, by Professor C. J. Everett, University of Wisconsin.

The speaker surveyed the principal results in the theory of irrationals, including the little known but important work of W. Maier and O. Ore. He advanced the opinion that available information now seems to warrant a modern attempt at a general theory of power series with rational coefficients. Several specific problems basic to such a project were cited.

The afternoon session was devoted to a discussion of post-war mathematics in the high school. Discussion leaders presented the papers listed below.

Professor C. C. MacDuffee presented a paper which is to be printed in a subsequent issue of this MONTHLY. Mr. Frank E. Baker, President of Milwaukee State Teachers College, discussed the philosophical, psychological, and practical values of mathematics. Mr. R. G. Chamberlin, Principal of Rufus King High School in Milwaukee, dwelt upon the concern of secondary school teachers and administrators over the question as to what position mathematics will occupy in the post-war period. This speaker itemized numerous factors which in his opinion have retarded the acceptance of mathematics.

Mr. Arpad E. Elo, research physicist for the Perfex Corporation in Milwaukee, presented industry's arguments in favor of a strong mathematics program in the high schools. He stated that the revolutionary methods of production created by the war emergency will be expanded in the future. This will demand

a higher level of technical proficiency in both skilled and unskilled workers. Our war-depleted ranks of scientific and technical personnel must be replenished from the group now in or entering high school. The opinion was advanced that "hand book engineers" suffer a great disadvantage in competition with individuals who thoroughly understand and appreciate analytical methods.

An active and valuable discussion followed the remarks of these speakers.

P. L. TRUMP, *Secretary*

THE SPRING MEETING OF THE MARYLAND, DISTRICT OF COLUMBIA, VIRGINIA SECTION

The spring meeting of the Maryland, District of Columbia, Virginia Section of the Mathematical Association of America was held at George Washington University on Saturday, May 12, 1945. Professor C. H. Wheeler III, Chairman of the Section, presided at the morning and afternoon sessions.

There were forty-four persons present, including the following twenty-five members of the Association: G. R. Clements, Abraham Cohen, E. L. Crow, Alexander Dillingham, J. A. Duerksen, P. J. Federico, Michael Goldberg, D. W. Hall, M. A. Hyman, F. E. Johnston, L. M. Kells, M. H. Martin, Florence M. Mears, T. W. Moore, W. K. Morrill, R. E. Root, E. D. Schell, M. F. Smiley, C. V. L. Smith, A. D. Sollins, J. H. Taylor, C. C. Torrance, F. J. Weyl, C. H. Wheeler III, R. H. Wilson, Jr.

At the business meeting the following officers were elected for the coming year: Chairman, J. B. Scarborough, United States Naval Academy; Secretary-Treasurer, E. J. Finan, Catholic University of America; Members of the Executive Committee, W. K. Morrill, Johns Hopkins University, D. W. Hall, University of Maryland; Regional Governor, Gillie A. Larew, Randolph-Macon Woman's College.

The morning program consisted of the following papers:

1. *Equations of motion of classical dynamical systems of variable mass*, by Lieut. Joseph Giarratana, U. S. Naval Academy.

The speaker derived a set of general equations of motion for classical dynamical systems of variable mass. The variation of mass considered was due to either a continuous deformation and motion of the defining surface, or to a motion of the material points of the system, or to both. Special cases of the general equations were also considered.

2. *Stresses in rectangular plates*, by Lieut. C. B. Lindquist, U. S. Naval Academy.

The problem considered was that of an isotropic rectangular plate stressed in its own plane uniformly by a tensile stress on one edge, and held in static equilibrium by uniform shear forces on the two adjacent edges. The stress function F (from which the stresses were obtained by differentiation) was expressed in terms of an infinite series constructed to satisfy the biharmonic equation

$\nabla^4 F=0$ and the boundary conditions on F and its normal derivatives. The conditions of the normal derivatives lead to an infinite system of linear equations in infinitely many unknowns, the unknowns being the undetermined constants appearing as coefficients in the series for the stress function F. This system of equations was shown to have a unique, bounded solution. A similar method was used to solve the problem of material having two directions of elastic symmetry parallel to the edges of the plate.

The afternoon speaker was Professor Tobias Dantzig of the University of Maryland, who gave an interesting talk on *The Bequest of the Greeks*.

W. K. MORRILL, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Hamline University in St. Paul, Minnesota, on Saturday, May 12, 1945. Sessions were held in the forenoon, at noon, at luncheon, and in the afternoon. Professors K. H. Bracewell, E. J. Camp, Dean W. H. Bussey, and Professor L. E. Bush presided at the respective sessions.

Thirty-five persons attended the meetings, including the following nineteen members of the Association: Walter Bartky, R. W. Brink, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, S. Elizabeth Carlson, Brother Louis De La Salle, I. C. Fischer, Gladys Gibbens, W. L. Hart, C. M. Jensen, J. M. H. Olmstead, Abraham Spitzbart, F. J. Taylor, Marian W. Thornton, Ella Thorp, K. W. Wegner, G. L. Winkelmann.

The following officers were elected for the coming year: Chairman, C. S. Carlson, St. Olaf College; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, J. M. H. Olmsted, University of Minnesota, K. W. Wegner, Carleton College, Brother Louis De La Salle, Saint Mary's College.

On a motion by Professor Brink, a résumé of the career of Professor Anthony L. Underhill of the University of Minnesota, who died on January 18, 1945, was entered on the minutes of the meeting.

At the noon session Dean Walter Bartky of the University of Chicago delivered a lecture on *Linear Systems of Differential Equations*. Dean Bartky applied matrices to the solution of the system

$$\frac{dx_i}{dt} = \sum_{=1}^{n} a_j x_j, \qquad (i = 1, 2, \cdots, n)$$

and showed how the labor of solving certain systems of this type which arise in industry could be greatly reduced.

In addition to the lecture by Dean Bartky, the following papers were presented:

1. *On Stieltjes' integral equations*, by Mr. Monroe D. Donsker, University of Minnesota.

The speaker reported on a paper appearing in the *Annals of Mathematics* in 1922 concerning the application of the classical Fredholm theory of integral equations to Stieltjes' integral equations of the type

$$u(x) = f(x) + \lambda \int_a^b u(y) d_y K(x, y).$$

2. *Hyperbolic inversion*, by Miss Esther Reizman, University of Minnesota, introduced by Dean Bussey.

The inverses of a point and of a curve with respect to a circle were defined in the usual way, and then defined again in terms of poles and polars. Generalized inversion with respect to any central conic was then defined on the basis of this second definition of circular inversion. Inversion with respect to the hyperbola $xy=1$ received special attention. Some facts about hyperbolic inverses of the general conic were presented. Some special cases and illustrations were described.

3. *An application of the theory of envelopes to a problem in light*, by Professor E. J. Camp, Macalester College.

It was the speaker's purpose to call attention to a class of problems from elementary physics which are applications of the theory of envelopes, but which are usually ignored in texts on differential equations.

Consider a family of parallel plane waves given by the equation $y=kx+b_n(t)$. Suppose that the y-axis is the cross section of a reflecting mirror, and that α_1 the inclination of the lines, is an obtuse angle. Then $\alpha_1-90°$ is the angle of incidence. Points where the incident waves strike the y-axis are to be regarded as centers of circular waves whose equations can be written in the form

$$x^2 + (y - \beta)^2 = \frac{[b_n(t) - \beta]^2}{k^2 + 1}\cdot$$

The envelope of this family of circles for a fixed t and a fixed n is a straight line with equation $y=-kx+b_n(t)$. Upon comparing this equation with the first equation in this paragraph, it is seen that the angle of incidence is equal to the angle of reflection. Other problems which yield to a similar analytic treatment can be obtained by considering incident waves with circular wave fronts.

4. *Matrices and quadric surfaces*, by Professor J. M. H. Olmsted, University of Minnesota.

Invariants of real quadratic equations in three variables are easily obtained by the use of matrices for distance preserving transformations, by means of the artifice of a fourth variable equal to 1. Professor Olmsted proved that although the characteristic roots of the fourth order matrix of a real quadratic form in three variables are not invariant, their signs are. This fact extends to higher dimensional spaces, and leads to a general characterization theorem for quadric hypersurfaces, involving the characteristic roots of the associated matrices of orders n and $n+1$.

5. *Analyzing objectives in the teaching of mathematics*, by Brother Louis De La Salle, Saint Mary's College.

The speaker discussed the advantages for the instructor and for the student of having a very explicit statement of objectives. He advocated a statement specific enough to enable the individual student to recognize his attainment. Excerpts from such a statement were distributed by way of illustration.

6. *Bernard Bolzano*, by Sister M. Thomas a Kempis, College of Saint Teresa.

In anticipation of the centenary of the death of Bernard Bolzano in 1848, the speaker presented a short biography which stressed his success in freeing mathematical analysis from intuitional treatment.

7. *The ancient quadratrix from a modern point of view*, by Dean W. H. Bussey, University of Minnesota.

The quadratrix was invented by Hippias (c. 425 B.C.) for the purpose of angle trisection, and was later used by Deinostratus (c. 350 B.C.) for the quadrature of the circle. Its equation may be written $x = y \cot (\pi y/2)$. The small part of the curve which is described in books on the history of elementary mathematics is a curved line from the point (0, 1) to (1/2, 1/2), and on to meet the x-axis at $(2/\pi, 0)$. When this curve and its equation were explained to a class in analytic geometry, one of the students inquired regarding the appearance of the rest of the curve. Dean Bussey gave a short summary of what he learned about the quadratrix in answering that question.

L. E. BUSH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-ninth Annual Meeting, Chicago, Ill., November 24–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

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NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the SECRETARY-TREASURER, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 10th of the month in which the change becomes effective.

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ELEMENTARY EVALUATION OF LAPLACE TRANSFORMS

D. E. RICHMOND, Williams College and Institute for Advanced Study

1. Introduction and evaluation of simple transforms. It is well known that Laplace transforms are increasingly applied to the solution of engineering problems, notably in the study of transients in filters and transmission lines. From a pedagogical point of view it is therefore of interest to observe that it is possible to build up a very extensive table of transforms by means within the reach of students familiar with the solution of elementary differential equations. It is even possible to go a considerable distance without performing a single integration. It is believed that the presentation of these methods may prove useful to teachers of undergraduate courses in mathematics.

A function $f(s)$ is said to be the Laplace transform of $F(t)$ if $f(s)=\int_0^\infty e^{-st}F(t)dt$. Hereafter this relationship between $f(s)$ and $F(t)$ will be abbreviated $f(s)=L\{F(t)\}$. The following theorems are well known and easily derived under suitable restrictions on $F(t)$.

If $f(s)=L\{F(t)\}$,

I. $sf(s)-F(0)=L\{F'(t)\}$,

II. $f'(s)=L\{-tF(t)\}$,

III. $f(s-a)=L\{e^{at}F(t)\}$.

Moreover, if $f_1(s)=L\{F_1(t)\}$ and $f_2(s)=L\{F_2(t)\}$,

IV. $c_1f_1(s)+c_2f_2(s)=L\{c_1F_1(t)+c_2F_2(t)\}$, where c_1 and c_2 are arbitrary constants. In particular, it follows that $0=L\{0\}$.

It is proposed in the following to make almost exclusive use of I–IV. In only one case will it be necessary to appeal to the integral representation. The student should be warned, however, that I and II fail to apply in the case of certain functions $F(t)$. He is perfectly safe if $F'(t)$ is continuous for $t>0$, $tF(t)\to 0$ as $t\to 0$ and $|F'(t)|<ce^{\alpha t}$ for some c and α, and for all t sufficiently large.

In the present section we shall develop some of the more elementary transforms from I–IV. We shall first find the Laplace transform of 1. If $x(s)=L\{1\}$, then by I, $sx(s)-1=L\{0\}=0$. Hence $x(s)=1/s$.

$$\frac{1}{s}=L\{1\}. \tag{1}$$

We have tacitly assumed that a given function of t, in this case 0, has only one transform, which of course is true.

In similar fashion, $1/s^2=L\{t\}$, and by induction

$$\frac{1}{s^{n+1}}=L\left\{\frac{t^n}{n!}\right\} \quad \text{for } n \text{ a positive integer.} \tag{2}$$

$$\frac{1}{s-a}=L\{e^{at}\}, \quad \text{using (1) and III.} \tag{3}$$

Similarly, if

$$x(s) = L\{\sin at\},$$
$$sx(s) = L\{a \cos at\},$$
$$s^2x(s) - a = L\{- a^2 \sin at\},$$

Hence

$$s^2x(s) - a = - a^2x(s), \qquad x(s) = \frac{a}{s^2 + a^2}$$

(4) $$\frac{a}{s^2 + a^2} = L\{\sin at\}, \quad \text{and by } I,$$

(5) $$\frac{s}{s^2 + a^2} = L\{\cos at\}.$$

From these results, one obtains by III,

(6) $$\frac{1}{(s - a)^{n+1}} = L\left\{\frac{t^n e^{at}}{n!}\right\},$$

(7) $$\frac{a}{(s - b)^2 + a^2} = L\{e^{bt} \sin at\},$$

(8) $$\frac{s - b}{(s - b)^2 + a^2} = L\{e^{bt} \cos at\};$$

and by II,

(9) $$\frac{2as}{(s^2 + a^2)^2} = L\{t \sin at\},$$

(10) $$\frac{s^2 - a^2}{(s^2 + a^2)^2} = L\{t \cos at\}.$$

The only explicit use of the integral representation which we shall need occurs in the case of the transform of t^α ($\alpha > -1$). We obtain $L\{t^\alpha\} = \Gamma(\alpha+1)/s^{\alpha+1}$ by substituting $x = st$ in $\int_0^\infty e^{-st}t^\alpha dt$. It will be useful to recall that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$. The method of this section is similar to the treatment of Churchill.*

2. More difficult transforms. To develop the less elementary transforms it is useful to prepare the following table† by successive use of I and II:

* Churchill, R. V. Modern Operational Mathematics in Engineering, McGraw-Hill, 1944.

† Equivalent to that given by van der Pol, B., On the operational solution of linear differential equations and an investigation of the properties of these solutions, Philosophical Magazine (7), vol. 8, 1929, p. 877. Except for the use of the asymptotic theorems, our method is the same as his.

$f(s)$	$F(t)$
x	X
$sx - X_0$	X'
$s^2x - sX_0 - X_1$	X''
$-x'$	tX
$-sx' - x$	tX'
$-s^2x' - 2sx + X_0$	tX''
x''	t^2X
$sx'' + 2x'$	t^2X'
$s^2x'' + 4sx' + 2x$	t^2X''

$X_0 = X(0)$

$X_1 = X'(0)$

Sufficient conditions for the validity of any entry are readily stated in consequence of those for Theorems I and II.

We shall also need the following asymptotic theorems:*

V. If $X(t) \sim At^\alpha$ $(\alpha > -1)$ as $t \to 0$,

$$x(s) \sim \frac{A\Gamma(\alpha + 1)}{s^{\alpha+1}} \quad \text{as} \quad s \to \infty.$$

VI. If $X(t) \sim Bt^\beta$ $(\beta > -1)$ as $t \to \infty$,

$$x(s) \sim \frac{B\Gamma(\beta + 1)}{s^{\beta+1}} \quad \text{as} \quad s \to 0.$$

By way of explanation of the notation,

$$F(t) \sim G(t) \text{ is equivalent to } \frac{F(t)}{G(t)} \to 1.$$

Example 1. To find the transform of $J_0(t)$. $J_0(t)$ is a solution of $tX'' + X' + tX = 0$. Using the table, we get at once the transformed equation:

$$s^2x' + x' + sx = 0.$$

By a simple integration, $x(s) = c/\sqrt{s^2+1}$. To determine c, note that $X(t) \sim 1$ as $t \to 0$. Hence $x(s) \sim 1/s$ as $s \to \infty$. But by inspection $x(s) \sim c/s$. Hence $c = 1$ and

$$\frac{1}{\sqrt{s^2 + 1}} = L\{J_0(t)\}. \tag{11}$$

Using $J_0'(t) = -J_1(t)$ and I,

* Doetsch, G. Theorie und Anwendung der Laplace-Transformation, Dover, 1943, pp. 188, 200.

$$\frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}} = L\{J_1(t)\}. \tag{12}$$

Example 2. Let $X(t)=\sin a\sqrt{t}$. By differentiating twice and eliminating, one readily obtains

$$4tX'' + 2X' + a^2X = 0.$$

Using $X_0=0$ and the table,

$$4s^2x' + (6s - a^2)x = 0, \quad \text{which gives}$$

$$x(s) = \frac{c}{s^{3/2}}e^{-a^2/4s}.$$

Now $X(t)\sim at^{1/2}$ as $t\to 0$. Hence $x(s)\sim a\sqrt{\pi}/2s^{3/2}$ as $s\to\infty$. By inspection $x(s)\sim c/s^{3/2}$, hence $c=a\sqrt{\pi}/2$ and

$$\frac{a\sqrt{\pi}}{2s^{3/2}}e^{-a^2/4s} = L\{\sin a\sqrt{t}\}, \quad \text{and by I,} \tag{13}$$

$$\frac{\sqrt{\pi}}{s^{1/2}}e^{-a^2/4s} = L\left\{\frac{\cos a\sqrt{t}}{\sqrt{t}}\right\}. \tag{14}$$

Example 3. $X(t)=J_0(a\sqrt{t})$ satisfies

$$4tX'' + 4X' + a^2X = 0.$$

The transformed equation is

$$4s^2x' + (4s - a^2)x = 0, \quad \text{which gives at once}$$

$$x(s) = \frac{ce^{-a^2/4s}}{s}.$$

Now $X(t)\sim 1$ as $t\to 0$, and $x(s)\sim 1/s$ as $s\to\infty$. Thus $c=1$ and

$$\frac{e^{-a^2/4s}}{s} = L\{J_0(a\sqrt{t})\}. \tag{15}$$

3. Another method. It is often easier to reverse this procedure, seeking the $X(t)$ which corresponds to a given $x(s)$. It will be assumed that $x(s)$ satisfies a linear differential equation with polynomial coefficients. The method consists in making such a linear combination of entries from the left side of the following table as to give this differential equation. The same linear combination of the right-hand entries gives the associated differential equation for $X(t)$, which may be found by integration and application of the asymptotic theorems.

$f(s)$	$F(t)$
x	X
x'	$-tX$
x''	t^2X
$sx - X_0$	X'
sx'	$-tX' - X$
sx''	$t^2X' + 2tX$
$s^2x - sX_0 - X_1$	X''
$s^2x' + X_0$	$-tX'' - 2X'$
s^2x''	$t^2X'' + 4tX' + 2X$

If it is necessary to use lines 4, 7, or 8, involving X_0 (and X_1), these quantities should be calculated by use of the known relations

$$X_0 = \lim_{s\to\infty} sx(s) \quad \text{and} \quad X_1 = \lim_{s\to\infty} [s^2x(s) - sX_0].$$

If line 7 is to be used, X_0 and X_1 must be 0. If line 4 or 8 occurs separately, X_0 must be 0; if both are used, sx and s^2x' must have equal coefficients. In case these conditions are not satisfied, simple modifications of the method may be made. (See example 6.)

Inverse transforms found by this method should be checked by the method of §2 to assure the validity of the results.

Example 4. Consider $x(s) = e^{-a\sqrt{s}}/\sqrt{s}$. By differentiation and elimination,

$$4sx'' + 6x' - a^2x = 0.$$

From the table,

$$4t^2X' + 2tX - a^2X = 0$$

and $X(t) = c/\sqrt{t}\, e^{-a^2/4t}$. Now $X(t) \sim c/\sqrt{t}$ as $t \to \infty$. Hence by VI,

$$x(s) \sim \frac{c\Gamma(1/2)}{\sqrt{s}} = \frac{c\sqrt{\pi}}{\sqrt{s}} \quad \text{as} \quad s \to 0.$$

By inspection, $x(s) \sim 1/\sqrt{s}$ so $c = 1/\sqrt{\pi}$ and

$$\frac{e^{-a\sqrt{s}}}{\sqrt{s}} = L\left\{\frac{1}{\sqrt{\pi t}} e^{-a^2/4t}\right\}. \tag{16}$$

The functions of t are such as to justify the argument by reversing the steps. The order chosen makes it necessary to solve only the simpler of the associated differential equations.

Example 5.
$$x(s) = \frac{(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}}$$
is suggested by (12). One easily obtains
$$(s^2 + 1)x'' + 3sx' + (1 - n^2)x = 0.$$
From the table,
$$t^2X'' + tX' + (t^2 - n^2)X = 0,$$
Bessel's equation of nth order.
$$\text{Now} \quad x(s) = \frac{1}{\sqrt{s^2+1}(\sqrt{s^2+1} + s)^n} \sim \frac{1}{2^n s^{n+1}} \quad \text{as} \quad s \to \infty.$$
We should choose $X(t)$ such that $X(t) \sim t^n/2^n n!$ as $t \to 0$. Thus $X(t) = J_n(t)$.

Example 6. $x(s) = e^{-1/s}/s^{1/2}$. Then $2s^2x' + (s-2)x = 0$. To get the equation for $X(t)$, we shall need to use lines 4 and 8 of the table, involving X_0. But $X_0 = \lim_{s\to\infty} s^{1/2}e^{-1/s} = \infty$, and the method fails. We use instead of $x(s)$, the function $e^{-1/s}/s^{3/2}$. The method will then work, giving
$$\frac{e^{-1/s}}{s^{3/2}} = L\left\{\frac{\sin 2\sqrt{t}}{\sqrt{\pi}}\right\} \quad cf.\ (13).$$
The answer to the original problem is now easily found as in (14).

It will be found that almost all the entries in tables of transforms in such standard texts as Churchill and Doetsch may be found by the use of this method or that of §2.

4. An important example.* A more difficult problem is to solve for $X(t)$
$$x(s) = \frac{e^{s-\sqrt{s^2+1}}}{\sqrt{s^2+1}} = L\{X(t)\}.$$
It is easy to show that $X_0 = 1$.

By repeated differentiation and a fairly difficult elimination, one obtains
$$(s^2 + 1)x''' - (3s^2 - 5s + 3)x'' + (2s^2 - 10s + 7)x' + (2s - 5)x = 0.$$
Adding to the table of §3 the entries
$$x''' = L\{-t^3X\} \quad \text{and} \quad s^2x''' = L\{-t^3X'' - 6t^2X' - 6tX\},$$
one gets the equation
$$(t + 1)(t^2 + 2t)X'' + (t^2 + 2t + 2)X' + (t + 1)^3X = 0. \tag{17}$$
Setting $u = t^2 + 2t$,

* A considerable simplification of this section was made as the result of a suggestion by the referee.

$$4u\frac{d^2X}{du^2} + 4\frac{dX}{du} + X = 0.$$

Comparing with Example 3, it will be seen that $J_0(\sqrt{u}) = J_0(\sqrt{t^2+2t})$ is the solution of (17) for which $X_0 = 1$. Consequently

$$\frac{e^{s-\sqrt{s^2+1}}}{\sqrt{s^2+1}} = L\{J_0(\sqrt{t^2+2t})\}. \tag{18}$$

A related result is widely used in the theory of transmission lines:

$$x(s) = \frac{e^{-\sqrt{s^2+1}}}{\sqrt{s^2+1}} = L\left\{\begin{matrix} 0, \ t < 1; \\ J_0(\sqrt{t^2-1}), \ t \geqq 1 \end{matrix}\right\}. \tag{19}$$

This result follows from (18) by the use of a well-known shift theorem, according to which

$$e^{-as}f(s) = L\left\{\begin{matrix} 0, t < a: \\ F(t-a), \ t \geqq a \end{matrix}\right\}$$

if $f(s) = L\{F(t)\}$. If our method is applied directly to (19), it is found that $x(s) \sim e^{-s}/s$ as $s \to \infty$. The asymptotic theorems do not apply immediately, but the result obtained suggests multiplication of $x(s)$ by e^s to bring the problem within the scope of our method. A similar device is applicable to a number of other known cases.

It is possible that the method of §3 may lead to the discovery of new transforms. Thus

$$x(s) = \frac{e^{\tan^{-1}s}}{\sqrt{1+s^2}}$$

yields $(1+s^2)x' + (s-1)x = 0$, the transformed equation of $tX'' + X' + (1+t)X = 0$. We desire the solution with $X_0 = e^{\pi/2}$. This solution may be written

$$X(t) = e^{\pi/2}[1 - t + t^3/9 + \cdots]$$

but seems not to be expressible in closed form.

Probability in Literature. Through chance, we are each a ghost to all the others, and our only reality; through chance, the huge hinge of the world, and a grain of dust; the stone that starts an avalanche, the pebble whose concentric circles widen across the seas.—Thomas Wolfe, *Look Homeward Angel.*

Three times he dropped a shot so close to the boat that the men at the oars must have been wet by the splashes . . . each shot deserved to be a hit he knew, but the incalculable residuum of variables in powder and ball and gun made it a matter of chance just where the ball fell in a circle of fifty yards radius, however well aimed.—C. S. Forester, *Captain Horatio Hornblower.*—*E. D. Schell.*

LIMITS FOR THE FIELD OF VALUES OF A MATRIX

A. B. FARNELL, West Point, N. Y.

1. Introduction. Let $A=(a_{rs})$ be a square matrix of order n with complex numbers as elements. The totality of numbers $\sum_{r,s}a_{rs}\bar{x}_r x_s$, where $\sum_r x_r\bar{x}_r$ has the value unity, comprise the *field of values* of the matrix A. The equation $|\lambda I-A|=0$ is called the *characteristic equation* fo the matrix A, and the roots λ_r, the *characteristic roots* of the matrix A. The maximum of the $|\lambda_r|$ is called the *dominant characteristic root.*

In 1900, Bendixson [3] obtained upper limits for the real and imaginary parts of the characteristic roots of a real matrix. In 1902, Hirsch [9] extended these results to matrices with complex numbers as elements. A limit was also given by Bromwich [4] in 1904. These limits were further refined by Browne [5] in 1930, and by Parker [10] in 1937.

In 1918, Toeplitz [11], using the results of Bendixson and Hirsch, showed that the field of values lies within a rectangle with sides parallel to the real and imaginary axis. Hausdorff [8] showed that the field of values is connected, bounded, closed, and convex.

If λ is a characteristic root of a matrix A of order n, there exist a set of numbers $(x_1, x_2, \cdots, x_n)$, such that

$$\sum_{r=1}^{n} x_r\bar{x}_r = 1,$$

which satisfy the relations

$$\lambda x_r = \sum_{s=1}^{n} a_{rs}x_s \qquad (r = 1, 2, \cdots, n). \tag{1}$$

If we multiply the rth equation in (1) by $\bar{x}_r$, and sum as to r, we obtain

$$\lambda = \sum_{r,s=1}^{n} a_{rs}\bar{x}_r x_s. \tag{2}$$

Hence the characteristic roots lie within the field of values of A. In the following, λ will be considered to be any value in this field, the conclusions holding a fortiori where λ is a characteristic root.

In a previous paper [6] the following theorem was established.

THEOREM 0. *If λ lies in the field of values of A,*

$$|\lambda| \leqq \left(\sum_{r,s}|a_{rs}|^2\right)^{1/2},$$

i.e., the absolute value of λ is not greater than the square root of the sum of the squares of the absolute values of the elements of A.

2. Some new limits. Let $\xi_r = |x_r|$. Then, using (2),

$$|\lambda| \leqq \sum_{r,s} |a_{rs}| \xi_r\xi_s = \sum_{r,s} d_{rs}\xi_r\xi_s,$$

where

$$d_{rs} = \tfrac{1}{2}(|a_{rs}| + |a_{sr}|), \qquad (d_{rs}) = D.$$

In view of this, the first two theorems obtained previously [6] may be restated as follows:

THEOREM 1. *Let $S_r = \sum_s d_{rs}$, $S = \max(S_r)$. Then, if λ lies in the field of values of A,*

$$|\lambda| \leqq S.$$

THEOREM 2. *If λ lies in the field of values of A, then,*

$$|\lambda| \leqq \left(\sum_{r,s} d_{rs}^2\right)^{1/2}.$$

3. Observations on these limits. Let $\bar{A}$ denote the conjugate of A, and write

$$B = (A + \bar{A}')/2, \qquad C = (A - \bar{A}')/2i.$$

Then, if $\lambda = \alpha + i\beta$, it can easily be shown that

$$\alpha = \sum_{r,s} b_{rs}\bar{x}_r x_s, \qquad \beta = \sum_{r,s} c_{rs}\bar{x}_r x_s,$$

i.e., α and β satisfy inequalities similar to those for λ.

It is interesting to note that two well-known results follow perhaps most easily from any theorem of the type given above. If A is Hermitian, C equals zero, and therefore the field of values lies on the real axis, and the characteristic roots are real. And if A is skew-Hermitian, *i.e.*, if $A + \bar{A}'$ equals zero, the field of values lies on the imaginary axis, and the characteristic roots are pure imaginary.

It might also be pointed out that if the quadratic form $\sum_{r,s} d_{rs}\xi_r\xi_s$ is positive [2], $|\lambda| \leqq \sum_r d_{rr}$. The quadratic form, or the symmetric matrix D, is positive if and only if every principal minor of D is positive [1]. Also since D is real symmetric there exists a real number γ such that $\gamma I + D$ is positive [1], and the characteristic roots of $\gamma I + D$ are $\lambda + \gamma$.

4. Convergence. R. Bellman noted that the approximations can sometimes be sharpened by using the fact that the roots of A^2 are λ^2. Still more can be established for Theorems 0 and 2.

LEMMA 1. *If A is symmetric, has real positive elements, and the determinants of every two-rowed principal minor is zero, then the determinant of any minor of the second order is zero.*

Consider the minor

$$\begin{pmatrix} a_{rs} & a_{rv} \\ a_{us} & a_{uv} \end{pmatrix}.$$

By hypothesis,

$$a_{jj}a_{kk} = a_{jk}a_{kj} = a_{jk}^2 = a_{kj}^2.$$

Therefore,

$$\begin{aligned} a_{rs}a_{uv} &= (a_{rr}a_{ss})^{1/2}(a_{uu}a_{vv})^{1/2} \\ &= (a_{rr}a_{vv})^{1/2}(a_{uu}a_{ss})^{1/2} \\ &= a_{rv}a_{us}. \end{aligned}$$

LEMMA 2. *If A satisfies the conditions of Lemma 1, then the characteristic roots are $\lambda_1 = \sum_r a_{rr}$, $\lambda_2 = \cdots = \lambda_n = 0$.*

Since the determinants of all two-rowed minors are zero, the determinants of all minors of higher order are zero. Therefore, the characteristic equation reduces to

$$\lambda^n - \left(\sum_r a_{rr} \right) \lambda^{n-1} = 0,$$

as desired.

LEMMA 3. *Applying Theorem 0 to $A\ exp\{2^{q-1}\}$, $q = 1, 2, \cdots$, yields a monotone convergent sequence of numbers $(\sum_{r,s} |a_{rs}^{(q)}|^2)^{1/2}$ where $a_{rs}^{(q)}$ is an element of the matrix $A\ exp\{2^{q-1}\}$. If A satisfies the conditions of Lemma 1, the upper limit obtained for $q+1$ is the same as the upper limit obtained for q.*

The characteristic roots of A^2 are λ_r^2. Also,

$$A^2 = \left(\sum_k a_{rk}a_{ks} \right).$$

Hence, using Theorem 0, and Schwarz's Inequality,

$$\begin{aligned} |\lambda|^4 &\leqq \sum_{r,s} \left| \sum_k a_{rk}a_{ks} \right|^2 \\ &\leqq \sum_{r,s} \sum_k |a_{rk}|^2 \cdot \sum_l |a_{ls}|^2 \\ &= \sum_r \sum_k |a_{rk}|^2 \cdot \sum_l \sum_s |a_{ls}|^2 \\ &= \left(\sum_{r,s} |a_{rs}|^2 \right)^2. \end{aligned}$$

Now, if A satisfies the conditions of Lemma 1, we have,

$$
\begin{aligned}
|\lambda| &\leqq \left(\sum_{r,s} a_{rs}^2\right)^{1/2} \\
&= (a_{11}^2 + a_{22}^2 + \cdots + a_{nn}^2 + 2a_{12}^2 + \cdots + 2a_{n-1,n}^2)^{1/2} \\
&= (a_{11}^2 + a_{22}^2 + \cdots + a_{nn}^2 + 2a_{11}a_{22} + \cdots + 2a_{n-1,n}a_{nn})^{1/2} \\
&= a_{11} + a_{22} + \cdots + a_{nn},
\end{aligned}
$$

which, by Lemma 2, is the exact value of the non-zero characteristic root. Hence in this case A^2 gives the same approximation as A.

The process can be reapplied to A^2 and A^4, *etc.*

LEMMA 4. *If $PA\overline{P}'=B$, $P\overline{P}'=I$, then*

$$\sum_{r,s} | a_{rs} |^2 = \sum_{r,s} | b_{rs} |^2.$$

Consider first the matrix equation

$$QC = E,$$

where $Q=(y_{rs}+iz_{rs})$, $C=(c_{rs}+id_{rs})$, and $E=(e_{rs})$, and $Q\overline{Q}'=I$, with the y_{rs}, z_{rs}, c_{rs}, and d_{rs} real numbers.

By a direct multiplication, and use of the fact that $\overline{Q}'Q=I$, it is fairly easily seen that

$$\sum_{r} | e_{rs} |^2 = \sum_{r} (c_{rs}^2 + d_{rs}^2),$$

and hence that

$$\sum_{r,s} | e_{rs} |^2 = \sum_{r,s} (c_{rs}^2 + d_{rs}^2).$$

Multiplying the matrix E on the right by $\overline{Q}'$ and proceeding in a similar manner, the lemma follows.

THEOREM 3. *The convergent sequence of Lemma 3 converges to the value of the dominant characteristic root.*

For there exists a unitary matrix P which transforms A into a triangular matrix T, and moreover displays the characteristic roots in any desired order along the principal diagonal. (We may therefore assume them to be arranged in order of descending magnitude along the principal diagonal.) Coupling this with the result of Lemma 4, it is seen that any result concerning the characteristic roots of T and $\sum_{r,s}|t_{rs}|^2$ will be equally valid for A.

Suppose $|t_{11}| = max(|\lambda_r|)$. It is sufficient to indicate the proof for a second order matrix. Then

$$T = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}.$$

Consider also

$$T'' = \begin{pmatrix} |t_{11}| & |t_{12}| \\ 0 & |t_{11}| \end{pmatrix}.$$

If the sequence of numbers obtained by applying Theorem 0 to T'' converges to $|t_{11}|$, then it will certainly converge to $|t_{11}|$ when applied to T, since $|t_{22}| \leqq |t_{11}|$.

Now if $p = 2^q$,

$$(T'')^p = \begin{pmatrix} |t_{11}|^p & p|t_{11}|^{p-1} \cdot |t_{12}| \\ 0 & |t_{11}|^p \end{pmatrix}, \qquad q = 1, 2, \cdots.$$

Applying Theorem 0, we have

$$|\lambda| \leqq (|t_{11}|^{2p} + p^2|t_{11}|^{2p-2}|t_{12}|^2 + |t_{11}|^{2p})^{1/2p}. \tag{3}$$

The right-hand member of (3) approaches $|t_{11}|$ as p approaches infinity, as desired.

The only difficulty in considering matrices T of higher order is the complexity of notation encountered in forming T^p.

THEOREM 4. *Applying Theorem 2 to the sequence of matrices A $exp\{2^{q-1}\}$, $q=1, 2, \cdots$, yields a sequence of numbers which converge to the dominant characteristic root.*

For evidently Theorem 2 applied to A yields a number not greater than that obtained using Theorem 0.

However the sequence is not necessarily monotone. Consider

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Theorem 2 applied to A and A^2 gives $|\lambda| \leqq 3$ and $|\lambda| \leqq (99)^{1/4}$, respectively.

5. Applications. As was pointed out in the previous paper [6], any number giving an upper limit for the dominant characteristic root will, when raised to the nth power, give an upper limit for the absolute value of the determinant.

In this connection we also state

THEOREM 5. *Let B_r denote the maximum absolute value of elements in the rth row, C_s for the sth column. Then*

$$|det(A)| \leq n! \prod_r B_r.$$

A similar inequality holds for the C_s.

This follows from the fact that there are $n!$ elements of the determinant of A of the form $a_{j1}a_{k2} \cdots a_{ln}$, and

$$|a_{j1}a_{k2} \cdots a_{ln}| \leqq B_1B_2 \cdots B_n.$$

Since the characteristic equation of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{pmatrix}$$

is

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0,$$

Theorem 4 can be used in calculating the dominant root of any polymonial equation.

Also Theorem 4 seems to be preferable to the method derived from the use of Sylvester's Theorem [7] in that all matrices are treated in exactly the same manner regardless of the types of characteristic roots involved.

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Arithmetic in Literature. He brought up all his heaviest artillery of logic and mathematics, drawing forth his pencil and notebook to figure just how much a given sum of money could be increased if it was shrewdly invested now in this or in that piece of property, and then sold when the time was right.—Thomas Wolfe, *You Can't Go Home Again.*

And having burnished his arms, he sat down patiently to compute how much a half a dollar per diem would amount to at the end of a six-months campaign; and when he had settled that problem, proceeded to the more abtruse calculations necessary for drawing up a brigade of two thousand men on the principle of extracting the square root.—Scott, *A Legend of Montrose.*

It was computed by an experienced arithmetician, that there was as much two penny ale consumed on the discussion as would have floated a first-rate man-of-war.—Scott, *Heart of Mid-Lothian.—E. D. Schell.*

A CIRCLE COVERING THEOREM

A. W. GOODMAN, Republic Aviation Corporation
R. E. GOODMAN, Queens College

1. Introduction. The following theorem was conjectured by Erdös:

THEOREM 1. *Let the circles $C_1, \cdots, C_n$, with radii $r_1, \cdots, r_n$, lie in a plane and have the following property: No line of the plane divides the circles into two non-empty sets without touching or intersecting at least one circle. Then the circles $C_1, \cdots, C_n$ can be covered by a circle of radius $r=\sum_{j=1}^n r_j$.*

The theorem is trivial when $n=2$. To the best of our knowledge, however, no proof has been given for any $n>2$. We shall give here an analytic proof of Theorem 1 which can easily be generalized to give the analogous theorem for euclidean space of k dimensions. This proof depends on the following lemma.

2. Lemma. *Let $I_1, \cdots, I_n$ be intervals* on a line L, of lengths $2r_1, \cdots, 2r_n$ and with midpoints $P_1, \cdots, P_n$, and such that $\sum_{j=1}^n I_j = I$ is a single interval AB. Let $\phi(P)=\sum_{j=1}^n \overline{PP}_j^2 r_j$, where P is any point of L. Then $\phi(P)$ has an absolute minimum value for some point $P=P_0$. Furthermore, I is covered by the interval J with P_0 as center and length $2\sum_{j=1}^n r_j$.*

Proof. Let A be regarded as the left end of the interval AB. Let $I_1, \cdots, I_n$ be numbered in the order in which the left ends of these intervals occur on L, proceeding from A to B. Let s_j be the distance AP_j, and let s be the distance AP. Then

$$PP_j = \pm (s - s_j),$$

so that

$$\phi(P) = \sum_{j=1}^n \overline{PP}_j^2 r_j = \sum_{j=1}^n (s - s_j)^2 r_j = g(s).$$

Thus

$$g'(s) = \sum_{j=1}^n 2(s - s_j) r_j = 0$$

when

$$s = s_0 = \sum_{j=1}^n s_j r_j \Big/ \sum_{j=1}^n r_j,$$

and $g(s_0)$ is the minimum of $g(s)$. Now

$$s_j \leqq 2r_1 + 2r_2 + \cdots + 2r_{j-1} + r_j,$$

so that

* In this paper all sets are considered as closed.

$$s_0 \leqq \sum_{j=1}^{n} r_j(2r_1 + \cdots + 2r_{j-1} + r_j) \Big/ \sum_{j=1}^{n} r_j$$
$$= \left(\sum_{j=1}^{n} r_j\right)^2 \Big/ \sum_{j=1}^{n} r_j = \sum_{j=1}^{n} r_j.$$

If now P_0 is the point such that $AP_0 = s_0$, then the above inequality implies that A is covered by the interval J.

By renumbering the intervals $I_1, \cdots, I_n$ in the order in which their right ends occur, proceeding from B to A, we prove similarly that B is covered by J.

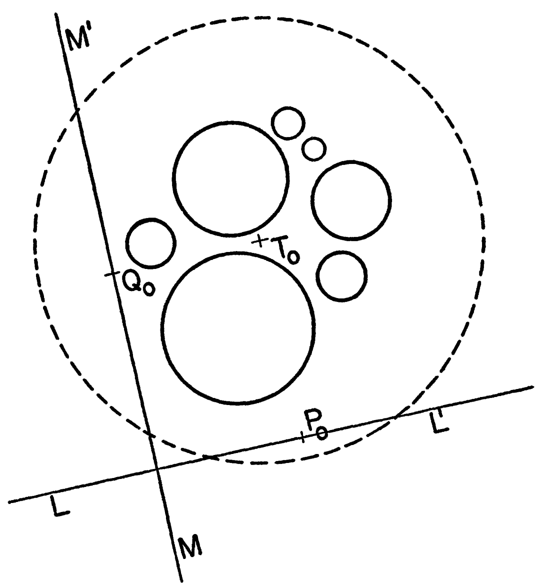

FIG. 1.

3. Proof of Theorem 1. Let $O_1, \cdots, O_n$ be the centers of $C_1, \cdots, C_n$. Define the function

$$f(T) = \sum_{j=1}^{n} \overline{TO}_j^2 r_j,$$

where T is any point in the plane. Let LL' and MM' be any two mutually perpendicular lines in the plane. (See Fig. 1.) Let P_j and Q_j be the projections of O_j on LL' and MM', respectively, and P and Q the projections of T on LL' and MM'. Then

$$\overline{TO}_j^2 = \overline{PP}_j^2 + \overline{QQ}_j^2,$$

so that

$$f(T) = \phi(P) + \psi(Q),$$

where

$$\phi(P) = \sum_{j=1}^{n} \overline{PP_j}^2 r_j,$$

$$\psi(Q) = \sum_{j=1}^{n} \overline{QQ_j}^2 r_j.$$

Since the circles $C_1, \cdots, C_n$ are nonseparable by any line in the plane, their projections on LL' and MM' are intervals satisfying the conditions of the lemma. Let P_0 on LL' and Q_0 on MM' be the points such that $\phi(P_0)$ and $\psi(Q_0)$ are the minimum values of $\phi(P)$ and $\psi(Q)$. Let T_0 be the point of intersection of the line through P_0 and parallel to MM' and the line through Q_0 and parallel to LL'. Let C be the circle with center at T_0 and radius r. By the lemma, the projection of C covers the projections of $C_1, \cdots, C_n$ on LL' and MM'.

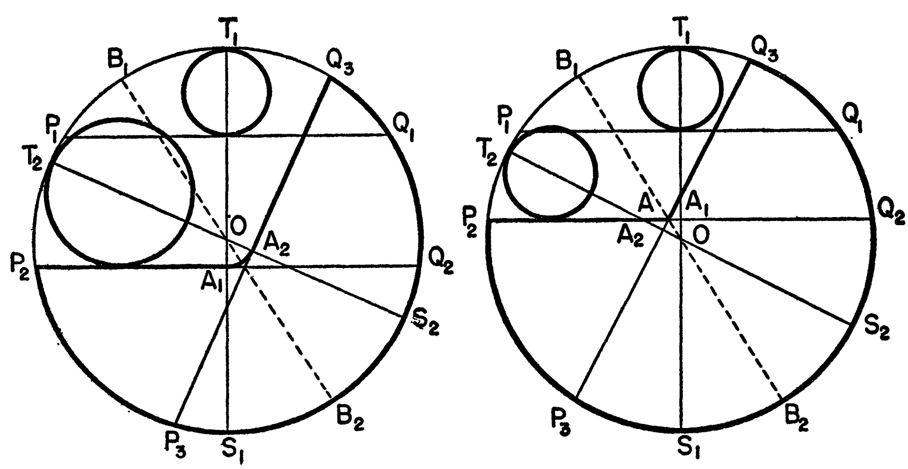

FIG. 2.

Furthermore, it is clear that the minimum of the function $f(T)$ is $f(T_0)$, and this minimizing property implies that the location of the point T_0 is independent of the particular pair of lines LL' and MM' used to construct T_0. Hence it is true that the projection of C on any line of the plane covers the projections of $C_1, \cdots, C_n$ on that line. Thus $C_1, \cdots, C_n$ are covered by every square circumscribed about C, whence $C_1, \cdots, C_n$ are covered by the product of these squares, that is, the circle C.

4. Geometric proof for a special case. For the special case $n=3$, we can give a purely geometric proof of Theorem 1, which seemingly cannot be generalized.

If C is the smallest circle which covers the given non-separable circles C_1, C_2, C_3, then either only two of the given circles are tangent to C or all three of the given circles are tangent to C. In the former case, the two points of tangency must lie at opposite ends of a diameter of C, and the theorem is trivial.

In the latter case, every semi-circumference of C contains one of the three points of tangency.

Let C_1, the smallest of the given circles, be tangent to C at T_1. (See Fig. 2.) Let chord P_1Q_1 be perpendicular to diameter T_1OS_1 of C, and tangent to C_1. Now P_1Q_1 cuts another circle, C_2. (All three given circles cannot lie in segment $P_1T_1Q_1$, since C_1 is the smallest of the given circles.) Let P_2Q_2 be the chord parallel to P_1Q_1 which is tangent to C_2 and does not intersect C_1, and let A_1 be the point of intersection of P_2Q_2 with T_1S_1. (Note, as in Fig. 2, that A_1 may be interior either to T_1O or to OS_1.) Let A_2 lie on T_2O with $T_1A_1 = T_2A_2$. Let chord $P_3A_2Q_3$ be perpendicular to diameter T_2OS_2. Now T_3, the point of tangency of C_3 with C, must lie on that arc S_1S_2 which does not contain T_1 and T_2. If $2r_3 \geqq A_1S_1$, the theorem is true. If $2r_3 < A_1S_1$, then C_3 does not touch the heavy boundary $P_2A_1A_2Q_3$ (or P_2AQ_3). But this boundary is symmetric about the bisector B_1B_2 of $\measuredangle T_1OT_2$, whence it is clear that C_3 can be separated from C_1 and C_2 by some line tangent to the arc A_1A_2 with O as center (or through A).

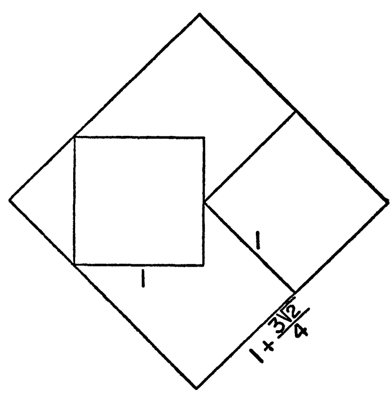

FIG. 3.

5. An analogous problem for convex bodies. Theorem 1 suggests consideration of the following problem:

Let $P_1, \cdots, P_n$ be given similar convex bodies with diameters $d_1, \cdots, d_n$. Let d be the diameter of the smallest convex body similar to the given convex bodies which can be placed so as to cover $P_1, \cdots, P_n$ whenever they are so situated in a plane that no line of the plane separates them into two non-empty sets without touching or intersecting at least one of them. Let $d = \lambda\sum_{j=1}^{n} d_j$. What is the value of λ?

Since $P_1, \cdots, P_n$ may always be placed with their diameters on a straight line and non-overlapping, it is clear that $\lambda \geqq 1$ in every case. Application of Theorem 1 shows at once that $\lambda \leqq \mu$ where μ is the ratio of the radius of the circumcircle to that of the incircle for the given set of convex bodies. It is clear that λ is arbitrarily large for sufficiently distorted figures. That λ may actually be >1, however, even for regular polygons is shown by the case of two congruent squares. From the arrangement in Fig. 3, $\lambda \geqq (4+3\sqrt{2})/8 > 1.03$.

If we specialize the above problem by requiring that $P_1, \cdots, P_n$ be not only similar but also homothetic, it seems reasonable to believe that $\lambda = 1$, although we have no proof of this conjecture.

The method of proof used for Theorem 1 immediately shows that $\lambda = 1$ for homothetic regular polygons for an even number of sides. Indeed, we may even relax the condition of non-separability of $P_1, \cdots, P_n$ to non-separability only by lines parallel to the sides of the given polygons, still obtaining the same covering polygon.

For homothetic regular polygons of an odd number of sides, however, the method of Theorem 1 does not produce results. And with the relaxed condition of non-separability for homothetic regular polygons of an odd number of sides, we find that $\lambda > 1$.

For let P be a regular polygon with vertices $V_1, \cdots, V_{2k+1}$. (See Fig. 4.) Let P_1 have vertices $U_1, \cdots, U_{2k+1}$, and let U_1U_2 lie on V_1V_2, with midpoints coinciding. Let the diameter of P_1 be large enough so that the line LL' through U_{k+3} and parallel to V_2V_3 and the line MM' through U_{k+2} and parallel to V_1V_2

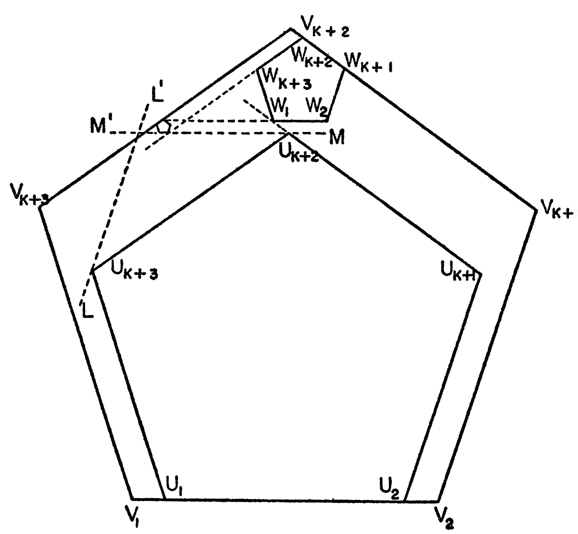

FIG. 4.

intersect outside P. Let P_2, with vertices $W_1, \cdots, W_{2k+1}$, have diameter d_2 equal to the distance between $V_{k+1}V_{k+2}$ and $U_{k+1}U_{k+2}$, and let P_2 be placed so that the distance between W_1W_2 and MM' is equal to the distance between $W_{k+2}W_{k+3}$ and $V_{k+2}V_{k+3}$, and call this distance d_3. Let P_3 have diameter d_3 and be placed with its base Y_1Y_2 on MM' and $Y_{k+2}Y_{k+3}$ on $V_{k+2}V_{k+3}$. It is clear, then, that P_1, P_2, P_3 cannot be separated by any line parallel to their sides and that P is the smallest polygon which will cover them. But $d > d_1 + d_2 + d_3$, so that $\lambda > 1$.

THE LAGRANGE IDENTITY AS A UNIFYING PRINCIPLE

LOUIS BRAND, University of Cincinnati

1. Introduction. Many important facts relating to linear differential equations of the second order can be systematized and unified by means of the *Lagrange identity* connecting adjoint differential expressions. If we write the reduced equation

$$P(u) = P_0(x)u'' + P_1(x)u' + P_2(x)u = 0, \tag{1}$$

the multipliers of $P(u)$ and the test for its exactness are at once apparent from this identity. When $P(u)$ is made self-adjoint by multiplication by a suitable factor $\lambda(x)$, the Lagrange identity for $\lambda P(u)$ leads at once to:

(a) The consideration of the Wronskian of two linearly independent solutions of the reduced equation;
(b) the Liouville differential equation satisfied by this Wronskian;
(c) the immediate deduction of a second solution of the reduced equation when a first particular solution is known;
(d) the general solution of the complete equation.

All these important elementary matters thus flow simply and naturally from the Lagrange identity, and without resort to artifices, such as "variation of parameters." This identity is also useful in dealing with equations of the Sturm-Liouville type.

2. Adjoint differential expressions. We assume that the $P_0(x)$, $P_1(x)$, $P_2(x)$ are continuous, twice-differentiable functions of the real variable x in the interval $a \leqq x \leqq b$, and that $P_0 \neq 0$ in this interval. By transforming the term of $vP(u)$ by the rule for differentiating a product we obtain the *Lagrange identity*

$$vP(u) - uQ(v) = \frac{d}{dx} R(u, v), \tag{2}$$

where

$$R(u, v) = P_0(u'v - uv') + (P_1 - P_0')uv \tag{3}$$

is a bilinear expression of the first order in u, v, and

$$Q(v) = (P_0 v)'' - (P_1 v)' + P_2 v \tag{4}$$

is known as the *adjoint* of $P(u)$.

From the *Lagrange identity* (2) it is readily shown that $Q(v)$ is the only differential expression which satisfies a relation of this form. Thus the identity establishes a unique association of differential expressions $P(u)$, $Q(v)$, which therefore are *adjoint to each other.*

If $P(u) = Q(u)$, $P(u)$ is said to be *self-adjoint*. In this case the Lagrange identity with $v = u$ requires that $R(u, u) = (P_1 - P_0')u^2$ be constant and hence

$$P_1 - P_0' = 0; \tag{5}$$

this necessary condition that $P(u)$ be self-adjoint is obviously sufficient.

If $P_1 - P_0' \neq 0$, a function $\lambda(x)$ which satisfies $(\lambda P_1) - (\lambda P_0)' = 0$ will make $\lambda P(u)$ self-adjoint; such a function is defined by

$$\lambda P_0 = \exp \int_c^x \frac{P_1}{P_0} dx \tag{6}$$

where c is a point of (a, b).

The Lagrange identity for the self-adjoint form $\lambda P(u)$ is

$$\lambda v P(u) - \lambda u P(v) = \frac{d}{dx} \{\lambda P_0(u'v - uv')\}. \tag{7}$$

3. Multipliers. A function $v(x)$ is said to be a *multiplier* of $P(u)$ if $vP(u)$ is identically equal to the derivative of a linear differential expression of the first order in u. For any v, the right member of the Lagrange identity (3) is such a derivative, but the term $uQ(v)$ on the left can not be so expressed. We may therefore state

THEOREM 1. *A function v is a multiplier of $P(u)$ when and only when $Q(v) = 0$.*

We pass now from differential *expressions* to differential *equations* and call $P(u) = 0$, $Q(v) = 0$ *adjoint equations.* Then we may say that the multipliers of $P(u) = 0$ are precisely those functions which satisfy its adjoint $Q(v) = 0$.

If $P(u)$ is expressible as a derivative, the equation $P(u) = 0$ is said to be *exact.* In this case $P(u)$ admits the multiplier 1 and hence $Q(1) = 0$. The Lagrange identity, with $v = 1$, now gives

$$P(u) = \frac{d}{dx} \{P_0 u' + (P_1 - P_0')u\}. \tag{8}$$

In this case the *complete* linear equation

$$P(u) = f(x) \tag{9}$$

may be solved by quadratures: for its first integral

$$P_0 u' + (P_1 - P_0')u = \int f(x) dx + C,$$

is a linear equation of the first order.

If $P(u) = 0$ is not exact, $vP(u) = 0$ will be exact if v is a solution of $Q(v) = 0$. Thus if any non-zero solution of $Q(v) = 0$ is known, the general solution of (9) may be obtained by quadratures.

4. The Wronskian. If u_1 and u_2 are two solutions of $P(u) = 0$, let $v = u_1$, $u = u_2$ in (7); then

$$\frac{d}{dx} (\lambda P_0 W) = 0 \quad \text{where} \quad W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

is the *Wronskian* of the two solutions. On integration we obtain

$$\lambda P_0 W = (\lambda P_0 W)_{x=c} = W(c) \tag{10}$$

since $\lambda P_0 = 1$ when $x = c$; hence, from (6),

$$W = W(c) \exp\left(-\int_c^x \frac{P_1}{P_0}\,dx\right). \tag{11}$$

5. Solution of adjoint equation. If u_1 and u_2 are linearly independent solutions of $P(u) = 0$, their Wronskian $W \neq 0$. From (7), λu_1 and λu_2 are multipliers of $P(u) = 0$ and therefore linearly independent solutions of the adjoint equation $Q(v) = 0$. From (10) we have $\lambda = W(c)/P_0 W$; hence the

THEOREM 2. *If u_1, u_2 are linearly independent solutions of $P(u) = 0$ whose Wronskian is W,*

$$v_1 = u_1/P_0 W, \qquad v_2 = u_2/P_0 W \tag{12}$$

are linearly independent solutions of the adjoint equation $Q(v) = 0$.

A simple calculation shows that

$$W(u_1, u_2) W(v_1, v_2) = 1/P_0^2. \tag{13}$$

6. Solution of complete equation. If u_1 is a non-zero solution of $P(u) = 0$, we have from (7), with $v = u_1$,

$$\lambda u_1 P(u) = \frac{d}{dx}\left\{\lambda P_0(u' u_1 - u u_1')\right\}; \tag{14}$$

thus λu_1 is a multiplier of $P(u)$. The complete equation (9) may be written

$$\frac{d}{dx}\left\{\lambda P_0(u' u_1 - u u_1')\right\} = \lambda u_1 f(x); \tag{15}$$

hence the fundamental

THEOREM 3. *The general solution of the complete equation can be found by quadratures if any non-zero solution of the reduced equation is known.*

A first integration of (15) gives

$$\lambda P_0(u' u_1 - u u_1') = \int_c^x \lambda(t) u_1(t) f(t)\,dt + B. \tag{16}$$

On dividing this by $\lambda P_0 u_1^2$ we have

$$\frac{d}{dx}\left(\frac{u}{u_1}\right) = \frac{1}{\lambda P_0 u_1^2}\int_c^x \lambda u_1 f\,dt + \frac{B}{\lambda P_0 u_1^2}.$$

A second integration yields the complete solution

$$u = u_1 \int_c^x \frac{ds}{\lambda P_0 u_1^2} \int_c^s \lambda u_1 f\, dt + B u_1 \int_c^x \frac{dt}{\lambda P_0 u_1^2} + A u_1. \tag{17}$$

This discloses that the reduced equation has the second particular solution

$$u_2(x) = u_1(x) \int_c^x \frac{dt}{\lambda P_0 u_1^2}. \tag{18}$$

Since $u_2'(c) = 1/u_1(c)$, $W(u_1, u_2) = 1$ when $x = c$.

The iterated integral in (17) may be written

$$U(x) = u_1 \int_c^x \left(\frac{u_2}{u_1}\right)' ds \int_c^s \lambda u_1 f\, dt. \tag{19}$$

On integration by parts it assumes the form

$$\begin{aligned} U(x) &= u_2(x) \int_c^x \lambda u_1 f\, dt - u_1(x) \int_c^x \lambda u_2 f\, dt \\ &= \int_c^x \lambda(t) f(t) \begin{vmatrix} u_1(t) & u_2(t) \\ u_1(x) & u_2(x) \end{vmatrix} dt \end{aligned} \tag{20}$$

where $\lambda(t) = 1/P_0(t)W(t)$ from (10). $U(x)$ is evidently a particular solution of the complete equation; it is determined by the conditions $U(c) = 0$, $U'(c) = 0$. The solution of the complete equation,

$$u = Au_1(x) + Bu_2(x) + U(x) \tag{21}$$

agrees with that obtained by Lagrange's method of "variation of parameters."

If we have two independent solutions u_1, u_2 of the reduced equation, we can deduce the solution $U(x)$ by obtaining two first integrals of (9) of the form (16) and eliminating u'. Putting $\lambda = W(c)/P_0 W$ in (16), we have

$$\frac{1}{W}(u_i u' - u_i' u) = \int_c^x \frac{u_i(t)}{P_0(t)W(t)} f(t) dt + C_i \qquad (i = 1, 2).$$

Elimination of u' from these equations again gives (21).

MAGIC RECTANGLES MODULO p

E. J. FINAN, Catholic University

1. Introduction. Since remote times magic configurations of integers have been of interest to mathematicians. There are magic cubes, rectangles, spheres, pencils, crosses, squares, *etc.* The best known and most interesting of these is the magic square which still merits attention. Its most obvious property of interest—its magic property—is that the sums of the elements in each row, each column, and each principal diagonal are the same.

The purpose of this paper is to discuss a rectangular array of positive integers which is associated with elementary number theory. One of the properties of the rectangle, although not the most useful, is that the sum of the integers in any row or column is zero modulo p. This justifies the term Magic Rectangle. Some of the properties which are listed below as theorems are restatements, special cases, or generalizations of well known theorems. Others are peculiar to this particular configuration.

The results of section 4 may be of assistance in performing some of the calculations one meets in elementary number theory. This is true in particular when one modulus is used at length since the rectangles $R_{t,k}$ and $A_{t,k}$ yield with very little work the primitive roots, the nth power residues, the reciprocals, and the solution of $ax \equiv b$ all modulo p. However if only one of the above quantities, for example a primitive root, is wanted, it is not advisable to construct $R_{t,k}$.

Unless otherwise stated all congruences are modulo p.

2. Law of formation. Let p be an odd prime for which $\phi(p) = p-1 = kt$, where $(k, t) = 1$ and k is even. Let α and β designate the smallest* positive integers that belong to k and t respectively mod p. The rectangular array whose element in the (i, j) cell or position is the least positive residue of $\beta^i \alpha^j$ mod $p(i = 1, 2, \cdots, t;$ $j = 1, 2, \cdots, k)$, will be called the magic rectangle $R_{t,k}$. It is uniquely determined by t and k for a given p.

For example, let $p = 43$, $t = 7$ and $k = 6$. Then $\alpha = 7$, $\beta = 4$, and $R_{7,6}$ is the first of the two rectangles appearing below.

28	24	39	15	19	4
26	10	27	17	33	16
18	40	22	25	3	21
29	31	2	14	12	41
30	38	8	13	5	35
34	23	32	9	20	11
7	6	42	36	37	1

1	7	6	42	36	37	1
4	28	24	39	15	19	4
16	26	10	27	17	33	16
21	18	40	22	25	3	21
41	29	31	2	14	12	41
35	30	38	8	13	5	35
11	34	23	32	9	20	11
1	7	6	42	36	37	1

3. Properties. Some of the interesting properties of such a rectangular array are included in the following theorems.

THEOREM 1. *Each of the numbers* $1, 2, \cdots, kt = p-1$ *appears in* $R_{t,k}$ *exactly once.*

---

* The restriction that k be even and that α and β be smallest are imposed for convenience in order to make $R_{t,k}$ unique and to make some of the proofs easier to word.

Since each of the kt cells is filled it is sufficient to show the entries are distinct. Assume $\beta^i\alpha^j \equiv \beta^r\alpha^s$ where $j \leqq s$. Then $\beta^i \equiv \beta^r\alpha^{s-j}$. After raising both sides of the last congruence to the tth power it is evident k divides $s-j$ because $(k, t)=1$ and α belongs to k. Hence $j=s$ and $\beta^i \equiv \beta^r$. But this is impossible unless $i=r$.

THEOREM 2. *All the elementary symmetric functions of the numbers in each row or column of $R_{t,k}$ excepting the product is congruent to zero. The product of the elements in the ith row is congruent to $-\beta^{ik}$ and the product of those in the jth column is congruent to α^{jt}.*

Since β belongs to t, the elements in the last row are $\alpha, \alpha^2, \cdots, \alpha^k=1$. They are roots of $x^k \equiv 1$. There are no other roots because a congruence with prime modulus cannot have more roots than its degree. The elements in the ith row are the products of the corresponding elements in the last row by β^i. Hence they are all of the roots of the congruence $x^k \equiv \beta^{ik}$. In the same way the elements of the jth column are all of the roots of the congruence $x^t \equiv \alpha^{jt}$. Theorem 2 follows from the above observations and the relations between the coefficients of an equation and the elementary symmetric functions of its roots.

COROLLARY. *If the product of the elements in the ith row of $R_{t,k}$ is $\theta_i (i=1, 2, \cdots, t)$ then the roots of $x^t \equiv -1$ are θ_i.*

These products are $-\beta^k, -\beta^{2k}, \cdots, -\beta^{tk}$, which satisfy $x^t \equiv -1$ because β belongs to t and t is odd.

When $k=p-1$ and $t=1$ the rectangle consists of one row. The corollary is then a statement of Wilson's Theorem. Hence Theorem 2 is a generalization of Wilson's Theorem unless $p=2^{2n}+1$. In this case it yields Wilson's Theorem.

COROLLARY. *If the product of the elements in the jth column of $R_{t,k}$ is $\pi_j (j=1, 2, \cdots, k)$ then the roots of $x^k \equiv 1$ are π_j.*

THEOREM 3. *Each row (column) of $R_{t,k}$ is a multiple of every row (column).*

For example, the product of the ith row by β^{k-i} is the kth row.

For the next theorems it is convenient to denote the rows and columns of $R_{t,k}$ as $r_i (i=1, 2, \cdots, t)$ and $c_j (j=1, 2, \cdots, k)$ respectively. Also define c_0 as an additional column consisting of t zeros. The sum c_r+c_s will be defined as follows. Select any two elements, one from c_r and one from c_s, lying in the same row. Let their sum modulo p be in c_m. Then $c_r+c_s=c_m$. It remains to show the sum is well defined, *i.e.*, in this instance, that it is unique. In equation form this amounts to showing that if $\beta^i\alpha^r+\beta^i\alpha^s=\beta^t\alpha^m$ then $\beta^k\alpha^r+\beta^k\alpha^s=\beta^n\alpha^m$ for some n. Multiply the first equation by β^{k-i}. This gives $\beta^k\alpha^r+\beta^k\alpha^s=\beta^{t+k-i}\alpha^m$ which is in c_m. Hence the c_m is determined uniquely. Since the elements in the last row of $R_{t,k}$ are the roots of $x^k \equiv 1$ where k is even, these roots may be paired so that the sum of each pair is zero modulo p. Consequently for any given column c_s there exists a column c_b so that $c_s+c_b=c_0$. Hence the $c_j (j=0, 1, \cdots, k)$ form a set of $k+1$ elements which is closed under addition, contains the identity,

contains the inverse of each element, but does not obey the associative law. For example, in the above illustration $(c_1+c_2)+c_6=c_4+c_6=c_5$ while $c_1+(c_2+c_6)=c_1+c_1=c_4$.

Similarly the product $c_r c_s$ of two columns may be defined as follows. Select two elements $\beta^i\alpha^r$ and $\beta^i\alpha^s$ lying in the same row of $R_{t,k}$. Let their product modulo p be in c_m. Then $c_r c_s=c_m$. Again one must show this product is unique. This amounts to proving that if $(\beta^i\alpha^r)(\beta^i\alpha^s)=\beta^l\alpha^m$ then $(\beta^q\alpha^r)(\beta^q\alpha^s)_b\beta^n\alpha^m$ for some n. However this is obvious since the exponent on α in both equations is $r+s$. Since $\alpha^k\equiv 1 \bmod p$ the exponents on α are reduced modulo k. Also since the product of two columns may be obtained by the addition modulo k of the exponents on α the $c_j(j=1, 2, \cdots, k)$ form a group under multiplication which is isomorphic with the cyclic additive group whose elements are $1, 2, \cdots, k$ with addition performed modulo k. Call this group C. In the same way it can be shown that the $r_i(i=1, 2, \cdots, t)$ form a group R under multiplication which is isomorphic with the cyclic additive group of order t.

One may form the symbols (r_i, c_j), $i=1, 2, \cdots t$; $j=1, 2, \cdots k$, and define the product of two such by the equation $(r_q, c_d)(r_m, c_s)=(r_{q+m}, c_{d+s})$ where the addition of the subscripts on r and c is performed modulo t and k respectively. These $kt=p-1$ symbols (r_i, c_j) form a group P which is the direct product* $R\times C$ of R and C. Since it may be generated by (r_1, c_1) it is cyclic. Also the $p-1$ elements of $R_{t,k}$ form a group G under multiplication modulo p. It may be generated by $\beta\alpha$. Hence P and G are isomorphic. One isomorphic correspondence between the two is $\beta^i\alpha^j\leftrightarrow(r_i, c_j)$. These results are stated in the next two theorems.

THEOREM 4. *The set $c_j(j=0, 1, \cdots, k)$ is closed under addition, contains the identity, contains the unique inverse of each element but does not obey the associative law.*

THEOREM 5. *The $r_i(i=1, 2, \cdots, t)$ and $c_j(j=1, 2, \cdots, k)$ form cyclic groups under multiplication. Their direct product is isomorphic with the multiplicative group modulo p composed of the elements of $R_{t,k}$.*

Let $A_{t,k}$ be a rectangular array of $t+1$ rows and $k+1$ columns formed from $R_{t,k}$ as follows. First place an additional row consisting of $\alpha, \alpha^2, \cdots, \alpha^k=1$ at the top of $R_{t,k}$. Then place an additional column consisting of $\beta, \beta^2, \cdots, \beta^t=1$ at the left of $R_{t,k}$. Finally place unity in the (1,1) position of $A_{t,k}$. The second rectangle displayed above is $A_{7,6}$. It is convenient to call the elements in the added row and column $\beta^0\alpha^j(j=0, 1, \cdots, k)$ and $\beta^i\alpha^0(i=0, 1, \cdots, t)$ respectively. Let O be the geometric center of the rectangle determined by $A_{t,k}$. If the reflection of $\beta^r\alpha^s$ in O is $\beta^b\alpha^a$ then $b+r=t$ and $a+s=k$. This proves

THEOREM 6. *The inverse modulo p of any number is its reflection in the center of $A_{t,k}$.*

* See, for example, MacDuffee, Introduction to Abstract Algebra, p. 67.

4. Applications. Once $A_{t,k}$ has been constructed for a given p, a good many of the elementary number theory results are easily obtained. This refers to both the theorems and calculations. A few illustrations will be given.

If a is in cell (i, j) and b is in cell (r, s) then ab is in cell $(i+r, j+s)$ where additions are performed modulo t and k respectively.

The inverse of a is obtained by inspection from $A_{t,k}$ as explained in Theorem 6.

The solution of $ax\equiv b \bmod p$ is $x=a^{-1}b$ which may be obtained from $A_{t,k}$.

THEOREM 7. *A necessary and sufficient condition that* $x^n\equiv\beta^i\alpha^j$ *be solvable is that* $d_1=(t, n)$ *divide* i *and* $d_2=(k, n)$ *divide* j. *There are then* d_1d_2 *solutions.*

This follows from the fact that the nth root of β^i exists if and only if there is an x such that $tx+i\equiv 0 \bmod n$. This x exists if and only if $d_1=(t, n)$ divides i and there are then d_1 values of x. In the same way there are $d_2=(k, n)$ nth roots of α^j if d_2 divides j. This yields a total of d_1d_2 solutions.

An equivalent result, Euler's Criterion,* may be stated as follows: "If $d=(n, p-1)$, a necessary and sufficient condition that $x^n\equiv b$ be solvable is that $b^{(p-1)/d}\equiv 1$. There are then exactly d roots." To show they are equivalent first assume d_1 divides i and d_2 divides j. Then $t=t_1d_1$, $k=k_2d_2$, $n=n_1d_1=n_2d_2$, $i=i_1d_1$, and $j=j_2d_2$ where $(d_1, d_2)=1$ since $(k, t)=1$. The congruence $b^{(p-1)/d}\equiv 1$ may then be written $(\beta^i\alpha^j)^{t_1k_2}\equiv 1$ or $\beta^{t_1k_2i_1d_1}\alpha^{t_1k_2j_2d_2}\equiv 1$ which is evident since the exponents on β and α are multiples of t and k respectively. Hence if d_1 divides i and d_2 divides j then $b^{(p-1)/d}\equiv 1$.

Now assume $b^{(p-1)/d}=(\beta^i\alpha^j)^{(p-1)/d}\equiv 1$ where $d=(n, kt)$ and $kt=p-1$. Since $(k, t)=1$ one can write $d=d_1d_2$ where $d_1=(t, n)$, $d_2=(k, n)$, and $(d_1, d_2)=1$. Then using same notation as above, from $(\beta^i\alpha^j)^{(p-1)/d}\equiv 1$ follows $\beta^{t_1k_2i}\equiv\alpha^{t_1k_2j}\equiv 1$. Hence there exists an integer s so that $t_1k_2i=st=st_1d_1$ from which d_1 divides i since $(d_1, k_2)=1$. In the same it follows that d_2 divides j.

Theorem 7 enables one to obtain the nth power residues.† For example, to obtain the cubic residues modulo 43 let $n=3$, $i=7$, and $k=6$. Since $(7, 3)=1$ the residues may be in any row. Since $(6, 3)=3$ then j must be a multiple of 3. Hence the cubic residues are the 14 numbers in the 3rd and last columns of $R_{7,6}$. Of course, the quadratic residues are those numbers in the 2nd, 4th and 6th columns.

The primitive roots‡ modulo p are those numbers $\beta^i\alpha^j$ for which $(i, t)=(j, k)=1$. For $p=43$ they are in the first and fifth columns and in all rows except the last of $R_{7,6}$.

By replacing p and $p-1$ by m and $\phi(m)$ respectively it is possible to construct $R_{t,k}$ using only those integers in a reduced residue system. The procedure parallels the above.

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* *Cf.* L. W. Reid, The Elements of the Theory of Algebraic Numbers, p. 115.

† *Cf.* Uspensky and Heaslet, Elementary Number Theory, p. 202 et seq.

‡ *Ibid.*, p. 232. The above method does not shorten the labor necessary to find a primitive root.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

RATIONAL VALUES OF TRIGONOMETRIC FUNCTIONS

J. M. H. OLMSTED, University of Minnesota

A recent article* pointed out that cos x is an algebraic number whenever x is equal to an integral number of seconds. An immediate extension of this is that all six standard trigonometric functions have algebraic values for angles measured rationally in degrees. An earlier article† gave a more advanced discussion of these algebraic numbers. It is the purpose of this note to present an elementary proof of the fact, apparently not widely known, that these numbers are scarcely ever rational—in fact only for the very familiar values associated with the sequence 0°, 30°, 45°, $\cdots$.

THEOREM. *If x is rational in degrees, then the only possible rational values of the trigonometric functions are: $\sin x$, $\cos x=0$, $\pm 1/2$, ± 1; $\sec x$, $\csc x=\pm 1$, ± 2; $\tan x$, $\cot x=0$, ± 1.*

The proof can obviously be limited to the cosine and tangent. Assume that cos x is rational but not equal to one of the given values, and that kx equals a multiple of 360°, where k is an integer. Then for any integer n, knx is a multiple of 360°. Now cos knx can be expressed as a polynomial in cos nx with integral coefficients and with leading coefficient equal to 2^{k-1}. Since cos nx is rational and satisfies the equation obtained by equating this polynomial to 1, therefore when cos nx is expressed as a rational fraction in lowest terms, the denominator must be a factor of 2^{k-1}, and incidentally a power of 2. A contradiction is obtained by showing that this denominator of cos nx can be made arbitrarily large. Accordingly, let $\cos \alpha=p/q$, where p is odd and q is a power of 2 greater than the first. Then $\cos 2\alpha=p'/q'$, where $p'=p^2-q^2/2$, an odd number, and $q'=q^2/2$, a power of $2>q$. Thus the terms of the sequence obtained by repeated doubling of the angle, cos x, cos $2x$, cos $4x$, $\cdots$, when expressed as rational fractions in lowest terms, have successively larger denominators.

Proof for the tangent follows essentially the same pattern. In this case tan knx can be expressed as the quotient of two polynomials in tan nx, each having integral coefficients, and the numerator having leading coefficient 1 or k. Since tan nx is rational (if finite) and satisfies the equation obtained by equating the numerator polynomial to 0, therefore when tan nx is expressed as a rational

* R. W. Hamming, The transcendental character of cos x, this MONTHLY, vol. 52, 1945, pp. 336–337.

† D. H. Lehmer, A note on trigonometric algebraic numbers, this MONTHLY, vol. 40, 1933, pp. 165–166.

fraction in lowest terms, the denominator must be a factor of k. Let $\tan \alpha = p/q$, where $p \neq \pm q$, $p \neq 0$, $q \neq 0$, and the fraction is in lowest terms. Then $\tan 2\alpha = p'/q'$, where $p' = 2pq \neq 0$ and $q' = q^2 - p^2 \neq 0$. It is impossible for p' to equal $\pm q'$ since the largest possible common factor of p' and q' is 2, this occurring only when p and q are both odd. In any case, when p'/q' is expressed in lowest terms, the resulting fraction is of the same type as p/q, and the new denominator is numerically greater than the first. Again a contradiction is provided by a sequence obtained by repeated doubling.

THE AREA OF A TRIANGLE AS A FUNCTION OF THE SIDES*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Historical remarks. The first mention of the rule giving the area of a triangle as a function of the three sides is found in the works of Heron of Alexandria (1st century). Although it is now believed that this rule pre-dates Heron, demonstrations of it are in his two works, *Metrics* and *Treatise on the Diopter*.

In the book of the three Arabian brothers, Mohammed, Ahmed, and Alhasan, (9th century) we encounter a new demonstration, the first which came to us in Europe. It was reproduced by Leonardo of Pisa in his *Practical Geometry* (1220) and then by Jordanus Nemorarius (13th century), and by most of the geometers of the Renaissance. It is curious that Heron, the Hindus, as well as all the authors we have cited, made an application of this rule to the triangle of sides 13, 14, and 15, whose area is 84. One is led to ask if these three numbers have a common origin, but, as Chasles had observed, the Greeks, the Hindus, and the Arabs may very well have separately become aware of the fact that 13, 14, 15 are the smallest integers which give a rational area for an acute angled triangle.

One finds, still later, other new proofs of the rule by Newton in his *Universal Arithmetic* (1707); by Euler in the *Recent Commentaries of Petersburg* (v. I, 1747, p. 48); by Boscovich in volume V of his *Works* concerning optics and astronomy (1785). This last demonstration is obtained by trigonometric considerations.

2. New demonstration. The author has previously given a very short geometrical demonstration of the formula under consideration (*Mathesis*, 1931, p. 27), and here is another equally simple.

Being given a triangle ABC, ($BC = a$, $CA = b$, $AB = c$, $a+b+c = 2p$), let (see figure) B' and B'', C' and C'' be the orthogonal projections of the vertices B and C on the bisectors AD and AD' of angle A. Rectangle $AB''MC'$ has dimensions equal to BB' and CC'', and is the sum of two rectangles, $AB''BB'$ and $B'BMC'$. The first of these rectangles is equal in area to triangle BAE, which has AB' for altitude and $BE = 2\ BB'$ for base. The second is equal in area to triangle BEC, which has $NC = B'C'$ for altitude and BE for base. Thus rectangle $AB''MC'$ is equal in area to triangle ABC. Similarly, rectangle $AC''NB'$ is equal in area to triangle ABC, for this rectangle is the difference of rectangles

* Translated from the French by Howard Eves.

$AC''CC'$ and $C'CNB'$, respectively equivalent to triangles CAF and CBF, the difference of which is triangle ABC. We thus have

$$BB'\cdot CC''\cdot CC'\cdot BB'' = (\text{area } ABC)^2. \tag{1}$$

Now, the hyperbola $(\mathcal{H})$ which has foci at B and C and which passes through A, has for tangent at A the interior bisector AD of the angle formed by the radii vectors AB and AC, and the ellipse $(\mathcal{E})$ having foci at B and C and passing through A has for tangent at A the exterior bisector of angle A. Therefore*

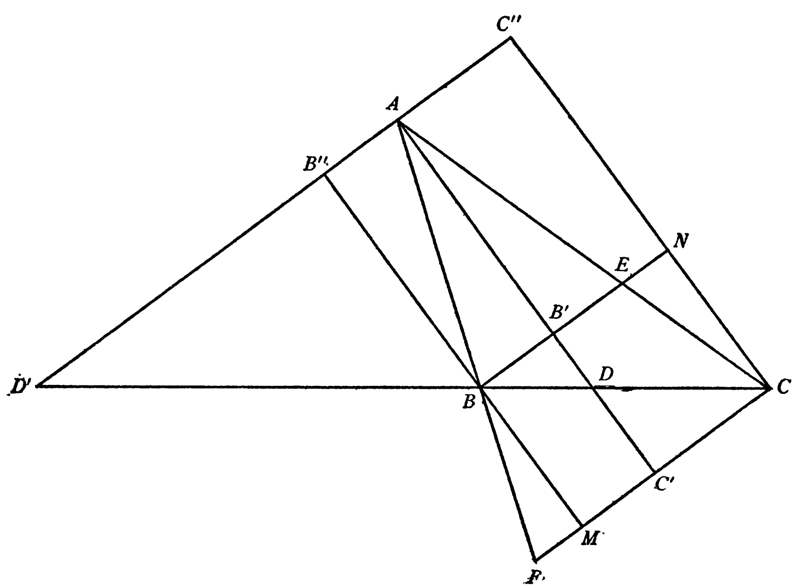
$$\begin{aligned} BB'\cdot CC' &= \tfrac{1}{4}[a^2 - (b-c)^2] \\ &= \tfrac{1}{4}(a+b-c)(a-b+c) = (p-c)(p-b), \end{aligned} \tag{2}$$

$$\begin{aligned} BB''\cdot CC'' &= \tfrac{1}{4}[(b+c)^2 - a^2] \\ &= \tfrac{1}{4}(b+c+a)(b+c-a) = p(p-a), \end{aligned} \tag{3}$$

whence, by virtue of (1),

$$(\text{area } ABC)^2 = p(p-a)(p-b)(p-c).$$

The consideration of the conics $(\mathcal{H})$ and $(\mathcal{E})$ remarkably simplifies the calculation of the products (2) and (3), which can also be obtained directly by the evaluation of the segments BB', BB'', CC', CC'' as functions of the sides a, b, c of triangle ABC.



* *The product of the focal perpendiculars on any tangent to a central conic,* $x^2/\alpha^2 \pm y^2/\beta^2 = 1$, *is constant and equal to* β^2. See almost any analytical geometry text, *e.g.*, Fine and Thompson, p. 85. A synthetic demonstration may be found in Macaulay's Geometrical Conics (2nd ed.), p. 112. (H. Eves)

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944–45

Delta-x, University of the City of Toledo

In the past year, *Delta-x* has maintained a membership of almost ninety despite the large number of members called to the armed forces. Papers of interest to students of mathematics were presented at each of the monthly meetings:

The trisection of angles, by Lois Martin

Greek contributions to mathematics, by Al Philop

A comparative history of mathematics, by Dorothy Wehde and Robert Dancer

Graphic solution of equations, by Richard Reisbach

The dozen system, by Virginia Webber

Mathematics used in production, by Sam Kory

The annual Get-Acquainted Roast opened the social activities of the club early in October. A Hard Times Party was held in November and a Christmas party was combined with the regular December meeting. A pot luck supper took place before the March business meeting. The annual *Delta-x* banquet was held in May followed by a talk entitled:

Solution of diophantine equations, by Dr. Lionel Fleischmann.

Delta-x yearbooks were presented to members at the banquet. Lois Martin was the editor. The club voted to place two concrete benches in front of the University as a memorial to its members in the armed services. The last function of the club was a picnic.

Officers elected for the coming year are: President, Albertine Krohn; Vice-President, Dorothy Wehde; Secretary-Treasurer, Lois Zeigler.

Pi Mu Epsilon, Duke University

Four regular meetings were held during the year 1944–45. At the fall initiation meeting, a talk was given on

The theory of ballistics, by Mr. John Kelly of the Aberdeen Proving Ground.

A tea dance and reception was held before Christmas to welcome the new members. At the spring initiation meeting the membership reached 153. Preceding the initiation the talk was given on

Prime numbers, by Professor Leonard Carlitz.

Election of the following officers for 1944–45 took place at the final meeting: President, Jacqueline Quinn; Vice-President, Jane Ammerman; Secretary, Ethelyn Smith; Treasurer, Joseph King.

Mathematics Club, Case School of Applied Science

At the present time the Mathematics Club has about 25 members who regularly attend the meetings and lectures held under the auspices of this group. Membership is open to all interested students of the college. During the past school year, 1944–45, the following talks were given:

Geometry of the complex plane, by George Springer

Fun with figures, by Gerald Levy

Congruences and congruent numbers, by Robert Smith

The algebra of plane and point sets, by Murray Ellis

Special methods for the use of infinite series in the solution of differential equations, an original paper by Professor Jesse Pierce

Approximate numerical solutions to algebraic equations of degree greater than three, by Robert Gall

Some geometrical constructions and impossibility proofs, by Donald Glaser

Introduction to the algebra of matrices, by George Springer

Some elementary deductions from the special theory of relativity by David Dutton.

In addition to these formal talks which were usually followed by general discussion and comment, several meetings of the club were given over to the discussion of special mathematical problems and their solutions. Many solutions to problems appearing in this MONTHLY were presented at these meetings.

For the past year the officers of the club have been: President, Robert Smith; Vice-President, Donald Glaser; Secretary, Robert Gall; and Sponsor, Professor Max Morris.

Pi Mu Epsilon, Hunter College

The topic discussed in the twelve papers which were presented at eight program meetings was:

The theory of functions of a complex variable, presented by the following members: Dorothy Geddes, Theo Gelbfarb, Gladys Heinlein, Marilyn Hochberg, Grace Lesser, Ann Muzyka, Hannahruth Moses, Leila Rubashkin, Joyce Rubin, Wanda Seglow, Marilynn Spanglet, and Alice Winzer.

Twenty-one students and one faculty member were elected to the chapter during the year. The main social event was the festive Twentieth Anniversary Dinner of the chapter, attended by 125 members at the Hotel Shelton, April 28, 1945, and presided over by Professor Jewell Hughes Bushey, Chairman. The address was entitled

Mathematics in everyday life, by the guest speaker, Professor Oystein Ore of Yale University.

Officers for 1944–45 were: President, Dorothy Geddes; Recording Secretary, Marilynn Spanglet; Corresponding Secretary, Rae Adelson; Treasurer, Mae Perlstein. Officers elected for 1945–46 are: President, Mary Greene; Recording Secretary, Carol Podell; Corresponding Secretary, Sally Rothstein; Treasurer, Leila Rubashkin; Director, Dr. Annita Tuller.

Mathematics Club, Hunter College

The Mathematics Club of Hunter College has had a very active year. The club was fortunate in having three very interesting talks on mathematics:

The theory of games, by Dr. Irving Kaplansky of Columbia University

Order of infinities, by Mr. Ernst Straus, Assistant to Professor Einstein

Mathematics and the social sciences, by Professor Jewell Hughes Bushey, Chairman of the Mathematics Department.

Student reports included:

Mathematics and music

Short cuts in mathematics

The history of measurements

The metric system

The history of calculus

Astronomy.

The club made two trips to the mathematics library of Columbia University, once to see the mathematical instruments and once to visit the Plimpton Library. Other activities of the year included learning how to use the slide rule and the solving of mathematical puzzles. Social activities consisted of a Christmas party and a Spring party. A hike is planned for early in September.

The officers of the club were: President, Carol Podell; Vice-President, Sally Rothstein; Secretary, Wilhelmina Fluhr; Treasurer, Paulette Gross; Publicity Manager, Dorothy Beck. Miss Isabel C. McLaughlin was the Faculty Adviser.

Mathematics Club, University of Wisconsin

In view of a wartime scarcity of advanced graduate students, the various mathematics clubs of the University of Wisconsin were merged into one during the academic year 1944–45, and speakers were asked to keep their talks at a fairly elementary level. This expedient proved happy indeed, and it was found worthwhile to hold meetings at intervals of about two weeks. Except where indicated below, the speakers were graduate students or members of the Mathematics Department. The following papers were presented:

A mathematical theory of traffic light signals, by Professor L. R. Wilcox, of the Illinois Institute of Technology

Functions of random variables, by Professor H. P. Evans

Hamilton and his quaternions, by Professor C. C. MacDuffee

Some properties of orthogonal matrices, by Mr. N. A. Wiegmann

Gibb's phenomenon, by Mr. W. G. Scobert

Current problems, by Professor C. J. Everett

Hamilton's principle, by Miss Gloria Olive

How the Fourier series was not discovered, by Professor R. E. Langer

How the Fourier series was discovered, by Professor R. E. Langer

A theorem of Markhoff, by Mr. L. H. Kanter

Applications of the theory of elasticity to plywood, by Professor H. W. March (on leave at Forest Products Laboratory)

The cluster, a generalization of the ring, by Mr. R. A. Good.

Pi Mu Epsilon, University of California at Berkeley

The California Beta chapter of *Pi Mu Epsilon* has been holding regular meetings during the past year and a quarter, at which many interesting and outstanding papers have been presented, as follows:

Higher curves, illustrated by Pollack models, presented in March 1944 by Mr. R. Wakerling

How to turn a line around, by Professor R. M. Robinson

Mathematical quotations from non-mathematical literature, by Professor Max Zorn of UCLA

Orthogonal polynomials, by A. Horn, former teaching assistant in the Mathematics Department

Surface and capacity of the ellipsoid, by Professor D. H. Lehmer

Competition in economics, by Professor G. C. Evans, Head of the Mathematics Department

Exterior ballistics, by Professor L. Swinford

Problems of runs, by Mr. W. M. Chen

Problems concerning the teaching of mathematics, by Mrs. S. L. McDonald

Just how big should a rocket be? by Dr. Sam Schaaf

Ideal numbers, by Dr. L. Walton

Restricted systems of equations, by Professor A. R. Williams

Finite algebras, by Bjarni Jonsson.

The year's activities were concluded at an initiation banquet, at which Allan Davis presided as toastmaster.

Officers for 1944–45 were: Director, Frank Massey; Vice-Director, Bjarni Jonsson; Secretary-Treasurer, Betty Shapiro; Librarian, Sarah Hallam. Officers for 1945–46 are: Director, Bjarni Jonsson; Vice-Director, Betty Shapiro; Secretary, Esther Bateman; Treasurer, Frank Davis; Librarian, Sarah Hallam.

Kappa Mu Epsilon, Central Michigan College of Education

The activities for the year consisted of eight monthly meetings, including two formal initiation programs and the annual spring picnic. Thirty-one new members were initiated on January 10, and thirty-six on May 2. Attendance was greatly increased by the presence of a Navy V-12 unit on campus. Papers presented during the year were as follows:

The value and determination of pi, by Betty Sack

Aeronautical navigation, by Mr. Lester Serier

Mathematical puzzles, by Pauline Nelson.

The fraternity gave donations to the American National Red Cross, and to the War Student Service Fund. Members are now working on a pamphlet for secondary schools on the mathematics courses students should have in high school in preparation for various vocations, such as engineering, law, medicine, pharmacy, nursing, laboratory technicians, and flying. Joyce Sherwood, graduating president, was given a membership in the National Council of the Teachers of Mathematics.

The officers for the year were: President, Joyce Sherwood; Vice-President, Pauline Nelson; Secretary, Emma May Skinner; Treasurer, Frances Teel; and Corresponding Secretary, Mr. L. C. Gray. Summer officers elected at the May meeting were: President, Wayne Munger; Vice-President, P. T. Austin; Secretary, Sol Jacobson; Treasurer, Mike Schweinburg; and Corresponding Secretary, Miss Nikoline Bye. The officers for the year 1945–46 are: President, Margaret Ketchum; Vice-President, Frances Teel; Secretary, Dorothy Michener; Treasurer, Dorothy Sharard; and Corresponding Sectetary, Mr. Lester Serier.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Statistics. By Hyman Levy and E. E. Preidel. New York, The Ronald Press Company, 1945. 7+184 pages. $2.25.

If any one is so benighted as to be still of the opinion, which apparently had some credence at one time, that anything can be proved by statistics, he will be enlightened by the first paragraph of the introduction to this little book. The authors point out that while it is, of course, possible to fake or doctor evidence, statistics properly used can detect and refute such evidence.

Many books on elementary statistics merely give methods of applying, in a mechanical manner, various statistical operations. This book is not of that type. It gives, by means of good discussions and simple examples, an insight into the meaning and purpose of fundamental statistical concepts, and the reader who carefully studies it will be able to make elementary statistical analyses and inferences in an intelligent fashion. A few exercises are provided at the ends of chapters and elsewhere.

The book deals with classic statistics and contains nothing, for example, about Student's distribution, chi square, or Fisher's analysis of variance. However, it does emphasize the importance of small samples. Also it contains a chapter on quality control.

It covers more material than the number of pages might indicate, since the print is small. This material is part of the work on statistics given at the Imperial College of Science, London, where the authors are located, to students of mathematics, physics, and biology during the past twenty years. Recently it has been added to the curriculum of first-year engineering students at that institution.

P. R. RIDER

Plane and Spherical Trigonometry. By F. M. Morgan. New York, American Book Company, 1945. 5+243+72 pages. $2.50.

As is stated in the preface "this brief presentation of Plane and Spherical Trigonometry emphasizes the numerical aspect and gives as much theory as is necessary for a thorough preparation for further work in mathematics." The book has been written with great care and considerable pedagogical insight. The numerous applications to mechanics and navigation will help to motivate the student to the study of trigonometry by impressing upon him the timeliness of the subject. The occasional historical notes will give him some knowledge of its development. The text as well as the problem lists contain many thought provoking questions. The problem lists contain tests of various types: drill exercises, true and false tests, completion tests and mastery tests.

In addition to the material commonly found in trigonometry texts, this book contains a chapter on DeMoivre's theorem, a section on sailing, and the Mercator map. The trigonometric functions of an acute angle are first defined, and sufficient applications to the solutions of the right triangle are given to familiarize the student before the functions of the general angle are introduced. The idea of the polar triangle is used to reduce the number of cases for the solution of the oblique spherical triangle. The Appendix contains a chapter on logarithms. Five place tables of logarithms of numbers, of logarithms of trigonometric functions, of logarithmic and natural haversines, and of natural trigonometric functions are included in the book. No list of answers to the problems is given.

The entire book is remarkably well written and the reviewer recommends it highly for use in high schools and colleges. The type and presswork are excellent.

H. P. THIELMAN

NEW BOOKS RECEIVED

Applied Mathematics for Radio and Communication Engineers. By C. E. Smith. New York and London, McGraw-Hill Book Co., Inc., 1945. 10+336 pages. $3.50.

An Historical and Analytical Bibliography of the Literature of Cryptology. By J. S. Galland. Evanston, Northwestern University, 1945. 8+209 pages. $5.00.

Hydrodynamics. By Horace Lamb. (6th edition; reprint). New York, Dover Publication, 1945. 15+738 pages. $4.95.

An Introduction to Mathematics for Teachers. By L. E. Boyer. New York, Henry Holt and Co., 1945. 17+478 pages. $3.25.

Note by the editor. A recent review of *The Education of T. C. Mits* (this MONTHLY, vol. 52, 1945, p. 270) might lead the reader to infer that the book was written by H. G. Lieber, with L. R. Lieber supplying the drawings. The purpose of the present note is to correct this impression; the text is due to L. R. Lieber and the drawings are by H. G. Lieber. H. P. E.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 691. *Proposed by E. K. Paxton, Washington and Lee University*

The top of a certain bridge is 215 feet above the ground and 100 feet wide. Neglecting air resistance, find the minimum initial velocity with which an arrow is shot in order to go over the bridge. At what distance from the base must the archer stand? What is the angle of elevation? How high does the arrow go? (Assume the arrow leaves the bow at a height of 5 feet above the ground.)

E 692. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider $2p+1$ consecutive integers. Show that: (1) the product of the first p of them diminished by the product of the last p gives an integer divisible by $p(p+1)$; (2) the product of the first p increased by the product of the last p gives an integer divisible by $p(p+1)$, unless $p+1$ is a prime factor of the remaining integer of the set.

(This is a generalization of problem 203 by Fitz-Patrick, p. 32 of G. de Longchamps' *Exercises d'arithmétique.*)

E 693. *Proposed by N. A. Court, University of Oklahoma*

Through a given point draw a line meeting three given planes in three points so that the anharmonic ratio of the four points shall have a given value.

E 694. *Proposed by J. D. Bankier, Sedbergh School, Montebello, Province of Quebec*

A sequence $\{x_n\}$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{n-1}{n} x_{n-1} + \frac{1}{n} x_{n-2}.$$

To what value does the sequence converge?

E 695. *Proposed by H. L. Lee, University of Tennessee*

Find triangles whose sides are integers in arithmetic progression, and whose areas are integers.

SOLUTIONS

A Generalization of E 255

E 657 [1945, 95]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the locus of centers of spheres passing through two given points and touching a given sphere.

Solution by Howard Eves, College of Puget Sound. Let O be the center and r the radius of the given sphere, let A and B be the two given points, $\angle OAB=\theta$, and let P be a point on the required locus. If the value of r lies between OA and OB, we easily see that there is no real locus. So we consider in turn the two other possibilities.

(a) Suppose both OA and OB are greater than r. Then we must have $|PO-PA|=r$, and one surface on which P lies is a hyperboloid of revolution of two sheets with foci O and A. The rotation angle, ϕ, of the asymptotic cone of the hyperboloid, is given by $\cos\phi=r/OA$. But P lies also on the plane perpendicularly bisecting the segment AB. Therefore the locus of P is

(1) a circle if $\theta=0$ or π,

(2) an ellipse if $0<\theta<\frac{1}{2}\pi-\phi$ or $\frac{1}{2}\pi+\phi<\theta<\pi$,

(3) a parabola if $\theta=\frac{1}{2}\pi\pm\phi$,

(4) a hyperbola if $\frac{1}{2}\pi-\phi<\theta<\frac{1}{2}\pi+\phi$.

(b) Suppose both OA and OB are less than r. Here $PO+PA=r$, and the locus of P is the intersection of an ellipsoid of revolution and a plane, *viz.*,

(1) a circle if $\theta=0$ or π,

(2) an ellipse otherwise.

Also solved by J. M. Feld, H. L. Lee, P. D. Thomas, J. T. Webster, and the proposer. N. A. Court draws attention to his *Modern Pure Solid Geometry* p. 196, art. 614.

The Conformal Points of a Projection

E 661 [1945, 159]. *Proposed by Howard Eves, College of Puget Sound*

A plane p is projected from a point L onto a plane p'. Find those points on p for which all angles on p having such a point for vertex are invariant under the projection.

Solution by the Proposer. We adopt the convention that angular directions on p (or p') are positive if they are counterclockwise when p (or p') is viewed from point L. We now make the

DEFINITIONS. A point on p will be called a *positive isocenter* or a *negative isocenter* on p according as all the angles on p having the point for vertex project respectively into equal and similarly directed or equal and oppositely directed angles on p'.

By an elementary synthetic treatment we shall show that when p and p' are nonparallel there is one and only one positive isocenter and one and only one negative isocenter on p. We shall also geometrically locate these isocenters.

THEOREM 1. *If P is a positive (or negative) isocenter on p, then its image, P', is a positive (or negative) isocenter on p'.*

THEOREM 2. *If p and p' are parallel, then every point on p is a positive or a negative isocenter according as L does not or does lie between p and p'.*

THEOREM 3. *If p and p' are not parallel, then p cannot have two positive isocenters.*

For suppose P and Q are two positive isocenters on p. Draw any circle s on p passing through P and Q, and let R be any other point on s. Then, since $\angle QPR = \angle Q'P'R'$ and $\angle PQR = \angle P'Q'R'$, it follows that $\angle PRQ = \angle P'R'Q'$, whence s projects into a circle s'. Thus all circles on p through P and Q project into circles on p' through P' and Q'. But this is impossible. For consider a circle on p, passing through P and Q and cutting the vanishing line on p. This circle must project into a hyperbola. Thus the original assumption of the existence of two positive isocenters is incorrect.

THEOREM 4. *Let p and p' be nonparallel and let O be the foot of the perpendicular from L onto p, and let V on p be the image of V', the foot of the perpendicular from L onto p'. Then any line on p perpendicular to VO is parallel to any line on p' perpendicular to $V'O'$.*

Draw VQ on p perpendicular to VO. Then

$$LQ^2 = LO^2 + OQ^2 = LO^2 + VO^2 + VQ^2 = LV^2 + VQ^2.$$

Therefore VQ is perpendicular to LV, and therefore to plane VLO. Hence any line on p perpendicular to VO is perpendicular to plane VLO. Similarly, any line on p' perpendicular to $O'V'$ is perpendicular to plane VLO. This proves the theorem.

THEOREM 5. *If p and p' are nonparallel, then p has exactly one positive isocenter, and it is the intersection with p of the bisector of the angle formed by the perpendiculars dropped from L onto p and p'.*

We adopt the notation of theorem 4. Let P be any point on p and let W be the foot of the perpendicular from P on VO. Then W' is the foot of the perpendicular from P' on $V'O'$ and, by Theorem 4, PW is parallel to $P'W'$. Let I be the intersection with VO of the bisector of angle VLO. Draw WK perpendicular to LV to cut LV in K and LI in J. Then

$$IW/I'W' = JW/I'W' = LW/LW' = WP/W'P'.$$

Hence $\angle WIP = \angle W'I'P'$. Similarly, if Q is any other point on p, $\angle QIW = \angle Q'I'W'$. Hence $\angle QIP = \angle Q'I'P'$, and I is a positive isocenter on p. By theorem 3, I is the only positive isocenter on p.

THEOREM 6. *If p and p' are nonparallel, then p has exactly one negative isocenter, and it is the intersection with p of the external bisector of the angle formed by the perpendiculars dropped from L on p and p'.*

We may give a proof of this theorem analogous to that given for theorem 5.

Note. The positive isocenter has long been known and employed in photogrammetry (the science of surveying by means of photographs), but the negative isocenter seems to have escaped notice. For an analytical treatment of the above and for an application to photogrammetry see "Analytical and Graphical Rectification of a Tilted Photograph," by Howard Eves, *Photogrammetric Engineering*, April-May-June issue, 1945.

Two Scales of Notation

E 662 [1945, 159]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A number is represented by a in the scale of α and by b in the scale of β ($\beta<\alpha$). Regarding both a and b as written in the scale of α, we write the difference $b-a=c$. Show how to determine the greatest possible value of a for given values of α, β, c; *e.g.*, when $\alpha=10$, $\beta=7$, and $c=3501$.

Solution by R. C. Buck, Harvard University. We have

$$N = \sum a_k\alpha^k = \sum b_k\beta^k$$

and

$$c = \sum b_k\alpha^k - \sum a_k\alpha^k = \sum b_k(\alpha^k - \beta^k).$$

Set $\gamma_k=\alpha^k-\beta^k$, $k=0, 1, 2, \cdots$. If c is expanded in terms of the γ_k, choosing always the greatest values for the coefficients of $\gamma_n, \gamma_{n-1}, \cdots$, in order, then the maximum a will be found. Thus

$$b_n = [c/\gamma_n],\ b_{n-1} = [(\text{remainder } c/\gamma_n)/\gamma_{n-1}], \cdots, b_0 = \beta - 1.$$

In the special case where $\alpha=10$, $\beta=7$, $c=3501$ we find

$$\{\gamma_k\} = \{0, 3, 51, 657\}, \qquad \{\beta_k\} = \{6, 4, 4, 5\}.$$

Therefore

$$a = 5(7^3) + 4(7^2) + 4(7) + 6(1) = 1945.$$

Also solved by Murray Barbour, D. H. Browne, F. M. Carpenter, Robert Hoskins, E. E. Osborne, E. D. Schell, E. P. Starke, and the proposer

Two Powers Differing by Unity

E 663 [1945, 159]. *Proposed by Irving Kaplansky, Columbia University*

If $2^n+1=p^r$, where p is a prime, prove that r is a power of 2 (including the possibility $r=2^0=1$).

I. *Solution by Joseph Rosenbaum, Bloomfield, Connecticut.* The supposition that $r=hk$, where k is odd and greater than 1 gives

$$(p^h - 1)[(p^h)^{k-1} + (p^h)^{k-2} + \cdots + p^h + 1] = 2^n.$$

This is impossible because the bracket is also odd and greater than 1. Hence $r = 2^m$.

However, when $r = 2^m$, the given equation can be written as

$$(p^{2^{m-1}} - 1)(p^{2^{m-1}} + 1) = 2^n,$$

and it is easily proved that this is possible only when $m = 0$ or 1, and that for the latter situation $p = 3$.

It is also seen that the hypothesis that p is a prime is superfluous.

II. *Solution by D. W. Alling, Rochester, New York.* We have $p^r \equiv 1 (\text{mod } 2^n)$, and it is clear that p belongs to the exponent r mod 2^n. But $\phi(2^n) = 2^{n-1}$ and $r | 2^{n-1}$. Hence r is a power of 2. If $r > 1$, then r is even and the integral factorization

$$(p^{r/2} - 1)(p^{r/2} + 1) = 2^n$$

is possible. Therefore

$$p^{r/2} - 1 = 2^a, \qquad p^{r/2} + 1 = 2^b, \qquad 2^a - 2^b = -2.$$

Solving we find, uniquely, $b = 2, a = 1, n = 1$.

Also solved by D. W. Alling (another solution), Murray Barbour, D. H. Browne, R. C. Buck, A. Charnes, Roy Dubisch, Paul Erdös, N. J. Fine, J. B. Kelly, E. D. Schell, Peter Scherk (two ways), E. P. Starke, and the proposer (two ways).

The solutions of Murray Barbour, Roy Dubisch, N. J. Fine, J. B. Kelly, E. D. Schell, and one of the solutions of the proposer are essentially like solution I above. The alternate solution of D. W. Alling and the solution of D. H. Browne utilize the fact that all primes are of the form $4k \pm 1$. The proposer offered a second solution using the theory of Galois groups. R. C. Buck established the more general theorem: If $q^n + 1 = a^r$, *q prime, then* $r = 1$ *except for the special case* $2^3 + 1 = 3^2$. *If, further, a is prime, then the only solutions are the Fermat primes,* $p = 2^{2^n} + 1$. Peter Scherk and Paul Erdös solved essentially Buck's extension. Most solvers noted the uniqueness of solution when $r > 1$, and several observed that p need not be restricted to a prime.

An Integral Related to the Gamma Function

E 664 [1945, 159]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that if $x < 1$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \int_1^{\infty} t^n e^{-t} dt = \frac{e^{x-1}}{1 - x}.$$

I. *Solution by N. J. Fine, Purdue University.* The problem should be stated with the condition $|x| < 1$. Set $B_n = 1/n! \int_1^\infty t^n e^{-t} dt$. $B_0 = e^{-1}$ and an integration by by parts shows that $B_n = e^{-1}/n! + B_{n-1}$, so $B_n = e^{-1} \sum_{k=0}^{n} (1/k!)$. Hence, if $|x| < 1$,

$$\sum_{n=0}^{\infty} B_n x^n = e^{-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^n}{k!} = e^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=k}^{\infty}x^n = e^{-1}\sum_{k=0}^{\infty}\frac{x^k}{k!}\cdot\frac{1}{1-x} = \frac{e^{x-1}}{1-x}.$$

II. *Solution by Harley Flanders, University of Chicago.* The problem should be stated with the condition $|x| < 1$. Set $c_n = \int_1^\infty t^n e^{-t}dt$. Then we have

$$n! = \int_0^\infty t^n e^{-t}dt > c_n = \int_0^\infty t^n e^{-t}dt - \int_0^1 t^n e^{-t}dt > n! - \int_0^1 e^{-t}dt = n! - 1 + e^{-1}.$$

Therefore

$$\sum_{n=0}^{\infty} x^n > \sum_{n=0}^{\infty} (c_n x^n)/n! > \sum_{n=0}^{\infty} [1 - (1 - e^{-1})/n!]x^n.$$

By the "ratio test" we see that both extreme series, and hence the given series, converge in the interval $|x| < 1$. Hence the given series converges uniformly in that interval by a known theorem on power series and we may interchange the operations:

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\int_1^\infty t^n e^{-t}dt = \int_1^\infty\left[\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right]e^{-t}dt = \int_1^\infty e^{(x-1)t}dt = \frac{e^{x-1}}{1-x},$$

(since $x-1<0$).

Also solved by D. W. Alling, Murray Barbour, Ellen Buck, Sidney Glusman, J. E. Hanson, J. B. Kelly, E. E. Osborne, E. P. Starke, C. W. Topp, and J. T. Webster.

J. B. Kelly obtained his solution from results of E 654. E. P. Starke showed, as an incidental note to his solution, that

$$ec_n/n! < e < (ec_n + 1)/n!,$$
$$ec_1 = 2, \qquad ec_{n+1} = 1 + (n+1)ec_n.$$

Since the integers ec_n are easy to compute we then have here a novel way of computing the numerical value of e.

Circles Covering a Given Curve

E 665 [1945, 159]. *Proposed by L. A. Santalo, Rosario, Argentina*

Let C be a closed convex plane curve with continuous radius of curvature R. Let R_M be the greatest value of R. Given $\lambda \geqq R_M$, show that the area F_λ covered by the centers of circles of radius λ which contain C in their interior is given by

$$F_\lambda = F - L\lambda + \pi\lambda^2,$$

where L and F are the length and area of C.

Solution by R. A. Rosenbaum, U.S.N.R. Problem E 630 [1945, 160] can be easily generalized so as to include the present problem. The generalization of E 630 is:

For any point P of a given closed convex curve C, let P' be that point on the normal to C at P for which $PP'=k$, a constant, taken as positive or negative according as P' lies on the exterior or the interior normal. The locus of P' is a curve C'. Let s, s' be the respective lengths of C, C', and A, A' the areas of these curves. Then

$$s' = s + 2\pi k,$$
$$A' = A + sk + \pi k^2.$$

The proof as given for E 630 applies to this generalization, and the present problem is seen to be a special case.

Also solved by R. H. Wilson, Jr. and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4176. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove the following two theorems in affine geometry of three dimensions:

(a) If all the faces of a convex polyhedron are parallelograms, their number is the product of two consecutive integers;

(b) If each face of a convex polyhedron has a center of symmetry, the whole polyhedron has a center of symmetry.

4177. *Proposed by P. R. Halmos, Syracuse University*

What is the smallest number of people sufficient to ensure that the probability that there be at least two with the same birthday is at least 1/2? It is to be assumed that any two days of the year are equally likely to be the birthday of a given individual and that there are no leap years.

4178. *Proposed by N. A. Court, University of Oklahoma*

If among the twelve points of intersection of a quadric surface with the edges of a tetrahedron there are six points located on the six edges, such that the three lines joining the points on opposite edges are concurrent, the remaining six points of intersection also have this property.

Note. This is a converse of a known proposition (see, for inst. Baker, *Principles of geometry*, vol. 3, p. 54, ex. 17, 1923): The transversal is drawn from a

point O to each pair of opposite joins of four points A, B, C, D, so giving rise to six points. Six other points, one on each join, are similarly obtained from another point O'. The twelve points lie on a quadric.

4179. *Proposed by J. R. Musselman, Western Reserve University*

The poles of the medians of a triangle $A_1A_2A_3$ as to its circumcircle are three points of a line; those points where the external bisectors of the angles of the tangential triangle of $A_1A_2A_3$ meet the opposite sides of this tangential triangle.

4180. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The product of k positive integers whose sum is N, where $N=kp+h$, is a maximum when h of the factors are equal to $p+1$ and the $k-h$ others to p.

Dedicated to E. P. Starke.

SOLUTIONS

Trigonometric Determinants

4125 [1944, 352]. *Proposed by Hüseyin Demir, Columbia University*

Prove that

$$\begin{vmatrix} \sin\theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots 0 & 0 \\ \sin\theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots 0 & 0 \\ \sin\theta_3 & 0 & e^{i\theta_3} & -e^{-i\theta_3} & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sin\theta_n & 0 & 0 & 0 & \cdots 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1+\theta_2+\cdots+\theta_n).$$

Solution by Mary L. Boas, Tufts College. Put each $\sin\theta=(e^{i\theta}-e^{-i\theta})/2i$ and remove the factor $1/2i$ outside the determinant. Subtract from each element of the first column the sum of all the other elements in its row. The determinant then becomes

$$\frac{1}{2i}\begin{vmatrix} e^{i\theta_1} & -e^{-i\theta_1} & 0 & \cdots 0 & 0 \\ 0 & e^{i\theta_2} & -e^{-i\theta_2} & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots e^{i\theta_{n-1}} & -e^{-i\theta_{n-1}} \\ -e^{-i\theta_n} & 0 & 0 & \cdots 0 & e^{i\theta_n} \end{vmatrix}.$$

Expand by elements of the first column. The minor of $e^{i\theta_1}$ is $e^{i(\theta_2+\theta_3+\cdots+\theta_n)}$ since all elements below the main diagonal of this minor are zero. The minor of $-e^{-i\theta_n}$ is $(-1)^{n-1}e^{-i(\theta_1+\theta_2+\cdots+\theta_{n-1})}$ since all elements above its main diagonal are zero. Therefore the determinant equals

$$\frac{1}{2i}\left[e^{i(\theta_1+\theta_2+\cdots+\theta_n)}-e^{-i(\theta_1+\theta_2+\cdots+\theta_n)}\right]=\sin(\theta_1+\theta_2+\cdots+\theta_n).$$

Solved also by E. F. Allen, Murray Barbour, C. B. Barker, Jr., Shepard Bartnoff, R. P. Boas, Jr., Mrs. R. C. Buck, Howard Eves, Clifford Gardner, P. C. Hammer, R. Hamming, J. F. Hofmann, L. M. Kelly, E. Lukacs, Norman Miller, Henry Nelson, Ivan Niven, H. N. Shapiro, Robert Steinberg, R. H. Wilson, Jr., and the proposer.

Editorial Note. About half of the solutions used induction proofs and about the same number used simple determinant transformations without induction. Hammer considered the transformation of the determinant and its value by replacing θ_j by $\pi/2-\theta_j$ which gives after reduction a determinant with $a_{j1}=\cos\theta_j$ in the first column and the principal diagonal $\cos\theta_1$, $e^{-i\theta_2}$, $e^{-i\theta_3}$, $\cdots$ and the parallel above it $e^{i\theta_1}$, $e^{i\theta_2}$, $e^{i\theta_3}$, $\cdots$ with zeros in the remaining places. He found for the value of the determinant $\cos\sum\theta_j$ if n is odd, and $-i\sin\sum\theta_j$ if n is even. A simpler procedure is to make the same change in the first column but to alter the principal diagonal to $\cos\theta_1$, $-e^{i\theta_2}$, $-e^{i\theta_3}$, $\cdots$, $-e^{i\theta_n}$ and leave the rest of the original determinant unaltered. The value of this determinant is the same as that for Hammer's determinant.

Spherical Loxodromes

4128 [1944, 409]. *Proposed by C. E. Springer, University of Oklahoma*

Consider the tangent planes to a sphere at three points A, B, C of a curve lying on the sphere. Let R be the limiting point of intersection of the planes as B and C move independently along the curve and approach coincidence with A. Each curve on the sphere through A has its corresponding R point. Prove that the curves through A, the locus of whose R points is a certain straight line lying in the tangent plane to the sphere at A, are the loxodromes through A.

Solution by the Proposer. The tangent plane at the point $A(u, v)$ on the sphere represented by

$$x=\sin u\cos v,\qquad y=\sin u\sin v,\qquad z=\cos u \tag{1}$$

has the equation

$$x\cos v+y\sin v+z\cot u=\csc u. \tag{2}$$

The radius of the sphere is taken as unity for convenience, without loss of generality. If the curve through A is represented by $u=u(s)$, $v=v(s)$, (the parameter s denoting length of arc), then the coordinates of its "R" point satisfy equation (2), together with equations

$$x\sin v\cdot v'-y\cos v\cdot v'+z\csc^2 u\cdot u'=\csc u\cot u\cdot u' \tag{3}$$

and

$$\begin{aligned} &x(\cos v\cdot v'^2+\sin v\cdot v'')+y(\sin v\cdot v'^2-\cos v\cdot v'')\\ &\qquad+z(\csc^2 u\cdot u''-2\csc^2 u\cot u\cdot u'^2)\\ &\qquad=\csc u[\cot u\cdot u''-u'^2(\csc^2 u+\cot^2 u)].\end{aligned} \tag{4}$$

Primes indicate differentiation with respect to s.

In solving equations (2), (3), (4), it is to be noted that the coordinates of R are given by

$$(5) \qquad x = -\frac{u'}{v'}\sin v \sec u, \qquad y = \frac{u'}{v'}\cos v \sec u, \qquad z = \sec u,$$

if, and only if, the functions $u(s)$, $v(s)$ satisfy the differential equation

$$(6) \qquad u'v'' - u''v' + u'^2v' \cot u = 0.$$

For all values of u'/v' the point R given by (5) lies on the line of intersection of the planes

$$(7) \qquad x\cos v + y \sin v = 0, \qquad z = \sec u,$$

and this line lies in the tangent plane to the sphere at A.

Equation (6) can be written in the form

$$(8) \qquad \frac{d^2v}{du^2} + \cot u \frac{dv}{du} = 0.$$

The curves whose equations satisfy (8) are given by

$$(9) \qquad v = c_1 \log \tan \frac{u}{2} + c_2$$

where c_1 and c_2 are constants. Equation (9) represents the loxodromes on the sphere. Therefore, the loxodromes through a point on a sphere have the property that the locus of their "R" points is a straight line lying in the tangent plane to the sphere at the point.

Note: The Proposer has published the differential equation defining *dual geodesics* on a general surface in metric space—*Dual Geodesics on a Surface*, Bull. Amer. Math. Soc., vol. 48, No. 12, pp. 901–906, December, 1942. For the unit sphere, this differential equation reduces to equation (8). Therefore, the result of this problem can be stated as follows: The dual geodesics on a sphere are the loxodromes.

Editorial Note. The curve γ lies on the sphere (O) with center O and unit radius, and passes through the points A, B_1, B_2; the plane $[A, B_1, B_2]$ through these three points meet (O) in a circle with center $\overline{C}$, the foot of the perpendicular from O to this plane. In the limit where $B_1, B_2 \rightarrow A$ the plane $[A, B_1, B_2] \rightarrow \pi$, the osculating plane for γ at A; the circle with center $\overline{C} \rightarrow$ the circle of plane curvature for γ at A with center C where OC is perpendicular to π. Let $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ be the unit vector tangent, principal normal, and binormal at A, forming a right hand system, then $\pi = [\mathbf{t}, \mathbf{n}]$ and OC is parallel to $\mathbf{b}$.

The tangent planes to (O) at A, B_1, B_2 meet in a point $\overline{\overline{C}}$, and $\overline{C}$, $\overline{\overline{C}}$ are inverse points re (O). Hence $\overline{\overline{C}} \rightarrow R$ and C, R are inverse points re (O); C lies on (OA)

the sphere with diameter OA, and this sphere is the inverse of the tangent plane to (O) at A. Now suppose that there is an infinite set of curves γ on (O) passing through A such that there is an infinite set of points R lying on a straight line l in the tangent plane at A, that is they have the R property. Then the plane $[l, O]$ meets (OA) in a fixed circle, the inverse of l, and C lies on this fixed circle and AC moves on a fixed cone with vertex A. For a chosen tangent $\mathbf{t}$ at A, the plane through OA perpendicular to $\mathbf{t}$ meets the fixed circle again in C where the vector $\overrightarrow{AC}=\rho\mathbf{n}$, $\rho=1/\kappa$, $\sigma=1/\tau$, κ, τ being the curvature and torsion for the γ considered. The position of $\mathbf{t}$ in the tangent plane is fixed by the angle α with an initial line defined below. Thus at A, $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, ρ, $\overrightarrow{OC}=\mathbf{c}$ are determined functions of α which enters only in trigonometric form; and two different values of α give different values to each of these five quantities.

The equations (1), (2), (3) in the solution of 4105 [1945, 221] give

$$(\sigma\rho')^2+\rho^2=1, \qquad (\sigma\rho')'+\tau\rho=0, \qquad \mathbf{c}=-\sigma\rho'\mathbf{b}. \tag{1}$$

Set $\bar{\mathbf{t}}=\mathbf{r}\times\mathbf{t}$, then by the methods of 4105 we get

$$\mathbf{n}=-\sigma\rho'\bar{\mathbf{t}}-\rho\mathbf{r}, \qquad \mathbf{b}=\rho\bar{\mathbf{t}}-\sigma\rho'\mathbf{r}. \tag{2}$$

With the parametric coordinates u, v in the above solution we have

$$d\mathbf{r}=\mathbf{r}_1du+\mathbf{r}_2dv, \quad (ds)^2=(du)^2+\sin^2 u(dv)^2, \quad \mathbf{r}_1=\mathbf{t}_1, \quad \mathbf{r}_2=\sin u\mathbf{t}_2, \tag{3}$$

with the origin of vectors at 0 and $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{r}$ form a right-hand orthogonal system of unit vectors. We may write

$$\begin{gathered}\mathbf{t}=\mathbf{t}_1\cos\alpha+\mathbf{t}_2\sin\alpha, \qquad \bar{\mathbf{t}}=-\mathbf{t}_1\sin\alpha+\mathbf{t}_2\cos\alpha,\\ u'=du/ds=\cos\alpha, \qquad v'=\sin\alpha\csc u, \qquad dv/du=\tan\alpha\csc u.\end{gathered} \tag{4}$$

For any curve on (O) it easily follows from a figure that

$$\mathbf{r}_{11}=-\mathbf{r}, \qquad \mathbf{r}_{12}=\cot u\mathbf{r}_2, \qquad \mathbf{r}_{22}=-\sin u(\mathbf{r}\sin u+\mathbf{r}_1\cos u). \tag{5}$$

Computation then gives

$$\kappa\mathbf{n}=-\mathbf{r}+(\alpha'+\sin\alpha\cot u)\bar{\mathbf{t}}, \qquad \kappa\mathbf{b}=(\alpha'+\sin\alpha\cot u)\mathbf{r}+\bar{\mathbf{t}}. \tag{6}$$

In order for the system of curves γ to have the R property at A, the vector $\kappa\mathbf{b}$ with its origin at O must lie in a fixed plane. This requires that α' be a linear homogeneous function of $\sin\alpha$ and $\cos\alpha$, $\alpha'=d_1\sin\alpha+d_2\cos\alpha$, where d_1 and d_2 may be functions of u and v. The fixed plane is perpendicular to the vector

$$d=(d_1+\cot u)\mathbf{t}_1-d_2\mathbf{t}_2+\mathbf{r}. \tag{7}$$

A system of curves γ which has the R property at each point of a member satisfies the differential equation

$$\frac{d^2v}{du^2}=d_1\sin^2 u\left(\frac{dv}{du}\right)^3+d_2\sin u\left(\frac{dv}{du}\right)^2+(d_1-\cot u)\frac{dv}{du}+d_2\csc u. \tag{8}$$

The simplest case is where the line l is the line at infinity in the tangent plane; the fixed circle reduces to the point O, and the curves γ are the geodesics, *i.e.*, great circles. Here $d_1 = -\cot u$, $d_2 = 0$. Another special case is where $\alpha' = 0$, $d_1 = d_2 = 0$, so that each curve γ has a constant α at each of its points: hence they are the loxodromes.

A Probability Identity

4130 [1944, 409]. *Proposed by J. R. Musselman, Western Reserve University*

Show that

$$\sum_{j=1}^{n} {}_nC_j \frac{(-1)^{j-1}}{j} = \sum_{j=1}^{n} \frac{1}{j}.$$

I. *Solution by G. B. Lang, University of Florida.* Denote the sum on the left by $f(n)$, then

$$\begin{aligned}\Delta f(\mathrm{n}) &= f(n+1) - f(n) = \frac{(-1)^n}{n+1} + \sum_{j=1}^{n} [{}_{n+1}C_j - {}_nC_j] \frac{(-1)^{j-1}}{j} \\ &= \frac{(-1)^n}{n+1} + \sum_{j=1}^{n} {}_nC_{j-1} \frac{(-1)^{j-1}}{j} = \frac{1}{n+1}\left[(-1)^n - \sum_{j=1}^{n} {}_{n+1}C_j(-1)^j \right] \\ &= \frac{1}{n+1} [-(-1)^{n+1} - (1-1)^{n+1} + (-1)^{n+1} + 1].\end{aligned}$$

Hence $\Delta f(n) = 1/(n+1)$, and summation of each member of this equation from $n=1$ to $n-1$, noting that $f(1)=1$, gives the desired result.

II. *Solution by G. T. Williams, Student, Harvard University.* Set

$$f(x) = \sum_{\nu=1}^{n} (-1)^{\nu-1} \binom{n}{\nu} \frac{x^\nu}{\nu}$$

then

$$\begin{aligned}f'(x) &= -\frac{1}{x}\left\{ -1 + \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x^\nu \right\} = \frac{1-(1-x)^n}{1-(1-x)} \\ &= \sum_{\nu=1}^{n} (1-x)^{\nu-1}.\end{aligned}$$

Then

$$f(x) = -\sum_{\nu=1}^{n} \frac{(1-x)^\nu}{\nu} + c, \qquad f(0) = 0 = -\sum_{\nu=1}^{n} \frac{1}{\nu} + c, \qquad f(1) = c,$$

$$f(1) = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{\nu} \binom{n}{\nu} = \sum_{\nu=1}^{n} \frac{1}{\nu}.$$

Here is a related inversion formula which may be of interest:

$$g(n) = \sum_{\nu=c}^{n} (-1)^{\nu} \binom{n}{\nu} f(\nu), \qquad n = c, c+1, c+2, \cdots,$$

$$f(n) = \sum_{\nu=c}^{n} (-1)^{\nu} \binom{n}{\nu} g(\nu), \qquad n = c, c+1, c+2, \cdots .$$

The proof depends upon the following Lemma:

$$\sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{s\nu}{\mu} = 0, \qquad \text{for } \mu < n,$$
$$= (-1)^n s^n, \quad \text{for } \mu = n.$$

Proof of the Lemma

$$\{(1+x)^s - 1\}^n = \left\{ sx + \binom{s}{2} x^2 + \cdots \right\}^n = s^n x^n + a_1 x^{n+1} + \cdots$$
$$= \sum_{\nu=0}^{\infty} \binom{n}{\nu} (-1)^{n-\nu} \sum_{\mu=0}^{\infty} \binom{s\nu}{\mu} x^{\mu}$$
$$= \sum_{\mu=0}^{\infty} (-1)^n x^{\mu} \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{n}{\nu} \binom{s\nu}{\mu}.$$

Since n is the lowest power of x the proof is complete. The proof of the inversion formula is as follows, taking $s=1$:

$$\sum_{\nu=c}^{n} (-1)^{\nu} \binom{n}{\nu} \sum_{\mu=c}^{\nu} (-1)^{\nu} \binom{\nu}{\mu} f(\mu) = \sum_{\mu=c}^{n} (-1)^{\mu} f(\mu) \sum_{\nu=\mu}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{\nu}{\mu}$$
$$= (-1)^n f(n) \binom{n}{n} \binom{n}{n} (-1)^n = f(n).$$

This completes the proof.

A related formula is as follows

$$\sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{2\nu}{\mu} = (-1)^n \binom{n}{\mu - n} 2^{2n-\mu}.$$

Proof

$$(x^2 + 2x)^n = x^n \sum_{\nu=0}^{n} \binom{n}{\nu} x^{\nu} 2^{n-\nu} = \sum_{\mu=n}^{2n} \binom{n}{\mu - n} x^{\mu} 2^{2n-\mu}.$$

Setting $s=2$ in the Lemma and equating coefficients of x the desired result follows.

Solved also by T. W. Anderson, Jr., Shepard Bartnoff, J. R. Britton, Howard Eves, A. M. Glicksman, G. Grossman, J. F. Hofmann, M. Home,

J. B. Kelly, M. S. Macphail, C. D. Olds, G. W. Petrie, E. D. Schell, N. C. Scholomiti, E. R. Smith, G. Szegö, and the proposer.

Editorial Note. Macphail and Scholomiti's solutions are essentially as follows: Set $(x+1)^{(-j)} = 1/(x+1)(x+2)\cdots(x+j)$, then $\Delta^m(x+1)^{(-1)} = (-1)^m m!/(x+1)^{(-m-1)}$; and for $x=0$, we have $\Delta^m(0+1)^{(-1)} = (-1)^m m!/(m+1)! = (-1)^m/(m+1)$. Hence

$$\sum_{x=0}^{n-1} \frac{1}{x+1} = \left(\sum_{x=0}^{n-1} U^x\right)\frac{1}{0+1} = \frac{U^n-1}{U-1}(0+1)^{(-1)} = \frac{(\Delta+1)^n-1}{\Delta}(0+1)^{(-1)}$$

$$= \sum_{j=1}^{n} {}_nC_j\Delta^{j-1}(0+1)^{(-1)} = \sum_{j=1}^{n} {}_nC_j(-1)^{j-1}/j.$$

This symbolic reasoning may be reversed.

Schell stated that this problem appears in Chrystal's Algebra, vol. 2, p. 9, Ex. 18; and Olds stated that a solution was given by Greenstreet, *Mathesis*, ser. 2, vol. 1, 1891, p. 104 using a generating function; also in Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 6, prob. 38. About half of the remaining solvers used a generating function and the rest used difference methods in various forms.

This problem is related to the probability average problem 3652 [1933, 610], proposed by A. F. Stevenson, University of Toronto, and solved 1935, pp. 118–123, which is as follows:

The practice of certain cigarette manufacturers of supplying playing cards, poker hands, *etc.*, with their cigarette packets, and of offering various articles in exchange for complete set of these cards, suggests the following problem:

Assuming that each packet of cigarettes contains one of a set of 52 cards, and that these cards are distributed among the packets at random (the number of packets available being infinite), what is the average minimum number of packets that must be purchased in order to obtain a complete set of cards?

Three solutions were printed with the result $52\sum_{n=1}^{52} 1/n$ as the required average, that is 236 purchases. It is interesting to compare the methods used. In the solution of 4108 [1945, 282] Olds cited a related probability problem.

Factorial Binomial Theorem

4120 [1944, 289]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given k and h as positive integers, show that

$$\sum_{n=0}^{k-1}(-1)^n \frac{(h-1)(h-2)\cdots(h-n)}{n!}\,\frac{k(k+1)\cdots(k+h-n-1)}{(h-n)!} = \binom{k}{h},$$

where for $n=0$ the summand is $k(k+1)\cdots(k+h-1)/h!$.

Solution by G. T. Williams, Student, Harvard University. This is essentially the known factorial binominal theorem

$$(a+b)_{(k)} = \sum_{n=0}^{k} \binom{k}{n} a_{(n)} b_{(k-n)},$$

where $a_{(n)} = a(a-1) \cdots (a-n+1) = (-1)^n(n-a-1)_{(n)}$. Writing $\sum$ for the sum in the left-hand member of the problem, we have

$$\begin{aligned} h!\sum &= \sum_{n=0}^{h-1} (-1)^n \binom{h}{n} (h-1)_{(n)} (-1)^{h-n} (-k)_{(h-n)} \\ &= (-1)^h (h-k-1)_{(h)} = k_{(h)}. \end{aligned}$$

This gives the desired result.

Solved also by H. S. M. Coxeter, J. B. Kelly, M. S. Knebelman, and Norman Miller.

Editorial Note. The factorial binomial theorem is easily proved. Let b have any fixed value and let a, r denote positive integers $a \geqq r$. Then we have

$$(a+b)^{(r)} = U^a b^{(r)} = (\Delta + 1)^a b^{(r)} = \sum_{j=0}^{r} {}_aC_j \Delta^j b^{(r)},$$

where $b^{(r)} = b(b-1) \cdots (b-r+1)$, $b^{(0)} = 1$, and $\Delta^j b^{(r)} = r^{(j)} b^{(r-j)}$, the latter being zero if $j > r$. Hence

$$(a+b)^{(r)} = \sum_{j=0}^{r} \frac{a^{(j)}}{j!} r^{(j)} b^{(r-j)} = \sum_{j=0}^{r} {}_rC_j a^{(j)} b^{(r-j)}.$$

Consider now the two polynomials of degree r in a in the first and third expressions above, where a denotes any variable. They are equal for all integral values of $a \geqq r$, and hence they are equal for any value of a.

Another method of proof follows which applies to the ordinary binomial theorem. The theorem is true for $r=1$; assume it true for a chosen r. Multiply each member of the assumed equation by $a+b-r$ giving on the left $(a+b)^{(r+1)}$, whereas on the right we set $a+b-r = (a-j)+(b-r+j)$ giving the addition of two sums

$$(a+b)^{(r+1)} = \sum_{j=0}^{r} {}_rC_j a^{(j+1)} b^{(r-j)} + \sum_{j=0}^{r} {}_rC_j a^{(j)} b^{(r-j+1)}.$$

Using ${}_rC_{j-1} + {}_rC_j = {}_{r+1}C_j$ we have

$$(a+b)^{(r+1)} = \sum_{j=0}^{r+1} {}_{r+1}C_j a^{(j)} b^{(r+1-j)},$$

and this completes the induction proof.

The problem identity may now be established without using the binomial theorem but by applying the two methods above. For the first method let h denote a positive integer, m a chosen non-negative integer, and k a number of any kind. Then we have

$$(m-k)^{(h)} = U^m(0-k)^{(h)} = (\Delta+1)^m(0-k)^{(h)} = \sum_{j=0}^{m} {}_mC_j\Delta^j(0-k)^{(h)}$$

$$= \sum_{j=0}^{m} {}_mC_j h^{(j)}(-k)^{(h-j)} = \sum_{j=0}^{m} {}_hC_j m^{(j)}(-k)^{(h-j)}.$$

In the final equality above set $m=h-1$ and we get

$$\sum_{j=0}^{h-1} {}_hC_j(h-1)^{(j)}(-k)^{(h-j)} = (h-1-k)^{(h)},$$

which easily reduces to the desired result.

The solution by Miller uses the induction proof as given above. Kelly denoted the given sum by $u_{k,h}$ and derived the difference equation $u_{k+1,h}-u_{k,h}=u_{k,h-1}$ and boundary conditions; these suffice to prove the desired result. The remaining solvers equated the coefficients of x^h in

$$(1+x)^{h-1}(1+x)^{-k} = \sum_{n=0}^{h-1}\binom{h-1}{n}x^n \sum_{j=0}^{\infty}\binom{-k}{j}x^j.$$

A second binomial formula is obtained by similar methods

$$(a+b)^{(-r)} = \sum_{j=0}^{a}\binom{-r}{j}a^{(j)}b^{(-r-j)}$$

where a, r are non-negative integers and b is any kind of number excluding integral values in the interval $-(r+a-1)\leqq b\leqq 0$, $b^{(-r)}=1/b(b+1)\cdots(b+r-1)$, $b^{(-0)}=1$. From this follows

$$(a+c)^{(a)} = \sum_{j=0}^{a}\binom{-r}{j}a^{(j)}(a+c+r)^{(a-j)}.$$

This last result follows also at once from the first binomial formula replacing in the left-hand member $a+c$ by $-r+(a+c+r)$, and simple alterations on the right.

Simple Continued Fractions

4123 [1944, 290]. *Proposed by V. Thébault, Tennie, Sarthe, France*

For what values of m is the product $(2m+1)(10m+1)$ a square of an integer? *Application.* In what systems of numbers is a number of the form $aabb$ the square of a number of two digits bb?

Solution by M. F. Smiley, Annapolis, Md. If $(2m+1)(10m+1)$ is a square, then each of the two factors is a square, since they are odd and relatively prime. We set $x^2=2m+1$, $y^2=10m+1$ to obtain $5x^2-y^2=4$. The transformation (of determinant unity)

$$x = t+2u, \qquad y = 2t+5u,$$

yields the equation $t^2-5u^2=4$ (see Dickson's *Introduction to the Theory of Numbers*, 1929, p. 114). Note that $x\equiv y\equiv t\equiv u$ (2) so that we must delete from the sets (t_k, u_k) given by

$$(1)\qquad (t_k+u_k\sqrt{5})/2=\left(\frac{3+\sqrt{5}}{2}\right)^k,\qquad k=\pm 1,\ \pm 2,\cdots,$$

those in which $t\equiv 0$ (2). To obtain an explicit formula for the deleted solutions, write $u=2U$ to obtain $t^2-20U^2=4$, whose general solution is given by

$$(t_j+U_j\sqrt{5})/2=(9+2\sqrt{5})^j,\qquad j=\pm 1,\ \pm 2,\cdots$$

Since $(9+4\sqrt{5})=[(3+\sqrt{5})/2]^3$, we find that if $k\not\equiv 0$ (3) in (1), then $t\not\equiv 0$ (2) and

$$m=2t_ku_k+(9u_k^2+3)/2,\qquad k=\pm 1,\ \pm 2,\cdots$$

gives the solution of our problem. For $t_1=3, u_1=1$, we get $m=12$, $(2m+1)(10m+1)=5^2 11^2$; for $t_2=7$, $u_2=3$, $m=84$, the product is $13^2 29^2$; *etc.*

Application (sic). Let $q>1$ be the radix. Then $a, b<q$ and

$$aq^3+aq^2+bq+b=b^2(q+1)^2$$

yields

$$aq^2+b=b^2q+b^2.$$

Setting $b^2=mq+n$ $(m, n<q)$ gives

$$aq^2+b=mq^2+(m+n)q+n.$$

Hence $n=b$, $m+n=q$, $a=q+1-b$, and $b^2=(q-b)q+b$. Setting $t=5b-2$, $u=2q-b$ in this last equation gives $t^2-5u^2=4$ again. The requirement that b and q be integers is equivalent to $t\equiv -2(5)$ as is easily seen. The least positive solution $(t, u)=(3, 1)$ satisfies this requirement but not $q>1$. It is quite simple to check that if in (1) we have $t_k\equiv -2(5)$, then $t_{k+1}\equiv 2(5)$; while if in (1) we have $t_k\equiv 2(5)$, then $t_{k+1}\equiv -2(5)$. Hence the formula

$$(t_k+u_k\sqrt{5})/2=[(3+\sqrt{5})/2]^k,\qquad k=3, 5, 7,\cdots$$

yields all values of (t, u) and thus (q, b) which satisfy our problem. The first four solutions are

t	18	123	843	5778
u	8	55	377	2584
b	4	25	169	1156
q	6	40	273	1870
a	3	16	105	715.

Solved also by Murray Barbour, H. N. Carleton, J. B. Kelly, E. P. Starke, and the proposer.

Editorial Note. The proposer and Barbour derived also non-integral values $m=3/2, 1155/2, \cdots$. To obtain in turn the solutions of $y^2-5x^2=-4$, Starke used the relations

$$y' = |9y \pm 20x|, \qquad x' = 9x \pm 4y,$$

where $(x, y)=(1, 1)$ gives $(x', y')=(5, 11), (13, 29)$; and referred to 4047 [1944, 103]. Barbour referred to Dickson's *History of the Theory of Numbers*, vol. 2, chaps. XII, XIII, in particular to the remarks on the work of S. Realis, p. 407. The solutions of $5t^2-4=\square$ are found to be $t=1, 2, 5, 13, 34, 89, \cdots$ which is the Fibonacci series $S_n=3S_{n-1}-S_{n-2}$. The required base r is the product of successive pairs of the additive series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, $\cdots$ $r=1\cdot 1, 2\cdot 3, 5\cdot 8, 13\cdot 21, \cdots$. Kelly also referred to the Fibonacci series.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending new items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

The National Research Council announces predoctoral fellowships in the mathematical, physical and biological sciences. These fellowships are intended to assist young men and women, whose graduate study has been prevented or interrupted by the war, to complete their work for the doctorate. The annual stipend will be $1,200 for single persons and $1,800 for married men plus an allowance for tuition fees. Those interested should write to the Secretary, Committee on Predoctoral Fellowships, National Research Council, 2101 Constitution Avenue N.W., Washington 25, D. C.

Professor H. M. Gehman, branch head of the Shrivenham American University in England, announces the following civilian staff: V. W. Adkisson, J. P. Ballantine, T. C. Benton, E. T. Browne, F. J. H. Burkett, D. R. Davis, W. M. Davis, P. D. Edwards, C. J. Latimer, L. L. Lowenstein, J. H. Neelley, P. R. Rider, N. E. Rutt, C. G. Stipe, J. I. Tracey. It is hoped that in a later number announcement can be made of the staff in each of the other European army university centers.

G. E. Stechert and Company have received a cable from Professor H. Steinhaus of Krakow informing them that he is well and that his address is Lidia Kott, Krakow Lea 93, Poland.

At the University of California the following promotions are announced: Assistant Professor A. L. Foster to an associate professorship; Dr. L. H. Swinford to an assistant professorship; Dr. Alfred Tarski to an associate professorship.

Associate Professor C. K. Alexander of Occidental College has been promoted to a professorship.

Dr. Florence E. Allen of the University of Wisconsin has been promoted to an assistant professorship.

Dr. K. J. Arnold has been appointed to an assistant professorship at the University of Wisconsin.

Assistant Professor R. A. Beaver of the New York State College for Teachers, Albany, has been promoted to a professorship.

Associate Professor H. F. Bohenblust of Princeton University has been appointed to a professorship at Indiana University.

Assistant Professor E. C. Brown of Worcester Polytechnic Institute has been promoted to a professorship.

Associate Professor O. E. Brown of Case School of Applied Science has been promoted to a professorship.

Associate Professor R. E. Brown of Rhode Island State College has been promoted to a professorship.

Eleanor Calkins of the College of William and Mary has been promoted to an assistant professorship.

Associate Professor Teresa Cohen of Pennsylvania State College has been promoted to a professorship.

Dr. W. J. Combellack of Northeastern University has been promoted to an assistant professorship.

Associate Professor A. T. Craig of the University of Iowa has been promoted to a professorship.

Dr. A. W. Davis of Iowa State College has been promoted to an assistant professorship.

James Edgar Davis has been appointed to an assistant professorship at the University of Illinois.

Assistant Professor F. L. Dennis of Ursinus College has been promoted to an associate professorship.

Professor C. E. Dimick of the U. S. Coast Guard Academy has retired.

N. E. Dodson of Newberry College, South Carolina, has been promoted to an associate professorship.

Assistant Professor W. C. Doyle of Rockhurst College, Kansas City, Missouri, has been promoted to an associate professorship.

Assistant Professor L. T. Dunlap of Pennsylvania State College has been promoted to an associate professorship.

Assistant Professor R. H. Fox of Syracuse University has been appointed to an assistant professorship at Princeton University.

Assistant Professor L. M. Garrison of Louisiana Polytechnic Institute, Ruston, Louisiana, has been promoted to an associate professorship.

Dr. G. D. Gore has been appointed to a professorship and the chairmanship of the department of mathematics and engineering science at Roosevelt College, Chicago, Illinois.

Dr. J. W. Green has been appointed to an assistant professorship at the University of California at Los Angeles.

Professor H. H. Hartzler of Goshen College has been appointed to a professorship at Bluffton College, Bluffton, Ohio.

Lieutenant C. C. Hurd of the U. S. Coast Guard Academy has been appointed dean of Allegheny College.

Assistant Professor J. L. Kelley of the University of Notre Dame has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor J. C. C. McKinsey of Montana State College has been appointed to an assistant professorship at the University of Nevada.

Dr. W. A. Mersman of the College of Agriculture of the University of California has been promoted to an assistant professorship.

Dr. R. v. Mises of Harvard University has been appointed Gordon McKay professor of aerodynamics and applied mathematics.

Dorothy J. Morrow of Bryn Mawr College has been appointed to an assistant professorship at George Washington University.

Professor L. W. Stark of William Jewell College has been appointed acting professor of mathematics at Atlantic Christian College, Wilson, North Carolina.

Assistant Professor Alexander Weinstein of the University of Toronto has been appointed to an associate professorship.

Dr. Max Wyman of the University of Alberta has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
Amherst College: Daniel Finkel, F. G. Graff, R. L. Wine
University of Arizona: Dr. H. S. Kieval
University of Illinois: Dr. J. S. Stubbe

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE COOPERATIVE COMMITTEE ON SCIENCE TEACHING

The Cooperative Committee on Science Teaching is a committee of the American Association for the Advancement of Science. Each scientific organization interested in the project nominates one or more representatives to serve upon the Committee; final appointment, then, is made by the Executive Committee of the A.A.A.S. At the present time, Dr. R. J. Havighurst, division of human relations, University of Chicago, is chairman of the Committee, and mathematicians are represented by Professor R. W. Schorling, University of Michigan (M.A.A.), and by Professor E. H. C. Hildebrandt, Northwestern University (National Council).

A preliminary report upon the work of the Cooperative Committee was published in *School Science and Mathematics*, October, 1942. The tentative proposals and recommendations of that article are still under discussion. Mathematicians concerned with the problem of secondary teaching will follow the work of the Committee with interest.

The problem with which the Committee has been especially concerned is the preparation of teachers for the small high school. The Committee has found that half of the high schools in the country have five teachers or less. Three-fourths of them have ten teachers or less. Yet the small school must offer courses in at least twelve subjects, and many of them offer fifteen to twenty subjects. Moreover, most new teachers commence their work in small high schools because of the fact that the larger schools require teaching experience of candidates for positions. The beginning teacher in a small school nearly always must teach at least three different subjects, and often four or five.

Consequently, the Cooperative Committee believes that the science and mathematics curriculum in the small high school, as well as the preparation of teachers properly trained to handle such a program, represents an urgent problem for investigation. It is planned in the near future to make some specific recommendations to colleges and universities training prospective secondary teachers.

A COURSE FOR PROSPECTIVE TEACHERS IN OHIO

A committee of the Ohio Section of the M.A.A., under the chairmanship of Professor I. A. Barnett of the University of Cincinnati, has completed a tentative course of study in mathematics for prospective teachers in the elementary schools of Ohio. The general objectives of the course of study have been described in the following terms: "To present to teachers the historical and logical

background requisite to the proper and interrelated understanding of the appropriate mathematical facts and processes; to feature the human significance of this material and its usefulness in concrete applications; and to integrate the insight and perspective thus gained with the teaching process."

Five special objectives are listed as follows:

1. *Symbolism.* To emphasize the basic importance of mathematical symbolism; symbols as a powerful shorthand of thought—their role in grasping and solving difficult problems.

2. *Vocabulary.* There is no more important goal in teaching mathematics than in training the students to talk accurately in the mathematical classroom. This attainment will facilitate immeasurably the teaching and understanding of mathematics. The lack of practice on the part of the student in talking mathematics accurately means necessarily that much of what is going on in the classroom has little meaning to the student.

3. *Justification of the processes of arithmetic.* Review of arithmetic with emphasis on the reasons for the underlying processes as well as on the attainment of mechanical proficiency.

4. *Arithmetic-based algebra.* To show that the consciousness of algebraic thinking naturally dawns and develops in arithmetic.

5. *Geometry.* The intuitive is to go hand in hand wherever possible with the logical. The training of the logical faculty is not dependent on the quantity of theorems. It must be assured through independent thinking, and, to attain this important end, quantity should be drastically sacrificed if necessary.

The instructor of such a training course has been given three suggestions; namely,

1. The presentation should be informal and favor methods which teachers could later use in their own classrooms. The student teachers should have a major part in the development of the course.
2. It is very desirable that the definitions should be related to concrete experience rather than merely quoted from the books. The process of generalization, one of the most important concepts in the whole domain of mathematics, is to be given prominence through precept and exercise.
3. The teacher's interest is to be stimulated by stressing the part played by the mathematics of the past in the development of civilization. Mathematics is to be viewed as a living and growing force in our daily lives and not as a dead and closed chapter of the past.

Anyone interested in the actual content of the proposed course should write to Professor Rufus Crane, Ohio Wesleyan University, Delaware, Ohio, to obtain a mimeographed copy of the tentative outline.

ACADEMIC INTERESTS OF DISCHARGED VETERANS

All surveys which have been made of the academic interests of discharged veterans who are anticipating a return to college indicate a pronounced trend toward the technical fields. Moreover, it is becoming increasingly evident that

most of the men are thinking only of those branches of engineering and technical science most obviously related to the conduct of the war. In fact, Mr. B. A. Thresher, Director of Admissions at the Massachusetts Institute of Technology, has warned of a possible dislocation of the nation's educational programs and technical needs if present trends are ignored.

Mr. Thresher has analyzed 1,102 applications for admission to M.I.T., which he has recently received from men in service. The survey indicates that the majority of prospective students interested in engineering are thinking only of electrical, mechanical, and aeronautical engineering. Moreover, there appears to be very little interest generally in such fields as biology, geology, mathematics, and architecture. Even such fields as chemistry and metallurgy are being neglected by the returning veteran.

Mr. Thresher sampled the applications to obtain some specific data upon the interest in mathematics. A prewar sample showed that 1.2% of the enrollment in the institution was in mathematics. This included both undergraduate and graduate work. In the sample of applications for postwar admission, 0.67% showed an interest in mathematics. The numbers involved in the sample were perhaps too small to permit any broad conclusions. The figures on mathematics, however, may be contrasted with a comparable sample in the case of electrical engineering which shows an increase in postwar registrations of over 150%; aeronautical engineering shows almost as great an increase. In summary, Mr. Thresher writes, "It appears that mathematics is one of several fields which are, relatively speaking, being neglected by men in the service because they have been in close contact with the particular branches of electronics and aeronautics."

Applicants generally, according to Mr. Thresher, seem to be interested in "narrow and intensive training." Moreover, they fail to realize the importance of basic subjects.

TEACHERS FOR THE VETERAN REHABILITATION PROGRAM

The Veteran Rehabilitation Program is offering opportunities for recreational workers, physical directors, and teachers of both academic and commercial subjects to aid in the adjustment of disabled war veterans to civilian life. According to Announcement No. 362 (revised August 13, 1945), the Civil Service Commission is seeking qualified persons for these positions. No written test is required. The salaries are $2,190 and $2,433 a year, including overtime pay. Information may be obtained by writing directly to the Civil Service Commission, Washington 25, D. C.

To qualify for this work, applicants must have had at least one year of responsible experience in teaching mathematics or other academic subjects. In addition, applicants must have successfully completed a full four-year course of study leading to a bachelor's degree in a college, university, or teacher-training institution of recognized standing, with a minimum of twelve semester hours in education, including at least one course in practice teaching, provided that an

additional year of the experience described above may be substituted for the required course in practice teaching.

A teacher selected for the Program will teach an academic subject or a combination of such subjects to patients in Veterans Administration hospitals upon recommendation of the attending physician. He also will plan, organize, and schedule classes for patients or arrange for individual instruction, and will make recommendations to the manager regarding the use of correspondence courses and make the necessary arrangements for such courses. He will be expected to coordinate both class and individual instruction with the vocational rehabilitation service.

IMPORTANT LEGISLATION BEFORE CONGRESS

In view of the publicity given to the report, scientists generally are familiar with the recommendations of Dr. Vannevar Bush, Director of the Office of Scientific Research and Development, in regard to the role of the federal government in supporting and encouraging scientific research. The complete report is entitled *Science, The Endless Frontier*, and it may be obtained from the Superintendent of Documents, Government Printing Office, Washington 25, D. C. The price is thirty cents per copy.

As soon as the Bush report was released, three identical bills were introduced into Congress following its recommendations. They were (S. 1285) by Senator Warren G. Magnuson, Washington, (H.R. 3582) by Representative Wilbur D. Mills, Arkansas, and (H.R. 3860) by Representative Jennings Randolph, West Virginia. The Senate bill was referred to the Committee on Commerce and the House bills to the Committee on Interstate and Foreign Commerce.

In addition to the recommendations of Dr. Bush, another report bearing upon the same general question was made on July 23 by the Subcommittee on War Mobilization of the Senate Military Affairs Committee under the chairmanship of Senator Harley M. Kilgore of West Virginia.

The Kilgore report states that "We have depended in the past very heavily on the basic research done in Germany with the support of the German government. Through the cartel system we have also presumably relied on Germany for a great deal of our applied research. It is quite clear now that we can no longer rely upon Germany for basic or applied research. It will be necessary to reduce the German leadership in science and technology which time after time has led to wars of aggression. Hereafter we must rely on ourselves for basic research. This is a field of public responsibility, because basic research is seldom immediately profitable, although the application of such research may be of enormous value. The Nation's universities, both before and during the war, have distinguished themselves for their work in basic research. Congress should provide for a broad and representative board to advise on the use of the university laboratories and of some government facilities in a well-planned program of basic research. Funds must be provided to carry on the work upon which our future progress in applied science and technology depends."

To implement the provisions of its report, the Kilgore Committee recommends the establishment of a National Science Foundation. It would be headed by a director appointed by the President and a National Science Board under the chairmanship of the Director. The board would be made up of the Secretaries of War, Navy, Interior, Agriculture, Commerce, and Labor, the Attorney General, the Federal Security Administrator, and eight members appointed by the President to represent the public.

The Foundation itself would not be a research agency, but it would have the function of stimulating scientific research at all levels and would encourage the development and training of scientific personnel. It is further recommended that 20 per cent of the funds appropriated shall be used for national defense research, 20 per cent for medical research, and the remaining 60 per cent for basic science and other research.

Senators Kilgore, Edwin C. Johnson of Colorado, and Claude Pepper of Florida are the sponsors of a bill (S. 1297), to establish a National Science Foundation as recommended by the Committee. The bill has been referred to the Committee on Military Affairs.

As a further consequence of the Bush and Kilgore reports, Senator Thomas of Utah introduced S. 1316 to give federal aid to the states for science education. The bill asserts that there is a "widespread lack of opportunity for youth enrolled in public secondary schools to receive effective instruction in natural science subjects." So the bill proposes to appropriate $4,000,000 in 1945, increasing to $20,000,000 by 1951, to help the states pay salaries of supervisors and teachers of natural sciences and to help defray the cost of supplies and equipment. This money would be allotted to the states on the basis of the secondary-school population, and all funds would be expended "through public agencies and under public control as determined by the legislatures of the respective states." The Thomas bill has been referred to the Committee on Education and Labor.

An Interesting Theorem. *All real numbers are uninteresting.* For the integer one is evidently uninteresting. (It has only trivial representations a sum of Mth powers of integers, and trivially divides every other number.) Now assume that there are interesting integers, and let N be the least such. Then N is greater than one. Hence $N-1$ is a *very* interesting integer; for it is the first uninteresting integer whose immediate successor is not interesting. This contradiction shows that no positive integer is interesting. But every positive rational is trivially expressible in an infinite number of ways as an ordered couple of uninteresting numbers and hence is uninteresting. Since every negative rational may be obtained from an uninteresting number by a mere change of sign, no negative rational is interesting. But zero and each of the remaining real numbers is representable as the least upper bound of a set of uninteresting numbers . . . . We conclude that no real numbers are interesting. The extension to complex numbers and Hilbert space is left as an exercise to the reader.—*Morgan Ward.*

NEW MEMBERS

The following fifty persons have been elected to membership on applications duly certified:

REV. J. H. ANTONY, A.B. (Univ. of Dayton) Head of Dept., Purcell High School, Cincinnati, Ohio

J. G. BACHMANN, B.S. (New York Univ.) President, Bachmann and Co., Inc., 8511 Sunset Blvd., Hollywood 46, Calif.

T. H. BEDWELL, A.B. (South Dakota) Instr., Physics, Univ. of South Dakota, Vermillion, S. D.

K. H. BRACEWELL, Ph.D. (Indiana) Prof., Math. and Physics, Hamline Univ., St. Paul, Minn.

T. S. BRODERICK, M.A., F.T.C.D. (Dublin) Prof., Trinity Coll., Dublin, Ireland

R. H. BRUCK, Ph.D. (Toronto) Asst. Prof., Univ. of Wisconsin, Madison 6, Wis.

L. H. CHACALOS. Senior in High School, Orange, N. J.

P. L. CHESSIN. Student, Univ. of Wisconsin, Madison, Wis.

M. P. DE REGT, B.S. (Webb Inst.) Supervisor, Hull Tech. Section, Todd-Pacific Shipyards, Tacoma, Wash.

J. A. H. DUFFIE, B.S. (Univ. of New Brunswick) Grad. student, Chemistry-Physics, Catholic Univ. of America, Washington 17, D. C.

REV. L. E. ERNSDORFF, M.S. (Univ. of Notre Dame) Instr., Loras Coll., Dubuque, Iowa

M. P. GAFFNEY, B.S. (Harvard) C Sp (x), U.S.N.R. Naval Research Lab., Washington, D. C.

H. W. GODDERZ, B.S. (Northern St.T.C.) Coll. of St. Thomas, St. Paul 1, Minn.

ARTHUR GRAD, A.M. (Columbia) Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.

BROTHER JOSEPHUS GREGORY, A.M. (Loyola Univ.) St. Mary's Coll., Winona, Minn.

MARGARET M. HANSMAN, Ph.D. (Illinois) Asst. Prof., Colorado Coll., Colorado Springs, Colo.

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D. L. HOLL, Ph.D. (Chicago) Prof., Iowa State Coll., Ames, Iowa

REV. J. A. HRATZ, B.S. (Iowa) Acting Head of Dept., St. Ambrose Coll., Davenport, Iowa

MEYER KARLIN, Ph.D. (Columbia) Instr., Brooklyn Coll. Evening Session, Brooklyn, N. Y.

NORBERT KAUFMAN. S 2/c, U. S. Navy

WINFIELD KECK, A.M. (Pennsylvania) Instr., Math. and Physics, Muhlenberg Coll., Allentown, Pa.

NATHAN KEYFITZ, B.S. (McGill Univ.) Statistician, Dominion Bureau of Statistics, Ottawa, Ont., Can.

R. J. KOCH. RT 3/c, U. S. Navy

WALTER LEIGHTON, Ph.D. (Harvard) Lecturer, Rice Inst., Houston, Tex. On leave. Dir., Applied Math. Group, Northwestern Univ.

ANNE L. LEWIS, Ph.D. (Chicago) Asst. Prof., Woman's Coll., Univ. of North Carolina, Greensboro, N. C.

REV. C. J. LEWIS, M.S. (Georgetown Univ.) Instr., St. Peter's Coll., 144 Grand St., Jersey City 2, N. J.

SISTER M. MECHTILDIS MALIN, M.S. (Marquette) Instr., Viterbo Coll., LaCrosse, Wis.

SISTER MARY MICHAEL MALONEY, A.M. (Catholic Univ.) Head of Dept., D'Youville Coll., 320 Porter Ave., Buffalo 1, N. Y.

C. R. MUMMERY, B.S. (Toronto) Head, Mech. and Cleaner Research Sec., The Hoover Co., North Canton, Ohio

SISTER NOEL MARIE, A.M. (Catholic Univ.) Head of Dept., Coll. of Saint Rose, Albany 3, N. Y.

P. L. NORIEGA. Student, Geneva Coll., Beaver Falls, Pa.

F. F. OTIS, B.Ed. (St. T.C., River Falls, Wis.) Instr., Engg., Marquette Univ., Milwaukee, Wis.

ELLI OTTESON, A.M. (Wisconsin) Chm. of Dept., High School, Eau Claire, Wis.

S. T. PARKER, M.A. (Univ. of British Columbia) Asst. Prof., Speed Scientific School, Univ. of Louisville, Louisville 8, Ky.

MURRAY PESHKIN. T/4, U. S. Army

A. P. RHODES, A.M. (Stanford) Asso. Civil Engr., U. S. Navy, Dept. of Public Works, San Diego, Calif.

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HARRY SCHOR, M.S. in Educ. (C.C.N.Y.) Chm. of Dept., Manhattan High School of Aviation Trades, New York, N. Y.

C. F. SECADA, B.S. (Peruvian Naval Acad.) Grad. student, Massachusetts Inst. of Tech., Cambridge 39, Mass. Lt. Comdr., Peruvian Navy

AARON SHAPIRO, A.M. (C.C.N.Y.) Teacher, Midwood High School, Brooklyn, N. Y.

L. R. SHOBE, M.S. (Kansas State Coll.) Tech. Instr., General Motors Inst., Flint 2, Mich.

ETHEL SUTHERLAND, Ph.D. (Columbia) Instr., East Carolina Teachers Coll., Greenville, N.C.

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R. D. WAGNER, Ph.M. (Wisconsin) Instr., Univ. of Wisconsin, Madison, Wis.

ETTA A. WAITE, A.M. (Columbia) Teacher, Girls Commercial High School, Brooklyn, N. Y.

H. E. WOODWARD, A.M. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex.

W. B. CARVER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Twenty-ninth Annual Meeting, Chicago, Ill., November 24–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

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By HAIM REINGOLD, Illinois Institute of Technology

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CLUBS AND ALLIED ACTIVITIES

Edited by J. S. FRAME, Michigan State College

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ACTIVITIES

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Edited by H. P. EVANS, University of Wisconsin

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REVIEWS

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Edited by OTTO DUNKEL, Washington University, ORRIN FRINK, JR., Pennsylvania State College, H. S. M. COXETER, University of Toronto, and HOWARD EVES, College of Puget Sound.

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Numbers in black face type refer to problems; those in light face to pages.

WAR INFORMATION

GENERAL INFORMATION

Edited by C. V. NEWSOM, Oberlin College

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Edited by B. W. Jones, Cornell University

PERSONAL MENTION

This section contains the names of persons taking any active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

NECROLOGY

REPORTS OF THE MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

Edited by N. B. CONKWRIGHT, University of Iowa

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF

COLLEGIATE MATHEMATICS

VOLUME 52 NUMBER 10

PART I

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DECEMBER 1945

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THIS MONTHLY WAS FOUNDED IN 1894 BY BENJAMIN F. FINKEL

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, L. R. FORD, Illinois Institute of Technology, Technology Center, Chicago 16, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the SECRETARY-TREASURER, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 10th of the month in which the change becomes effective.

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ON THE NATURE OF MATHEMATICAL TRUTH

C. G. HEMPEL, Queens College

1. The problem. It is a basic principle of scientific inquiry that no proposition and no theory is to be accepted without adequate grounds. In empirical science, which includes both the natural and the social sciences, the grounds for the acceptance of a theory consist in the agreement of predictions based on the theory with empirical evidence obtained either by experiment or by systematic observation. But what are the grounds which sanction the acceptance of mathematics? That is the question I propose to discuss in the present paper. For reasons which will become clear subsequently, I shall use the term "mathematics" here to refer to arithmetic, algebra, and analysis—to the exclusion, in particular, of geometry [1].

2. Are the propositions of mathematics self-evident truths? One of the several answers which have been given to our problem asserts that the truths of mathematics, in contradistinction to the hypotheses of empirical science, require neither factual evidence nor any other justification because they are "self-evident." This view, however, which ultimately relegates decisions as to mathematical truth to a feeling of self-evidence, encounters various difficulties. First of all, many mathematical theorems are so hard to establish that even to the specialist in the particular field they appear as anything but self-evident. Secondly, it is well known that some of the most interesting results of mathematics—especially in such fields as abstract set theory and topology—run counter to deeply ingrained intuitions and the customary kind of feeling of self-evidence. Thirdly, the existence of mathematical conjectures such as those of Goldbach and of Fermat, which are quite elementary in content and yet undecided up to this day, certainly shows that not all mathematical truths can be self-evident. And finally, even if self-evidence were attributed only to the basic postulates of mathematics, from which all other mathematical propositions can be deduced, it would be pertinent to remark that judgments as to what may be considered as self-evident are subjective; they may vary from person to person and certainly cannot constitute an adequate basis for decisions as to the objective validity of mathematical propositions.

3. Is mathematics the most general empirical science? According to another view, advocated especially by John Stuart Mill, mathematics is itself an empirical science which differs from the other branches such as astronomy, physics, chemistry, *etc.*, mainly in two respects: its subject matter is more general than that of any other field of scientific research, and its propositions have been tested and confirmed to a greater extent than those of even the most firmly established sections of astronomy or physics. Indeed, according to this view, the degree to which the laws of mathematics have been borne out by the past experiences of mankind is so overwhelming that—unjustifiably—we have come to think of mathematical theorems as qualitatively different from the well-confirmed hypotheses or theories of other branches of science: we consider them as

certain, while other theories are thought of as at best "very probable" or very highly confirmed.

But this view, too, is open to serious objections. From a hypothesis which is empirical in character—such as, for example, Newton's law of gravitation—it is possible to derive predictions to the effect that under certain specified conditions certain specified observable phenomena will occur. The actual occurrence of these phenomena constitutes confirming evidence, their non-occurrence disconfirming evidence for the hypothesis. It follows in particular that an empirical hypothesis is theoretically disconfirmable; *i.e.*, it is possible to indicate what kind of evidence, if actually encountered, would disconfirm the hypothesis. In the light of this remark, consider now a simple "hypothesis" from arithmetic: $3+2=5$. If this is actually an empirical generalization of past experiences, then it must be possible to state what kind of evidence would oblige us to concede the hypothesis was not generally true after all. If any disconfirming evidence for the given proposition can be thought of, the following illustration might well be typical of it: We place some microbes on a slide, putting down first three of them and then another two. Afterwards we count all the microbes to test whether in this instance 3 and 2 actually added up to 5. Suppose now that we counted 6 microbes altogether. Would we consider this as an empirical disconfirmation of the given proposition, or at least as a proof that it does not apply to microbes? Clearly not; rather, we would assume we had made a mistake in counting or that one of the microbes had split in two between the first and the second count. But under no circumstances could the phenomenon just described invalidate the arithmetical proposition in question; for the latter asserts nothing whatever about the behavior of microbes; it merely states that any set consisting of $3+2$ objects may also be said to consist of 5 objects. And this is so because the symbols "$3+2$" and "5" denote the same number: they are synonymous by virtue of the fact that the symbols "2," "3," "5," and "+" are *defined* (or tacitly understood) in such a way that the above identity holds as a consequence of the meaning attached to the concepts involved in it.

4. The analytic character of mathematical propositions. The statement that $3+2=5$, then, is true for similar reasons as, say, the assertion that no sexagenarian is 45 years of age. Both are true simply by virtue of definitions or of similar stipulations which determine the meaning of the key terms involved. Statements of this kind share certain important characteristics: Their validation naturally requires no empirical evidence; they can be shown to be true by a mere analysis of the meaning attached to the terms which occur in them. In the language of logic, sentences of this kind are called analytic or true a priori, which is to indicate that their truth is logically independent of, or logically prior to, any experiential evidence [2]. And while the statements of empirical science, which are synthetic and can be validated only a posteriori, are constantly subject to revision in the light of new evidence, the truth of an analytic statement can be established definitely, once and for all. However, this characteristic "theoretical

certainty" of analytic propositions has to be paid for at a high price: An analytic statement conveys no factual information. Our statement about sexagenarians, for example, asserts nothing that could possibly conflict with any factual evidence: it has no factual implications, no empirical content; and it is precisely for this reason that the statement can be validated without recourse to empirical evidence.

Let us illustrate this view of the nature of mathematical propositions by reference to another, frequently cited, example of a mathematical—or rather logical—truth, namely the proposition that whenever $a=b$ and $b=c$ then $a=c$. On what grounds can this so-called "transitivity of identity" be asserted? Is it of an empirical nature and hence at least theoretically disconfirmable by empirical evidence? Suppose, for example, that a, b, c, are certain shades of green, and that as far as we can see, $a=b$ and $b=c$, but clearly $a\neq c$. This phenomenon actually occurs under certain conditions; do we consider it as disconfirming evidence for the proposition under consideration? Undoubtedly not; we would argue that if $a\neq c$, it is impossible that $a=b$ and also $b=c$; between the terms of at least one of these latter pairs, there must obtain a difference, though perhaps only a subliminal one. And we would dismiss the possibility of empirical disconfirmation, and indeed the idea that an empirical test should be relevant here, on the grounds that identity is a transitive relation by virtue of its definition or by virtue of the basic postulates governing it [3]. Hence, the principle in question is true a priori.

5. Mathematics as an axiomatized deductive system. I have argued so far that the validity of mathematics rests neither on its alleged self-evidential character nor on any empirical basis, but derives from the stipulations which determine the meaning of the mathematical concepts, and that the propositions of mathematics are therefore essentially "true by definition." This latter statement, however, is obviously oversimplified and needs restatement and a more careful justification.

For the rigorous development of a mathematical theory proceeds not simply from a set of definitions but rather from a set of non-definitional propositions which are not proved within the theory; these are the postulates or axioms of the theory [4]. They are formulated in terms of certain basic or primitive concepts for which no definitions are provided within the theory. It is sometimes asserted that the postulates themselves represent "implicit definitions" of the primitive terms. Such a characterization of the postulates, however, is misleading. For while the postulates do limit, in a specific sense, the meanings that can possibly be ascribed to the primitives, any self-consistent postulate system admits, nevertheless, many different interpretations of the primitive terms (this will soon be illustrated), whereas a set of definitions in the strict sense of the word determines the meanings of the definienda in a unique fashion.

Once the primitive terms and the postulates have been laid down, the entire theory is completely determined; it is derivable from its postulational basis in

the following sense: Every term of the theory is definable in terms of the primitives, and every proposition of the theory is logically deducible from the postulates. To be entirely precise, it is necessary also to specify the principles of logic which are to be used in the proof of the propositions, *i.e.* in their deduction from the postulates. These principles can be stated quite explicitly. They fall into two groups: Primitive sentences, or postulates, of logic (such as: If p and q is the case, then p is the case), and rules of deduction or inference (including, for example, the familiar modus ponens rule and the rules of substitution which make it possible to infer, from a general proposition, any one of its substitution instances). A more detailed discussion of the structure and content of logic would, however, lead too far afield in the context of this article.

6. Peano's axiom system as a basis for mathematics. Let us now consider a postulate system from which the entire arithmetic of the natural numbers can be derived. This system was devised by the Italian mathematician and logician G. Peano (1858–1932). The primitives of this system are the terms "0," "number," and "successor." While, of course, no definition of these terms is given within the theory, the symbol "0" is intended to designate the number 0 in its usual meaning, while the term "number" is meant to refer to the natural numbers 0, 1, 2, 3 $\cdots$ exclusively. By the successor of a natural number n, which will sometimes briefly be called n', is meant the natural number immediately following n in the natural order. Peano's system contains the following 5 postulates:

P1. 0 is a number
P2. The successor of any number is a number
P3. No two numbers have the same successor
P4. 0 is not the successor of any number
P5. If P is a property such that (a) 0 has the property P, and (b) whenever a number n has the property P, then the successor of n also has the property P, then every number has the property P.

The last postulate embodies the principle of mathematical induction and illustrates in a very obvious manner the enforcement of a mathematical "truth" by stipulation. The construction of elementary arithmetic on this basis begins with the definition of the various natural numbers. 1 is defined as the successor of 0, or briefly as $0'$; 2 as $1'$, 3 as $2'$, and so on. By virtue of P2, this process can be continued indefinitely; because of P3 (in combination with P5), it never leads back to one of the numbers previously defined, and in view of P4, it does not lead back to 0 either.

As the next step, we can set up a definition of addition which expresses in a precise form the idea that the addition of any natural number to some given number may be considered as a repeated addition of 1; the latter operation is readily expressible by means of the successor relation. This definition of addition runs as follows:

D1. (a) $n+0=n$; (b) $n+k'=(n+k)'$.

The two stipulations of this recursive definition completely determine the sum of any two integers. Consider, for example, the sum $3+2$. According to the definitions of the numbers 2 and 1, we have $3+2=3+1'=3+(0')'$; by D1 (b), $3+(0')'=(3+0')'=((3+0)')'$; but by D1 (a), and by the definitions of the numbers 4 and 5, $((3+0)')'=(3')'=4'=5$. This proof also renders more explicit and precise the comments made earlier in this paper on the truth of the proposition that $3+2=5$: Within the Peano system of arithmetic, its truth flows not merely from the definition of the concepts involved, but also from the postulates that govern these various concepts. (In our specific example, the postulates P1 and P2 are presupposed to guarantee that 1, 2, 3, 4, 5 are numbers in Peano's system; the general proof that D1 determines the sum of any two numbers also makes use of P5.) If we call the postulates and definitions of an axiomatized theory the "stipulations" concerning the concepts of that theory, then we may say now that the propositions of the arithmetic of the natural numbers are true by virtue of the stipulations which have been laid down initially for the arithmetical concepts. (Note, incidentally, that our proof of the formula "$3+2=5$" repeatedly made use of the transitivity of identity; the latter is accepted here as one of the rules of logic which may be used in the proof of any arithmetical theorem; it is, therefore, included among Peano's postulates no more than any other principle of logic.)

Now, the multiplication of natural numbers may be defined by means of the following recursive definition, which expresses in a rigorous form the idea that a product nk of two integers may be considered as the sum of k terms each of which equals n.

D2. (a) $n\cdot 0=0$; (b) $n\cdot k'=n\cdot k+n$.

It now is possible to prove the familiar general laws governing addition and multiplication, such as the commutative, associative, and distributive laws ($n+k=k+n$, $n\cdot k=k\cdot n$; $n+(k+l)=(n+k)+l$, $n\cdot(k\cdot l)=(n\cdot k)\cdot l$; $n\cdot(k+l)=(n\cdot k)+(n\cdot l)$).—In terms of addition and multiplication, the inverse operations of subtraction and division can then be defined. But it turns out that these "cannot always be performed"; *i.e.*, in contradistinction to the sum and the product, the difference and the quotient are not defined for every couple of numbers; for example, $7-10$ and $7\div 10$ are undefined. This situation suggests an enlargement of the number system by the introduction of negative and of rational numbers.

It is sometimes held that in order to effect this enlargement, we have to "assume" or else to "postulate" the existence of the desired additional kinds of numbers with properties that make them fit to fill the gaps of subtraction and division. This method of simply postulating what we want has its advantages; but, as Bertrand Russell [5] puts it, they are the same as the advantages of theft over honest toil; and it is a remarkable fact that the negative as well as the ra-

tional numbers can be obtained from Peano's primitives by the honest toil of constructing explicit definitions for them, without the introduction of any new postulates or assumptions whatsoever. Every positive and negative integer—in contradistinction to a natural number which has no sign—is definable as a certain set of ordered couples of natural numbers; thus, the integer $+2$ is definable as the set of all ordered couples (m, n) of natural numbers where $m=n+2$; the integer -2 is the set of all ordered couples (m, n) of natural numbers with $n=m+2$.—Similarly, rational numbers are defined as classes of ordered couples of integers.—The various arithmetical operations can then be defined with reference to these new types of numbers, and the validity of all the arithmetical laws governing these operations can be proved by virtue of nothing more than Peano's postulates and the definitions of the various arithmetical concepts involved.

The much broader system thus obtained is still incomplete in the sense that not every number in it has a square root, and more generally, not every algebraic equation whose coefficients are all numbers of the system has a solution in the system. This suggests further expansions of the number system by the introduction of real and finally of complex numbers. Again, this enormous extension can be effected by mere definition, without the introduction of a single new postulate [6]. On the basis thus obtained, the various arithmetical and algebraic operations can be defined for the numbers of the new system, the concepts of function, of limit, of derivative and integral can be introduced, and the familiar theorems pertaining to these concepts can be proved, so that finally the huge system of mathematics as here delimited rests on the narrow basis of Peano's system: Every concept of mathematics can be defined by means of Peano's three primitives, and every proposition of mathematics can be deduced from the five postulates enriched by the definitions of the non-primitive terms [6a]. These deductions can be carried out, in most cases, by means of nothing more than the principles of formal logic; the proof of some theorems concerning real numbers, however, requires one assumption which is not usually included among the latter. This is the so-called axiom of choice. It asserts that given a class of mutually exclusive classes, none of which is empty, there exists at least one class which has exactly one element in common with each of the given classes. By virtue of this principle and the rules of formal logic, the content of all of mathematics can thus be derived from Peano's modest system—a remarkable achievement in systematizing the content of mathematics and clarifying the foundations of its validity.

7. Interpretations of Peano's primitives. As a consequence of this result, the whole system of mathematics might be said to be true by virtue of mere definitions (namely, of the non-primitive mathematical terms) provided that the five Peano postulates are true. However, strictly speaking, we cannot, at this juncture, refer to the Peano postulates as propositions which are either true or false, for they contain three primitive terms which have not been assigned any specific meaning. All we can assert so far is that any specific interpretation of the primi-

tives which satisfies the five postulates—*i.e.*, turns them into true statements—will also satisfy all the theorems deduced from them. But for Peano's system, there are several—indeed, infinitely many—interpretations which will do this. For example, let us understand by 0 the origin of a half-line, by the successor of a point on that half-line the point 1 cm. behind it, counting from the origin, and by a number any point which is either the origin or can be reached from it by a finite succession of steps each of which leads from one point to its successor. It can then readily be seen that all the Peano postulates as well as the ensuing theorems turn into true propositions, although the interpretation given to the primitives is certainly not the customary one, which was mentioned earlier. More generally, it can be shown that every progression of elements of any kind provides a true interpretation, or a "model," of the Peano system. This example illustrates our earlier observation that a postulate system cannot be regarded as a set of "implicit definitions" for the primitive terms: The Peano system permits of many different interpretations, whereas in everyday as well as in scientific language, we attach one specific meaning to the concepts of arithmetic. Thus, *e.g.*, in scientific and in everyday discourse, the concept 2 is understood in such a way that from the statement "Mr. Brown as well as Mr. Cope, but no one else is in the office, and Mr. Brown is not the same person as Mr. Cope," the conclusion "Exactly two persons are in the office" may be validly inferred. But the stipulations laid down in Peano's system for the natural numbers, and for the number 2 in particular, do not enable us to draw this conclusion; they do not "implicitly determine" the customary meaning of the concept 2 or of the other arithmetical concepts. And the mathematician cannot acquiesce at this deficiency by arguing that he is not concerned with the customary meaning of the mathematical concepts; for in proving, say, that every positive real number has exactly two real square roots, he is himself using the concept 2 in its customary meaning, and his very theorem cannot be proved unless we presuppose more about the number 2 than is stipulated in the Peano system.

If therefore mathematics is to be a correct theory of the mathematical concepts in their intended meaning, it is not sufficient for its validation to have shown that the entire system is derivable from the Peano postulates plus suitable definitions; rather, we have to inquire further whether the Peano postulates are actually true when the primitives are understood in their customary meaning. This question, of course, can be answered only after the customary meaning of the terms "0," "natural number," and "successor" has been clearly defined. To this task we now turn.

8. Definition of the customary meaning of the concepts of arithmetic in purely logical terms. At first blush, it might seem a hopeless undertaking to try to define these basic arithmetical concepts without presupposing other terms of arithmetic, which would involve us in a circular procedure. However, quite rigorous definitions of the desired kind can indeed be formulated, and it can be shown that for the concepts so defined, all Peano postulates turn into true state-

ments. This important result is due to the research of the German logician G. Frege (1848–1925) and to the subsequent systematic and detailed work of the contemporary English logicians and philosophers B. Russell and A. N. Whitehead. Let us consider briefly the basic ideas underlying these definitions [7].

A natural number—or, in Peano's term, a number—in its customary meaning can be considered as a characteristic of certain *classes* of objects. Thus, *e.g.*, the class of the apostles has the number 12, the class of the Dionne quintuplets the number 5, any couple the number 2, and so on. Let us now express precisely the meaning of the assertion that a certain class C has the number 2, or briefly, that $n(C)=2$. Brief reflection will show that the following definiens is adequate in the sense of the customary meaning of the concept 2: There is some object x and some object y such that (1) $x\epsilon C$ (*i.e.*, x is an element of C) and $y\epsilon C$, (2) $x\neq y$, and (3) if z is any object such that $z\epsilon C$, then either $z=x$ or $z=y$. (Note that on the basis of this definition it becomes indeed possible to infer the statement "The number of persons in the office is 2" from "Mr. Brown as well as Mr. Cope, but no one else is in the office, and Mr. Brown is not identical with Mr. Cope"; C is here the class of persons in the office.) Analogously, the meaning of the statement that $n(C)=1$ can be defined thus: There is some x such that $x\epsilon C$, and any object y such that $y\epsilon C$, is identical with x. Similarly, the customary meaning of the statement that $n(C)=0$ is this: There is no object such that $x\epsilon C$.

The general pattern of these definitions clearly lends itself to the definition of any natural number. Let us note especially that in the definitions thus obtained, the definiens never contains any arithmetical term, but merely expressions taken from the field of formal logic, including the signs of identity and difference. So far, we have defined only the meaning of such phrases as "$n(C)=2$," but we have given no definition for the numbers 0, 1, 2, $\cdots$ apart from this context. This desideratum can be met on the basis of the consideration that 2 is that property which is common to all couples, *i.e.*, to all classes C such that $n(C)=2$. This common property may be conceptually represented by the class of all those classes which share this property. Thus we arrive at the definition: 2 is the class of all couples, *i.e.*, the class of all classes C for which $n(C)=2$.—This definition is by no means circular because the concept of couple—in other words, the meaning of "$n(C)=2$"—has been previously defined without any reference to the number 2. Analogously, 1 is the class of all unit classes, *i.e.*, the class of all classes C for which $n(C)=1$. Finally, 0 is the class of all null classes, *i.e.*, the class of all classes without elements. And as there is only one such class, 0 is simply the class whose only element is the null class. Clearly, the customary meaning of any given natural number can be defined in this fashion [8]. In order to characterize the intended interpretation of Peano's primitives, we actually need, of all the definitions here referred to, only that of the number 0. It remains to define the terms "successor" and "integer."

The definition of "successor," whose precise formulation involves too many

niceties to be stated here, is a careful expression of a simple idea which is illustrated by the following example: Consider the number 5, *i.e.*, the class of all quintuplets. Let us select an arbitrary one of these quintuplets and add to it an object which is not yet one of its members. 5′, the successor of 5, may then be defined as the number applying to the set thus obtained (which, of course, is a sextuplet). Finally, it is possible to formulate a definition of the customary meaning of the concept of natural number; this definition, which again cannot be given here, expresses, in a rigorous form, the idea that the class of the natural numbers consists of the number 0, its successor, the successor of that successor, and so on.

If the definitions here characterized are carefully written out—this is one of the cases where the techniques of symbolic, or mathematical, logic prove indispensable—it is seen that the definiens of every one of them contains exclusively terms from the field of pure logic. In fact, it is possible to state the customary interpretation of Peano's primitives, and thus also the meaning of every concept definable by means of them—and that includes every concept of mathematics—in terms of the following 7 expressions, in addition to variables such as "x" and "C": *not, and, if—then*; *for every object x* it is the case that . . . ; *there is some object x* such that . . . ; x is an *element* of class C; *the class of all things x such that* . . . . And it is even possible to reduce the number of logical concepts needed to a mere four: The first three of the concepts just mentioned are all definable in terms of "*neither—nor*," and the fifth is definable by means of the fourth and "*neither—nor*." Thus, all the concepts of mathematics prove definable in terms of four concepts of pure logic. (The definition of one of the more complex concepts of mathematics in terms of the four primitives just mentioned may well fill hundreds or even thousands of pages; but clearly this affects in no way the theoretical importance of the result just obtained; it does, however, show the great convenience and indeed practical indispensability for mathematics of having a large system of highly complex defined concepts available.)

9. The truth of Peano's postulates in their customary interpretation. The definitions characterized in the preceding section may be said to render precise and explicit the customary meaning of the concepts of arithmetic. Moreover—and this is crucial for the question of the validity of mathematics—it can be shown that the Peano postulates all turn into true propositions if the primitives are construed in accordance with the definitions just considered.

Thus, P1 (0 is a number) is true because the class of all numbers—*i.e.*, natural numbers—was defined as consisting of 0 and all its successors. The truth of P2 (The successor of any number is a number) follows from the same definition. This is true also of P5, the principle of mathematical induction. To prove this, however, we would have to resort to the precise definition of "integer" rather than the loose description given of that definition above. P4 (0 is not the successor of any number) is seen to be true as follows: By virtue of the definition of "successor," a number which is a successor of some number can apply only to

classes which contain at least one element; but the number 0, by definition, applies to a class if and only if that class is empty.—While the truth of P1, P2, P4, P5 can be inferred from the above definitions simply by means of the principles of logic, the proof of P3 (No two numbers have the same successor) presents a certain difficulty. As was mentioned in the preceding section, the definition of the successor of a number n is based on the process of adding, to a class of n elements, one element not yet contained in that class. Now if there should exist only a finite number of things altogether then this process could not be continued indefinitely, and P3, which (in conjunction with P1 and P2) implies that the integers form an infinite set, would be false. Russell's way of meeting this difficulty [9] was to introduce a special "axiom of infinity," which stipulates, in effect, the existence of infinitely many objects and thus makes P3 demonstrable. The axiom of infinity can be formulated in purely logical terms and may therefore be considered as a postulate of logic; however, it certainly does not belong to the generally recognized principles of logic; and it thus introduces a foreign element into the otherwise unexceptionable derivation of the Peano postulates from pure logic. Recently, however, it has been shown [10] that a suitable system of logical principles can be set up which is even less comprehensive than the rules of logic which are commonly used [11], and in which the existence of infinitely many objects can be proved without the need for a special axiom.

10. Mathematics as a branch of logic. As was pointed out earlier, all the theorems of arithmetic, algebra, and analysis can be deduced from the Peano postulates and the definitions of those mathematical terms which are not primitives in Peano's system. This deduction requires only the principles of logic plus, in certain cases, the axiom of choice. By combining this result with what has just been said about the Peano system, the following conclusion is obtained, which is also known as *the thesis of logicism concerning the nature of mathematics*:

Mathematics is a branch of logic. It can be derived from logic in the following sense:

a. All the concepts of mathematics, *i.e.* of arithmetic, algebra, and analysis, can be defined in terms of four concepts of pure logic.

b. All the theorems of mathematics can be deduced from those definitions by means of the principles of logic (including the axiom of choice).

In this sense it can be said that the propositions of the system of mathematics as here delimited are true by virtue of the definitions of the mathematical concepts involved, or that they make explicit certain characteristics with which we have endowed our mathematical concepts by definition. The propositions of mathematics have, therefore, the same unquestionable certainty which is typical of such propositions as "All bachelors are unmarried," but they also share the complete lack of empirical content which is associated with that certainty: The propositions of mathematics are devoid of all factual content; they convey no information whatever on any empirical subject matter.

11. On the applicability of mathematics to empirical subject matter. This result seems to be irreconcilable with the fact that after all mathematics has proved to be eminently applicable to empirical subject matter, and that indeed the greater part of present-day scientific knowledge has been reached only through continual reliance on and application of the propositions of mathematics.—Let us try to clarify this apparent paradox by reference to some examples.

Suppose that we are examining a certain amount of some gas, whose volume v, at a certain fixed temperature, is found to be 9 cubic feet when the pressure p is 4 atmospheres. And let us assume further that the volume of the gas for the same temperature and $p=6$ *at.*, is predicted by means of Boyle's law. Using elementary arithmetic we reason thus: For corresponding values of v and p, $vp=c$, and $v=9$ when $p=4$; hence $c=36$: Therefore, when $p=6$, then $v=6$. Suppose that this prediction is borne out by subsequent test. Does that show that the arithmetic used has a predictive power of its own, that its propositions have factual implications? Certainly not. All the predictive power here deployed, all the empirical content exhibited stems from the initial data and from Boyle's law, which asserts that $vp=c$ for *any* two corresponding values of v and p, hence also for $v=9$, $p=4$, and for $p=6$ and the corresponding value of v [12]. The function of the mathematics here applied is not predictive at all; rather, it is analytic or explicative: it renders explicit certain assumptions or assertions which are included in the content of the premises of the argument (in our case, these consist of Boyle's law plus the additional data); mathematical reasoning reveals that those premises contain—hidden in them, as it were,—an assertion about the case as yet unobserved. In accepting our premises—so arithmetic reveals—we have—knowingly or unknowingly—already accepted the implication that the p-value in question is 6. Mathematical as well as logical reasoning is a conceptual technique of making explicit what is implicitly contained in a set of premises. The conclusions to which this technique leads assert nothing that is *theoretically new* in the sense of not being contained in the content of the premises. But the results obtained may well be *psychologically new*: we may not have been aware, before using the techniques of logic and mathematics, what we committed ourselves to in accepting a certain set of assumptions or assertions.

A similar analysis is possible in all other cases of applied mathematics, including those involving, say, the calculus. Consider, for example, the hypothesis that a certain object, moving in a specified electric field, will undergo a constant acceleration of 5 feet/sec^2. For the purpose of testing this hypothesis, we might derive from it, by means of two successive integrations, the prediction that if the object is at rest at the beginning of the motion, then the distance covered by it at any time t is $\frac{5}{2}t^2$ feet. This conclusion may clearly be psychologically new to a person not acquainted with the subject, but it is not theoretically new; the content of the conclusion is already contained in that of the hypothesis about the constant acceleration. And indeed, here as well as in the case of the compression of a gas, a failure of the prediction to come true would be considered as indica-

tive of the factual incorrectness of at least one of the premises involved (*f.ex.*, of Boyle's law in its application to the particular gas), but never as a sign that the logical and mathematical principles involved might be unsound.

Thus, in the establishment of empirical knowledge, mathematics (as well as logic) has, so to speak, the function of a theoretical juice extractor: the techniques of mathematical and logical theory can produce no more juice of factual information than is contained in the assumptions to which they are applied; but they may produce a great deal more juice of this kind than might have been anticipated upon a first intuitive inspection of those assumptions which form the raw material for the extractor.

At this point, it may be well to consider briefly the status of those mathematical disciplines which are not outgrowths of arithmetic and thus of logic; these include in particular topology, geometry, and the various branches of abstract algebra, such as the theory of groups, lattices, fields, *etc.* Each of these disciplines can be developed as a purely deductive system on the basis of a suitable set of postulates. If P be the conjunction of the postulates for a given theory, then the proof of a proposition T of that theory consists in deducing T from P by means of the principles of formal logic. What is established by the proof is therefore not the truth of T, but rather the fact that T is true provided that the postulates are. But since both P and T contain certain primitive terms of the theory, to which no specific meaning is assigned, it is not strictly possible to speak of the truth of either P or T; it is therefore more adequate to state the point as follows: If a proposition T is logically deduced from P, then every specific interpretation of the primitives which turns all the postulates of P into true statements, will also render T a true statement.—Up to this point, the analysis is exactly analogous to that of arithmetic as based on Peano's set of postulates. In the case of arithmetic, however, it proved possible to go a step further, namely to define the customary meanings of the primitives in terms of purely logical concepts and to show that the postulates—and therefore also the theorems—of arithmetic are unconditionally true by virtue of these definitions. An analogous procedure is not applicable to those disciplines which are not outgrowths of arithmetic: The primitives of the various branches of abstract algebra have no specific "customary meaning"; and if geometry in its customary interpretation is thought of as a theory of the structure of physical space, then its primitives have to be construed as referring to certain types of physical entities, and the question of the truth of a geometrical theory in this interpretation turns into an *empirical* problem [13]. For the purpose of applying any one of these non-arithmetical disciplines to some specific field of mathematics or empirical science, it is therefore necessary first to assign to the primitives some specific meaning and then to ascertain whether in this interpretation the postulates turn into true statements. If this is the case, then we can be sure that all the theorems are true statements too, because they are logically derived from the postulates and thus simply explicate the content of the latter in the given interpretation.—In their application to empirical subject matter, therefore, these mathematical theories no less than

those which grow out of arithmetic and ultimately out of pure logic, have the function of an analytic tool, which brings to light the implications of a given set of assumptions but adds nothing to their content.

But while mathematics in no case contributes anything to the content of our knowledge of empirical matters, it is entirely indispensable as an instrument for the validation and even for the linguistic expression of such knowledge: The majority of the more far-reaching theories in empirical science—including those which lend themselves most eminently to prediction or to practical application—are stated with the help of mathematical concepts; the formulation of these theories makes use, in particular, of the number system, and of functional relationships among different metrical variables. Furthermore, the scientific test of these theories, the establishment of predictions by means of them, and finally their practical application, all require the deduction, from the general theory, of certain specific consequences; and such deduction would be entirely impossible without the techniques of mathematics which reveal what the given general theory implicitly asserts about a certain special case.

Thus, the analysis outlined on these pages exhibits the system of mathematics as a vast and ingenious conceptual structure without empirical content and yet an indispensable and powerful theoretical instrument for the scientific understanding and mastery of the world of our experience.

References

1. A discussion of the status of geometry is given in my article, Geometry and Empirical Science, American Mathematical Monthly, vol. 52, pp. 7–17, 1945.

2. The objection is sometimes raised that without certain types of experience, such as encountering several objects of the same kind, the integers and the arithmetical operations with them would never have been invented, and that therefore the propositions of arithmetic do have an empirical basis. This type of argument, however, involves a confusion of the logical and the psychological meaning of the term "basis." It may very well be the case that certain experiences occasion psychologically the formation of arithmetical ideas and in this sense form an empirical "basis" for them; but this point is entirely irrelevant for the logical questions as to the *grounds* on which the propositions of arithmetic may be accepted as true. The point made above is that no empirical "basis" or evidence whatever is needed to establish the truth of the propositions of arithmetic.

3. A precise account of the definition and the essential characteristics of the identity relation may be found in A. Tarski, Introduction to Logic, New York, 1941, Ch. III.

4. For a lucid and concise account of the axiomatic method, see A. Tarski, *l.c.*, Ch. VI.

5. Bertrand Russell, Introduction to Mathematical Philosophy, New York and London, 1919, p. 71.

6. For a more detailed account of the construction of the number system on Peano's basis, *cf.* Bertrand Russell, *l.c.*, esp. Chs. I and VII.—A rigorous and concise presentation of that construction, beginning, however, with the set of all integers rather than that of the natural numbers, may be found in G. Birkhoff and S. MacLane, A Survey of Modern Algebra, New York 1941, Chs. I, II, III, V.—For a general survey of the construction of the number system, *cf.* also J. W. Young, Lectures on the Fundamental Concepts of Algebra and Geometry, New York, 1911, esp. lectures X, XI, XII.

6*a*. As a result of very deep-reaching investigations carried out by K. Gödel it is known that arithmetic, and *a fortiori* mathematics, is an incomplete theory in the following sense: While all those propositions which belong to the classical systems of arithmetic, algebra, and analysis can

indeed be derived, in the sense characterized above, from the Peano postulates, there exist nevertheless other propositions which can be expressed in purely arithmetical terms, and which are true, but which cannot be derived from the Peano system. And more generally: For any postulate system of arithmetic (or of mathematics for that matter) which is not self-contradictory, there exist propositions which are true, and which can be stated in purely arithmetical terms, but which cannot be derived from that postulate system. In other words, it is impossible to construct a postulate system which is not self-contradictory, and which contains among its consequences all true propositions which can be formulated within the language of arithmetic.

This fact does not, however, affect the result outlined above, namely, that it is possible to deduce, from the Peano postulates and the additional definitions of non-primitive terms, all those propositions which constitute the classical theory of arithmetic, algebra, and analysis; and it is to these propositions that I refer above and subsequently as the propositions of mathematics.

7. For a more detailed discussion, *cf.* Russell, *l.c.*, Chs. II, III, IV. A complete technical development of the idea can be found in the great standard work in mathematical logic, A. N. Whitehead and B. Russell, Principia Mathematica, Cambridge, England, 1910–1913.—For a very precise recent development of the theory, see W. V. O. Quine, Mathematical Logic, New York 1940.—A specific discussion of the Peano system and its interpretations from the viewpoint of semantics is included in R. Carnap, Foundations of Logic and Mathematics, International Encyclopedia of Unified Science, vol. I, no. 3, Chicago, 1939; especially sections 14, 17, 18.

8. The assertion that the definitions given above state the "customary" meaning of the arithmetical terms involved is to be understood in the logical, not the psychological sense of the term "meaning." It would obviously be absurd to claim that the above definitions express "what everybody has in mind" when talking about numbers and the various operations that can be performed with them. What is achieved by those definitions is rather a "logical reconstruction" of the concepts of arithmetic in the sense that if the definitions are accepted, then those statements in science and everyday discourse which involve arithmetical terms can be interpreted coherently and systematically in such a manner that they are capable of objective validation. The statement about the two persons in the office provides a very elementary illustration of what is meant here.

9. *Cf.* Bertrand Russell, *l.c.*, p. 24 and Ch. XIII.

10. This result has been obtained by W. V. O. Quine; *cf.* his Mathematical Logic, New York, 1940.

11. The principles of logic developed in Quine's work and in similar modern systems of formal logic embody certain restrictions as compared with those logical rules which had been rather generally accepted as sound until about the turn of the 20th century. At that time, the discovery of the famous paradoxes of logic, especially of Russell's paradox (*cf.* Russell, *l.c.*, Ch. XIII) revealed the fact that the logical principles implicit in customary mathematical reasoning involved contradictions and therefore had to be curtailed in one manner or another.

12. Note that we may say "hence" by virtue of the rule of substitution, which is one of the rules of logical inference.

13. For a more detailed discussion of this point, *cf.* the article mentioned in reference 1.

TWO DIMENSIONS

J. L. COOLIDGE, Harvard University

1. The difficulty. A number of years ago I heard a distinguished geometer deliver the dictum, "In geometry the only functions in which we are interested are analytic functions, and the only interesting theorems are those which are true in any number of dimensions." I think that the two parts of this statement can be treated separately, as I cannot see any connection between them.

I heartily agree with the first part. The geometer does not like to be bothered with questions of analyticity; let the analysts worry about them. Some differential geometers, in an effort to appear liberal, change the formula and say, "We assume the existence of all derivatives which appear." This would seem fairly obvious to most persons. In any case, if the geometer has three or four good derivatives he does not much care whether there is a discontinuity in the fifth.

The case seems to me very different with regard to the second part of the statement: "No theorem is interesting unless it hold in any number of dimensions." Of course what is involved here is a definition of "interesting." A mathematician hesitates to say what he means by "interesting" and he carefully refrains from asking a non-mathematician what *he* means by "interesting" in mathematics, as he knows only too well what the answer would be. If one grants that part of the definition of "interesting" is that the statement holds in any number of dimensions, there is nothing more to be said. I am interested in the Pythagorean theorem and the four color problem; my opponent is not. Let us at least part as friends. To me it is a necessary condition for something to be interesting that it should be specific enough to present a definite and clear-cut picture to the mind, and that it should be general enough to include a fairly large variety of material.

What are the sufficient conditions? I don't know. But I will not accept application to any number of dimensions as necessary, but will maintain that geometry in two dimensions is not only simpler than geometry in a higher space, but frequently richer and more significant. Let me try to justify this assumption.

2. Examples. Let us begin with projective geometry. We have a set of objects in one to one correspondence with the values of the ratios $n+1$ homogeneous coordinates, not all zero. These we call "points of an n-space" or S_n. Points whose coordinates are linearly dependent on those of two given points form a line, those dependent on three linearly independent ones a plane. If there be $k+1$ linearly independent points we have a k-spread or S_k. The coordinates of the points of a k-spread are connected by $n-k$ homogeneous linear equations. The points whose coordinates are connected by a single quadratic equation, say $a_{ij}x^ix^j=0$ are said to form a *hyperquadric*. Hyperquadrics are classified under the projective group by the rank of the matrix $\|a_{ij}\|$.

So far all serene. There is no virtue in any one number of dimensions, but $\cdots$. In two dimensions we have the theorems of Pascal and Brianchon

for hexagons inscribed or circumscribed to conics. There is nothing really comparable in higher space, and for that reason the study of the conics is richer than that of the other hyperquadrics, even if we raise our eyebrows at the usual statement that Pascal discovered more than four hundred corollaries to his theorem.

At this point an opponent can come back at me rather hard with the remark that Desargues' two-triangle theorem can be derived for planes contained in a three-space from the simplest assumptions about points, lines and planes in the space, whereas it cannot be derived from assumptions about points and lines in the plane itself. This is perfectly true. I am willing to concede that a plane that can be contained in a three-space is richer than one that cannot, but I cannot see that the main thesis is affected. I think also that Desargues' figure of two triangles is more generally important that Stephanos' configuration of three desmic tetrahedra, the nearest three-dimensional analogue.

Let us next look at algebraic curves. The most teasing thing about these is their singular points. These can be of various sorts and the earlier English geometers invented a variety of names: acnodes, crunodes, ceratoid cusps and the like. But Noether showed that any singular point of a plane algebraic curve could be looked upon as a union of infinitely near ordinary singular points, or as the limit of such points as they approached to final positions, a singular point being ordinary when the tangents are distinct. The numbers here involved are not at all arbitrary but appear in perfectly definite fashion in the power series development of the curve in the vicinity of the singularity. This concept draws the whole theory together and enables us to handle it with confidence.

I am not aware that there is anything at all corresponding to this in the study of singularities of other algebraic varieties. Step up but one dimension. An algebraic surface may have isolated singular points, or singular curves, or singular points of singular curves or pinch points on curves. I don't think the possibilities have all been classified or sub-divided. In the birational theory we transform to surfaces with nothing worse than double curves with triple points on them, like the corners of a room, but in the projective theory there seems no simple guiding principle.

A Cremona transformation of a plane or space is one that is algebraic and rational, with a rational algebraic inverse. Now Noether has shown that in the plane any Cremona transformation can be factored into linear and quadratic transformations, but in 1912 Hilda Hudson proved that there was no corresponding reduction of Cremona transformations in space. In a sense this makes the space theory richer, and the number of studies of it is much larger, but Noether's theorem gives an invaluable central motif.

Let us compare a circle and a sphere. Each is the locus of points at a given distance from a fixed point of the space in question. In each case the tangent variety is perpendicular to the radius, the polar plane is perpendicular to the line joining the pole with the centre. The fundamental relation of two circles or two spheres is orthogonal intersection; linear systems are composed of circles or spheres cutting others orthogonally. The theory of inversion is the same in

both cases and the general circle or sphere transformation can be factored into a number of inversions and reflections in planes. The identity seems complete. Not quite. All angles inscribed in the same circular arc are equal; the sphere has no corresponding property. The results of this are incalculable. The geometry of the tetrahedron is meager indeed compared with the fantastically rich geometry of the triangle. The constancy of the inscribed angle is much the most useful property of the circle. It is curious to observe that it depends on the euclidean parallel postulate; had our customary geometry been Lobachevskian or Riemannian we should have been without it.

Let us finally look at conformal transformations. A transformation is conformal if it is a point to point transformation which keeps angles invariant. A conformal transformation of space carries a point to a point and keeps invariant the angle of intersection of two surfaces. It will therefore carry two surfaces which intersect at right angles into two other such surfaces, and so carry a triply orthogonal surface system into another such. But Dupin's theorem tells us that in a triply orthogonal system the various surfaces meet in their lines of curvature. Hence a conformal transformation will carry lines of curvature into lines of curvature, and so carry a surface, all of whose curves are lines of curvature, into another such surface. But the only surfaces with this property are spheres and planes. This gives us the theorem due, I think, to Liouville, that the only conformal transformations of three-space are sphere transformations.

Now look at two dimensions. Here it is well to use minimal coordinates

$$z = x + iy \qquad \bar{z} = x - iy.$$

A conformal transformation of the plane must carry a self-perpendicular direction into a self-perpendicular direction. The differential equation for such is

$$dz d\bar{z} = 0$$

so this must be invariant. Let our transformation be

$$z' = z'(z, \bar{z}) \qquad \bar{z}' = \bar{z}'(z, \bar{z})$$

$$dz' d\bar{z}' = F(z, \bar{z}) dz d\bar{z}$$

$$\frac{\partial z'}{\partial z} \frac{\partial \bar{z}'}{\partial z} = 0; \qquad \frac{\partial z'}{\partial \bar{z}} \frac{\partial \bar{z}'}{\partial \bar{z}} = 0.$$

Then either

$$z' = z'(z); \qquad \bar{z}' = \bar{z}'(\bar{z}')$$

or

$$z' = z'(\bar{z}) \qquad \bar{z}' = \bar{z}'(\bar{z}).$$

Here we have the most striking case of the divergence of plane geometry from the geometry of higher space. The general conformal transformation of three-space depends on ten parameters; the most general conformal transformation of the plane depends on an infinite number of parameters, that is, on an arbitrary

function of the complex variable. It is worth mentioning in this connection that there is a prodigious amount of literature of conformal transformations of the plane, generally called by those most interested Conforme Abbildung.

I will also mention in passing a curious and little known theorem of Clifford's which comes from the use of these same minimal parameters, and which, of course, has no analogue in higher dimensions. Suppose that we have n lines through each of the circular points at infinity (not including the infinite line itself). They will have n^2 intersections. A set of n points, each of which lies on just one line of each set shall be called a "group" of intersections. There will be $n!$ such groups. Then the product of the distances of *any* point in the plane from the intersections of a group is independent of the group chosen. For instance, the product of the distances of any point in the plane from the real foci of an ellipse is equal to the product of its distances from the two imaginary foci.

3. Underlying reasons. It would not be difficult to find other examples of this sort of thing. It is more significant to try to see if there be any inner reason for the phenomenon. I can only grope.

Let me give two different proofs of two of the theorems. First, Liouville's for conformal transformations of three-space. In such a transformation, a self-perpendicular direction will go into a self-perpendicular direction. Hence on any surface the minimal curves, those whose tangents are self-perpendicular, will go into the corresponding curves on another surface. Hence a surface which has but a single set of such curves will go into another surface of this sort. But the only surfaces with a single set of minimal curves is a minimal developable, one whose generators are minimal straight lines. Hence a minimal developable will go into a minimal developable. It follows then that a set of minimal developables tangent at one point will go into a similar set. But such a set of developables have a common minimal line. Hence, finally, a conformal transformation of three-space will carry a minimal line into a minimal line.

In the plane there is only a one-parameter set of minimal lines to be carried, but a conformal transformation of space must keep invariant a whole three-parameter family of these lines, and this great invariance hampers the freedom of the system seriously.

Secondly, let us look at the equality of all angles in a circular arc. Suppose that in three-space we have five points which lie on a sphere $P_1P_2 \cdots P_5$. Let the distance of P_iP_j be d_{ij}. Then it is well known that

$$\begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 \\ d_{21}^2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{51}^2 & d_{52}^2 & d_{53}^2 & d_{54}^2 & 0^2 \end{vmatrix} = 0.$$

This equation becomes rather complicated when multiplied out. There is a similar equation for any number of dimensions. But in the single case of two dimensions, when it is a question of four points lying on a circle, the determinant is factorable, and we have

$$(d_{12}d_{34} + d_{13}d_{24} + d_{14}d_{23})(-d_{12}d_{34} + d_{13}d_{24} + d_{14}d_{23})(d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23})$$
$$(d_{12}d_{34} + d_{13}d_{24} - d_{14}d_{23}) = 0.$$

If we set the appropriate one of these factors equal to zero, we have Ptolemy's theorem that the necessary and sufficient condition that the vertices of a quadrilateral lie on a circle is that the sum of the products of the opposite sides shall be equal to the product of the diagonals, from which the equality of the angles is easily deduced.

A correspondent kindly calls to my attention the following example which is simpler, if perhaps less instructive. If we have a triangle the lengths of whose sides are a, b, c, the square of the area, when expressed in determinant form is

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix}.$$

When this is expanded, it can be factored into the factors which appear in Hero's classical formula. The expression for the volume of a tetrahedron in terms of the lengths of its edges is unfactorable.

I conclude from these cases that when the data are comparatively simple a greater number of techniques are available, and so a greater variety of interesting results may be found. Or when there are too many dimensions or too many parameters involved, the great generality excludes the use of interesting methods which are available in simpler cases. I am not sure that this by any means covers the matter, but it seems to me a first suggestion.

I think it is also true that this same phenomenon appears in other parts of mathematics. In analysis the theory of functions of a single complex variable is a peculiarly compact and well integrated subject. The two corner stones here appear to be Cauchy's integral, and the Riemann surfaces, both of which involve the topology of the plane. Nothing so helpful is available when there are more variables. In the theory of differential equations we need not consider equations of higher than the second order to have most of the important material, nor does this seem to spring from the fact that we need only one differentiation for velocities and two for accelerations.

In algebra determinants, which are essentially two dimensional arrays, are of incalculable use. Various mathematicians have occupied themselves with three or more dimensional determinants; I could never learn that such things were of any real importance. The underlying fact here seems to be that when we are occupied with a certain number of functions of a certain number of variables

the question of their independence is vital, and this depends on the rank of a matrix, that is to say, the vanishing or non-vanishing of determinants.

The conclusion of the matter seems rather bleak. Modern mathematics is extremely abstract and extremely general. That is excellent. We always wish to know the extreme limits to which any truth will stretch. But I personally have the depressing feeling again and again, "Well, where do we go from here?" and I don't find the answer. Historically, progress is always from the particular to the general. I do not know of a single interesting theorem dealing with the geometry of the plane which was first found as a special case of something that holds in higher space, and there are plenty of interesting plane theorems that do not step up. With Fermat's and Goldbach's theorems still mocking us and the four color problem still unsolved there still remain some simple puzzles to tax our ingenuity.

AN ELEMENTARY CONSTRUCTIVE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

P. C. ROSENBLOOM, Brown University

1. Introduction. In this paper we shall give a method of constructing from a given polynomial $P(z)$ by rational operations a sequence of complex numbers $\{z_n\}$ which satisfies the Cauchy convergence criterion and such that $P(z_n)$ converges to zero. All of this part of the proof is constructive and completely elementary. The existence of a root of $P(z)$ then follows immediately from the definition of the real number system by means of Cantor's regular sequences. (See for example [4].) Brouwer and deLoor [1] gave an "intuitionistic" proof (see also [2]), but made use of the properties of continuous curves and functions, while Weyl's proof [5] makes use of integration, and in particular, the formula

$$\frac{1}{2\pi i}\int_C \frac{f'(z)dz}{f(z)} = n$$

for the number of zeros of the polynomial $f(z)$ enclosed by the contour C. If these preliminaries were developed in full, their proofs would be much more complicated than they appear to be. In our proof the only appearance of limiting processes is in the very definition of the real number system.

Our main device is the use of summing over the lattice points of squares instead of integration. This idea can be used to replace other function-theoretic proofs by elementary algebraic proofs. We suspect that, for example, the Schottky-Landau theorem can be proved for polynomials in this way. Pedagogically, however, it would be better to preface our present proof by the following function-theoretic argument after which it is modeled. If the polynomial $P(z)$ of degree N had no zeros, then $P'(z)/P(z)$ would be regular everywhere, and therefore

$$\frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}dz = 0$$

for the circle C: $|z| = R$. But

$$\left|\frac{1}{2\pi i}\int_C \frac{P'(z)dz}{P(z)} - \frac{1}{2\pi i}\int_C \frac{Ndz}{z}\right| = O(R^{-1})$$

as $R \to \infty$, and

$$\frac{1}{2\pi i}\int_C \frac{Ndz}{z} = N.$$

These three results are incompatible, and therefore $P(z)$ must have a root. The minor details of the computations involved in the following argument will be more perspicuous if this skeleton key is kept in mind.

2. Sums over lattice points. If $f(z)$ is any function of the complex variable z defined on the square $C_a(z_0)$ with center at z_0 and sides equal to $2a$ and parallel to the coordinate axes, we define the nth sum of $f(z)$ over $C_a(z_0)$ as

$$\begin{aligned} S_n(f; z_0, a) = \frac{2a}{n}\sum_{k=1}^{n} [\{f(z_0 - a - ai + (2k-1)h) - f(z_0 + a + ai - (2k-1)h)\} \\ + i\{f(z_0 + a - ai + i(2k-1)h) - f(z_0 - a + ai - i(2k-1)h)\}], \end{aligned}$$

where $h = a/n$. This is obtained by dividing each side of $C_a(z_0)$ into n equal segments, denoting by $z_1, \cdots, z_{4n+1} = z_1$ the points of subdivision taken in counterclockwise order and by $\zeta_k = (z_k + z_{k+1})/2$ the midpoint of the kth segment, and forming the sum

$$S_n(f; z_0, a) = \sum_{k=1}^{4n} f(\zeta_k)(z_{k+1} - z_k).$$

If we worked with circles instead of squares, we would have to introduce the exponential function and other foreign matter.

The following properties of this summing operation are almost trivial:

$$\begin{aligned} S_n(c; z_0, a) &= 0 \qquad (c = \text{constant}); \\ S_n(z; z_0, a) &= 0; \\ S_n(1/z; 0, a) &= \sum_{k=1}^{n} \frac{8ni}{n^2 + (n+1-2k)^2} = 2\pi_n i; \\ S_n(cf; z_0, a) &= cS_n(f; z_0, a) \qquad (c = \text{constant}); \\ S_n(f+g; z_0, a) &= S_n(f; z_0, a) + S_n(g; z_0, a); \\ S_n(f; z_0, a) &= \sum_{p,q=1}^{n} S_1\left(f; z_{pq}, \frac{a}{n}\right), \end{aligned}$$

where the z_{pq}'s are the centers of the n^2 squares with sides equal to $2a/n$ obtained by dissecting $C_a(z_0)$ by lines parallel to the coordinate axes. We note that

$$\pi_n > \frac{n \cdot 4n}{n^2 + n^2} = 2.$$

Also, if $|f(z)| \leqq M$ at the points ζ_k, then

$$(1) \qquad | S_n(f; z_0, a) | \leqq 8aM.$$

3. Cauchy's theorem. We now derive an analogue to Cauchy's theorem for rational functions. The details will be clearer if the classical proof of the latter, say as presented in Knopp [3], is borne in mind.

Let $f(z) = Q(z)/P(z)$, where $P(z) = \sum_{k=0}^{N} a_k z^k$, $Q(z) = \sum_{k=0}^{N} b_k z^k$; let $A = \max|a_k|$ and $B = \max|b_k|$ for $k = 0, 1, \cdots, N$. We shall maintain these notations throughout the rest of the paper.

LEMMA 1. *If* $|P(z)| \geqq m > 0$ *on the lattice points in and on the square* $C_a(0)$ *with the mesh* $h = a/n$, *and* $a < 2^{-1/2}$, *then*

$$| S_n(f; 0, a) | < \frac{64A^2Ba^3}{m^3(1 - \sqrt{2}a)}.$$

Proof. Let $\epsilon(z) = f(z) - f(0) - zf'(0)$. Then we have $S_n(f; 0, a) = S_n(\epsilon; 0, a)$. But

$$\begin{aligned} \epsilon(z) &= \frac{a_0^2 Q(z) - a_0 b_0 P(z) - (a_0 b_1 - b_0 a_1) z P(z)}{a_0^2 P(z)} \\ &= \frac{\sum_{k=2}^{N+1} (a_0^2 b_k - a_0 a_k b_0 - a_0 a_{k-1} b_1 + a_1 a_{k-1} b_0) z^k}{a_0^2 P(z)}, \end{aligned}$$

where $a_{N+1} = b_{N+1} = 0$. Therefore

$$| \epsilon(z) | \leqq \frac{4A^2B}{m^3} \sum_{k=2}^{N+1} (\sqrt{2}a)^k < \frac{8a^2BA^2}{m^3(1 - \sqrt{2}a)}$$

on the lattice points of $C_a(0)$. The lemma now follows from (1).

We must now eliminate the special position of the point 0 in this lemma.

LEMMA 2. *If* $|P(z)| \geqq m > 0$ *on the lattice points of* $C_a(z_0)$ *with the mesh* $h = a/n$, *and* $a < 2^{-1/2}$, $|z_0| \leqq R$, $R \geqq 1$, *then*

$$| S_n(f; z_0, a) | \leqq \frac{64\gamma^3 R^{3N} B A^2 a^3}{m^3(1 - \sqrt{2}a)},$$

where γ *is the binomial coefficient* $C_{N+1,q}$ *with* $q = [(N+1)/2]$.

Proof. Let $P_1(z) = P(z_0 + z) = \sum_{k=0}^{N} a_{k1} z^k$, $Q_1(z) = Q(z_0 + z) = \sum_{k=0}^{N} b_{k1} z^k$, $A_1 = \max|a_{k1}|$, and $B_1 = \max|b_{k1}|$. Then

$$(2)\qquad |a_{k1}| = \left|\frac{P^{(k)}(z_0)}{k!}\right| = \left|\sum_{n=0}^{N-k} a_{n+k}C_{n+k,k}z_0^n\right| \leqq AR^{N-k}\sum_{n=0}^{N-k} C_{n+k,k} \leqq \gamma AR^N.$$

Hence $A_1 \leqq \gamma AR^N$, and similarly $B_1 \leqq \gamma BR^N$. Then the above estimate follows by Lemma 1 applied to $f_1(z) = f(z_0+z)$.

THEOREM 1. *Let*

$$m = \min_{|k|,|l|\leqq n} \left|P\left(\frac{(k+il)a}{n}\right)\right|,$$

and let $\sqrt{2}a \leqq R$, $1 \leqq R$, $\sqrt{2}a < n$. *Then*

$$|S_n(f;0,a)| \leqq \frac{64\gamma^3R^{3N}BA^2a^3}{m^3(n-\sqrt{2}a)}.$$

Proof. We have

$$|S_n(f;0,a)| \leqq \sum_{p,q=1}^{n}\left|S_1\left(f;z_{pq},\frac{a}{n}\right)\right| = n^2\cdot\frac{64\gamma^3R^{3N}BA^2a^3}{m^3(1-(\sqrt{2}a)/n)n^3}.$$

This is the analogue of Cauchy's theorem, which is the limiting case as $n\to\infty$.

4. Estimates for m. We shall establish the following theorem.

THEOREM 2. *Let* $a_N = 1$ (*so that* $A \geqq 1$), $a \geqq 5NA$, $n > 2\sqrt{2}a$, *and let m be as in Theorem* 1. *Then*

$$m^3 < \frac{K}{n}, \quad \textit{where} \quad K = 2^{(3N/2)+6}\gamma^3A^3a^{3N+3}.$$

Proof. Let $f(z) = P'(z)/P(z)$, so that $Q(z) = P'(z)$, and $B = \max|ka_k| \leqq NA$. We can take $R = \sqrt{2}a$ in Theorem 1 and obtain $|S_n(f;0,a)| < K_1/m^3n$, where $K_1 = 2^{(3N/2)+7}\gamma^3NA^3a^{3N+3}$. But

$$2\pi_nNi = S_n(N/z;0,a) = S_n(f;0,a) + S_n((NP-zP')/zP;0,a).$$

Now

$$|NP - zP'| = \left|\sum_{k=0}^{N-1}(N-k)a_kz^k\right| \leqq NA\sum_{k=0}^{N-1}|z|^k \leqq N^2A|z|^{N-1}$$

if $|z| \geqq 1$, and

$$|P(z)| \geqq |z|^N - \sum_{k=0}^{N-1}|a_k||z|^k \geqq |z|^N - NA|z|^{N-1}.$$

Therefore

$$\left|\frac{NP-zP'}{zP}\right| \leqq \frac{N^2A}{|z|(|z|-NA)}$$

if $|z| > \max(1, NA)$. Hence

$$|S_n((NP - zP')/zP; 0, a)| \leqq \frac{8aN^2A}{a(a - NA)} = \frac{8N^2A}{a - NA}.$$

Therefore

$$4N < |2\pi_n Ni| < \frac{K_1}{m^3 n} + \frac{8N^2A}{a - NA},$$

and

$$m^3 < \frac{K_1(a - NA)}{4Nn(a - 3NA)} \leqq \frac{K_1}{2Nn} \doteq \frac{K}{n}.$$

Theorem 2 is already sufficient for our purpose. We mention, however, that a much better estimate for m immediately follows from Theorem 2.

COROLLARY 1. *If $a \geqq 5NA$, and $\epsilon > 0$, then there is a point z_ϵ inside $C_a(0)$ at which $|P(z_\epsilon)| < \epsilon$.*

We need only take $n > K/\epsilon^3$ in Theorem 2.

The numbers $P((k+il)a/n)$ can be calculated in a finite number of steps by rational operations only, and therefore the number z_ϵ in the above corollary can be found constructively. Since the points $(k+il)a/n$ for a given n can be ordered, say by letting $(k_1+il_1)a/n$ precede $(k_2+il_2)a/n$ if $k_1 < k_2$ or $k_1 = k_2$ and $l_1 < l_2$, then if for a given n we always choose the first such point which satisfies the inequality $|P(z)| < \epsilon$, no element of arbitrary choice is involved.

5. The fundamental theorem of algebra. Theorem 2 is the crux of this paper; from here on we can proceed in many ways. The following argument was suggested by the proof of Brouwer and deLoor but is perhaps more direct.

LEMMA 3. *Let $a_N = 1$, and $0 < \epsilon < 1$. Then we can find points $z_1, \cdots, z_N$ such that*

$$|P(z_i)| < \epsilon, \qquad i = 1, \cdots, N,$$

and such that if $|P(z)| < \delta$, where $\epsilon \leqq \delta < 1$, then

$$\min_{1 \leqq i \leqq N} |z_i - z| < 2\delta^{1/2^N}.$$

Proof. We shall actually prove the lemma with the factor $2^{1-1/2^N}$ instead of 2. First we note that if $|z| \geqq 1 + NA$, then

$$(3) \qquad |P(z)| \geqq |z|^N - NA|z|^{N-1} = |z|^{N-1}(|z| - NA) > 1.$$

The lemma is obvious if $N = 1$. Suppose that it is true for $N - 1$. Let $0 < \epsilon_1 < \epsilon$, and by Theorem 2 we find a point z_1 such that $|P(z_1)| < \epsilon_1$. By (3), $|z_1| < 1 + NA$. Now

$$P(z) = P(z_1) + (z - z_1)P_1(z),$$

where

$$P_1(z) = \sum_{m=0}^{N-1} b_m z^m,$$

and

$$|b_m| = \left| \sum_{k=m+1}^{N} a_k z_1^{k-1-m} \right| < NA(1 + NA)^{N-1} = A_1.$$

By assumption we can find points $z_2, \cdots, z_N$ such that $|P_1(z_i)| < \epsilon_1, i = 2, \cdots, N$, and such that

$$|P_1(z)| < \delta, \quad \text{where} \quad \epsilon_1 \leqq \delta < 1,$$

implies

$$\min_{2 \leqq i \leqq N} |z_i - z| < 2^{1-1/2^{N-1}} \delta^{1/2^{N-1}}.$$

By (3) applied to $|P_1(z)|$, we have $|z_i| < 1 + (N-1)A_1, i = 2, \cdots, N$. Hence

$$|P(z_i)| < \epsilon_1 + |z_i - z_1| \epsilon_1 < C\epsilon_1,$$

where

$$C = 1 + 1 + (N - 1)A_1 + 1 + NA.$$

Now if $|P(z)| < \delta$, then

$$|(z - z_1)P_1(z)| < \delta + \epsilon_1 \leqq 2\delta.$$

Therefore either

$$|z - z_1| < (2\delta)^{1/2} < 2^{1-1/2^N} \delta^{1/2^N}, \tag{4}$$

or

$$|P_1(z)| < (2\delta)^{1/2}.$$

In the latter case,

$$\min_{2 \leqq i \leqq N} |z - z_i| < 2^{1-1/2^{N-1}} (2\delta)^{1/2^N}. \tag{5}$$

The required inequality follows from (4) and (5). If we choose $\epsilon_1 = \epsilon/C$, all the assertions in the lemma follow.

THEOREM 3. *We can find a convergent sequence $\{z_n\}$ such that $P(z_n) \to 0$.*

Proof. Let $\epsilon_n = 2^{-n2^N}$, $n = 1, 2, \cdots$. By Lemma 3 we can find points $z_{1n}, \cdots, z_{Nn}$, such that $|P(z_{\nu n})| < \epsilon_n$, $\nu = 1, \cdots, N$, and such that if $|P(z)| < \delta$, where $\epsilon_n \leqq \delta < 1$, then

$$\min_{1 \leq \nu \leq N} |z - z_{\nu n}| < 2\delta^{1/2^N}.$$

Let $z_1 = z_{11}$. If z_n has already been chosen as one of the points $z_{\nu n}$, $\nu = 1, \cdots, N$, then

$$(6) \qquad |P(z_n)| < \epsilon_n,$$

and $\epsilon_{n+1} \leq \epsilon_n < 1$. Hence there is a point $z_{\mu,n+1}$ such that

$$|z_n - z_{\mu,n+1}| < 2\epsilon_n^{1/2^N} = 2^{1-n}.$$

We choose z_{n+1} to be that one for which μ is smallest. Hence we have constructed the sequence $\{z_n\}$ in such a way that

$$|z_n - z_{n+1}| < 2^{1-n},$$

and, by (6), $P(z_n) \to 0$. It follows immediately that $\{z_n\}$ is convergent, for

$$\begin{aligned} |z_n - z_{n+p}| &\leq |z_n - z_{n+1}| + \cdots + |z_{n+p-1} - z_{n+p}| \\ &< 2^{1-n} + \cdots + 2^{1-(n+p-1)} < 2^{2-n}, \end{aligned}$$

which can be made arbitrarily small.

6. Another approach. Theorem 3 is, of course, the fundamental theorem of algebra. We believe that it is worth while to give another approach which may be more desirable for, say, pedagogical purposes. If $P(z)$ is relatively prime to $P'(z)$, that is, if the discriminant of $P(z)$ is different from zero, we can construct a root by Newton's method of approximation. Since Newton's method is not usually discussed in the complex domain, we shall, for the sake of completeness, give a proof of its validity. This proof is essentially the same as that of Brouwer and deLoor, except that we carry the computations out in more detail in order to get explicit (but crude) estimates for the constants involved.

Let $P(z)$ be relatively prime to $P'(z)$. Then we can find by a finite number of rational operations (see [4], p. 91) polynomials $C(z)$ and $D(z)$ of degree at most N satisfying

$$C(z)P(z) + D(z)P'(z) \equiv 1.$$

Let $C(z) = \sum_{k=0}^{N} c_k z^k$, $D(z) = \sum_{k=0}^{N} d_k z^k$, $C = \max|c_k|$, and $D = \max|d_k|$; let $E = NA + 1$, and let $a_N = 1$. We can assume, of course, that $N \geq 2$.

LEMMA 4. *If*

$$|P(z)| < \alpha = \min\left(\frac{1}{2NCE^N}, \frac{1}{4NDE^N}, 1, \frac{\lambda^2}{g}\right), \text{ where } \lambda = \frac{1}{2NDE^N} \text{ and}$$
$$g = 2\gamma AE^N,$$

then

$$|P'(z)| > \lambda,$$

and

$$\left|P\left(z - \frac{P(z)}{P'(z)}\right)\right| < \frac{g}{\lambda^2}|P(z)|^2 < |P(z)|.$$

Proof. By (3), we have $|z| < E$. Then $|C(z)| < NCE^N$, $|D(z)| < NDE^N$, and

$$1 < NCE^N\alpha + NDE^N|P'(z)|,$$

so that

$$|P'(z)| > \lambda.$$

Let $z_1 = z - P(z)/P'(z) = z + h$; then

$$|h| = \left|-\frac{P(z)}{P'(z)}\right| < \frac{|P(z)|}{\lambda} < \frac{1}{2}.$$

Now

$$\begin{aligned} P(z_1) &= P(z) + hP'(z) + \sum_{k=2}^{N} h^k \frac{P^{(k)}(z)}{k!} \\ &= h^2 \sum_{k=2}^{N} h^{k-2} \frac{P^{(k)}(z)}{k!}. \end{aligned}$$

By (2),

$$\left|\frac{P^{(k)}(z)}{k!}\right| < \gamma A E^N,$$

and therefore

$$|P(z_1)| < |h|^2 \sum_{k=0}^{N-2} |h|^k \gamma A E^N < 2\gamma A E^N |h|^2 < \frac{g}{\lambda^2}|P(z)|^2 < |P(z)|.$$

THEOREM 4. *If $|P(z_1)| < \alpha$, then the sequence $\{z_n\}$ defined by*

$$z_{n+1} = z_n - \frac{P(z_n)}{P'(z_n)}, \qquad n = 1, 2, \cdots,$$

is convergent and $P(z_n) \to 0$.

Proof. Let

$$w_n = P(z_n), \qquad h_n = -\frac{P(z_n)}{P'(z_n)} = z_{n+1} - z_n, \qquad n = 1, 2, \cdots.$$

Then we can easily show by induction from Lemma 4 that

$$|w_n| < \frac{\lambda^2}{g}\theta^{2^{n-1}}, \tag{7}$$

and

$$|h_n| < \frac{\lambda}{g}\theta^{2^{n-1}}$$

where

$$\theta = \frac{g|w_1|}{\lambda^2} < 1 \cdot$$

Then

$$\begin{aligned} |z_{n+p} - z_n| &= |h_n + \cdots + h_{n+p-1}| \\ &< \frac{\lambda}{g}(\theta^{2^{n-1}} + \cdots + \theta^{2^{n+p-2}}) \\ &< \frac{\lambda\theta^{2^{n-1}}}{g(1-\theta)}, \end{aligned}$$

which shows that $\{z_n\}$ satisfies the Cauchy convergence criterion. The rest of the theorem follows from (7).

If the Euclidean algorithm can be performed to obtain the greatest common divisor of $P(z)$ and $P'(z)$, then even if the discriminant of $P(z)$ vanishes, we can construct a root of $P(z)$ by Newton's method. For if $P(z)$ is divided by this greatest common divisor, we obtain a polynomial which is relatively prime to its derivative, to which Theorem 4 applies. In general, however, the Euclidean algorithm cannot be performed in a finite number of steps since to carry out the second division we must know the degree of the remainder in the first division; this requires that we know the highest power of z whose coefficient is different from zero. But in general it is impossible to determine constructively in a finite number of steps whether a given real number is zero or not. If the coefficients are in a field like that of the rational numbers or of the algebraic numbers where such a constructive process always exists, Newton's method can always be used. We make a final remark that the application of Theorem 4 depends on Theorem 2 which shows how a point z_1 satisfying $|P(z_1)| < \alpha$ can be found.

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DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SIMPLE PROOF OF FEUERBACH'S THEOREM

H. F. SANDHAM, Trinity College, Dublin, Ireland

Let A, B, C denote the vertices of a triangle, a, b, c the lengths of the opposite sides whose mid points are L, M, N, and X, Y, Z the points of contact of these sides with the inscribed circle.

It is readily proved

$$MN = a/2, \qquad NL = b/2, \qquad LM = c/2,$$
$$LX = \pm (b - c)/2, \qquad MY = \pm (c - a)/2, \qquad NZ = \pm (a - b)/2,$$

and therefore

$$\pm LX\cdot MN \pm MY\cdot NL \pm NZ\cdot LM = 0. \tag{1}$$

Hence, by the converse of Ptolemy's theorem, there is a point P concyclic with L, M, N whose distances from these points are proportional to LX, MY, NZ—the tangents from L, M, N to the inscribed circle. Since the locus of points from which the tangents to two circles are in a constant ratio is a coaxal circle, therefore the circumcircle of L, M, N is coaxal with the inscribed circle and the point circle P. That is P is a limiting point, and since it lies on one of them, the circumcircle of L, M, N and the inscribed circle therefore touch at P.

The relation (1) holds if X, Y, Z are the points of contact of an escribed circle and Feuerbach's theorem is thus proved for this case also.

HYPERSPACIAL FIGURATE PROGRESSIONS

WILLIAM FUNKENBUSCH, Michigan College of Mining and Technology

We shall say that a set of numbers forms a figurate progression if they are the h to $(h+k)$th figurate numbers inclusive for a constant dimension n and a constant linear difference d. An arithmetic progression is then a linear figurate progression. Having recently developed a method for finding products of numbers in arithmetical progression by logarithms, I looked at the figurate progression for a generalization of the procedure.

The mth figurate number of dimension n and difference d is given by

$$M_{n,d} = \frac{1}{n!}\,\frac{(m+n-2)!}{(m-1)!}\,[(m-1)d + n].$$

Of course for linear figurate numbers (numbers in arithmetical progression) we obtain

$$M_{1,d} = (m - 1)d + 1.$$

We also notice that the sum of the first m terms of the progression gives the mth figurate number of dimension $(n+1)$ and difference d:

$$M_{n+1,d} = \sum_{d=1}^{m} M_{n,d}.$$

The product of the h to the $(h+k)$th terms inclusive of an n-space figurate progression of linear difference d is given by

$$(1) \qquad {}_{h,k}F_{n,d} = \prod_{m=h}^{h+k} \frac{1}{n!} \frac{(m+n-2)!}{(m-1)!} [(m-1)d + n].$$

Let us now introduce bifactorial a,

$$a!! = 1!2!3! \cdots a!.$$

By use of bifactorial notation, (1) may be written

$$(2) \qquad {}_{h,k}F_{n,d} = \left(\frac{1}{n!}\right)^{k+1} \frac{(h+k+n-2)!!}{(h+n-3)!!} \cdot \frac{(h-2)!!}{(h+k-1)!!} \left[{}_{h,k}A_{n,d}\right].$$

where ${}_{h,k}A_{n,d}$ is the arithmetical progression product

$${}_{h,k}A_{n,d} = [(h-1)d + n][hd + n][(h+1)d + n] \cdots [(h+k-1)d + n].$$

We may write

$${}_{h,k}A_{n,d} = d^{k+1} \frac{(h-1)d + n}{d} \cdot \frac{hd + n}{d} \cdot \frac{(h+1)d + n}{d} \cdots \frac{(h+k-1)d + n}{d}$$

and by use of fractional factorials as defined by the author*

$$\left(\frac{b}{c}\right)! = \frac{b}{c}\left(\frac{b}{c} - 1\right)!, \quad b \text{ and } c \text{ positive and rational, } b > c,$$

$$\left(\frac{b}{c}\right)! = \frac{b}{c}, \qquad b \text{ and } c \text{ positive and rational, } b < c,$$

$$(3) \qquad {}_{h,k}A_{n,d} = d^{k+1} \frac{\left[\dfrac{(h+k-1)d + n}{d}\right]!}{\left[\dfrac{(h-2)d + n}{d}\right]!}.$$

It is of course apparent that

* The λ Function, copyright 1944.

$$_{h,k}F_{1,d} = {}_{h,k}A_{1,d}.$$

By using (3) in (2) we get the desired formula

$$(4)\qquad _{h,k}F_{n,d} = \left[\frac{d}{n!}\right]^{k+1} \frac{(h+k+n-2)!!(h-2)!!\left[\dfrac{(h+k-1)d+n}{d}\right]!}{(h+n-3)!!(h+k-1)!!\left[\dfrac{(h-2)d+n}{d}\right]!}.$$

The F's obviously lend themselves to logarithmic calculation. The author possesses the only tables of logs of bifactorial and fractional factorials of which he knows.

Example. Obtain the product of the 2nd to the 5th four dimensional figurate numbers inclusive, of linear difference 5. These numbers are of course 9, 35, 95 and 210 and their product is 6,284,250. The logarithmic solution here of course is just to illustrate method.

Solution:

$$_{2,3}F_{4,5} = \left[\frac{5}{4!}\right]^4 \frac{7!!0!!\left(\dfrac{24}{5}\right)!}{3!!4!!\left(\dfrac{4}{5}\right)!}$$

$$\begin{aligned}
\log 5 &= 10.69897-10\\
-\log 4! &= \underline{\ \ 1.38021\qquad\quad}\\
&\ \ 9.31876-10\\
&\ \underline{\qquad\qquad\qquad 4}\\
\log\left[\frac{5}{4!}\right]^4 &= \ \ 7.27504-10\\
\log 7!! &= 11.09833\\
\log 0!! &= \ \ 0.00000\\
\log\left(\frac{24}{5}\right)! &= \underline{\ \ 1.86655\qquad\quad}\\
\log \text{Num.} &= 10.23992\\
-\log \text{Den.} &= \underline{\ \ 3.44166\qquad\quad}\\
\log {}_{2,3}F_{4,5} &= \ \ 6.79826\\
{}_{2,3}F_{4,5} &= \ \ 6{,}284{,}300
\end{aligned}
\qquad\qquad
\begin{aligned}
\log 3!! &= 1.07918\\
\log 4!! &= 2.45939\\
\log\left(\frac{4}{5}\right)! &= \underline{9.90309-10}\\
\log \text{Den.} &= 3.44166
\end{aligned}$$

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Engineering Preview. By L. E. Grinter, H. N. Holmes, H. C. Spencer, Rufus Oldenburger, Charles Harris, R. G. Kloeffler, and V. M. Faires. New York, Macmillan Co., 1945. 10+581 pages. $4.50.

Engineering Preview is designed to orient the reader in fields of science and engineering. Each of seven authors has contributed a section of the book in which each writer discusses the tools of the engineer, mathematics, physics, chemistry and engineering drawing.

The headings and authors of the chapters are: (1) Outlines of Science and Engineering—Orientation, by L. E. Grinter; (2) Chemistry—The Science of Matter and Molecules, by H. N. Holmes; (3) Technical Drawing—The Language of the Engineer, by H. C. Spencer; (4) Mathematics—A Universal Tool of Engineering, by R. Oldenburger; (5) The Slide Rule—The Engineer's Coat of Arms, by C. O. Harris; (6) Light and Electricity—Hallmarks of Civilization, by R. G. Kloeffler; and (7) Mechanics and Thermodynamics, by V. M. Faires. Dr. Holmes is from Oberlin College, Ohio; Professors Kloeffler and Faires from Kansas State College and Texas A. and M. College respectively and the remaining writers are from the Illinois Institute of Technology. The book appends fifty pages of tables adapted from the Macmillan Tables.

The first section is devoted entirely to orientation and includes a simple self scoring "parlor test" giving the reader a rough measure of his technological traits. Roughly one hundred pages are devoted to rapid surveys of subject matter in each of the sections on chemistry, drawing, and mathematics. It is not intended that this survey is to serve as an initial textbook even though a large number of problems is included. If the reader is partially familiar with some of the material, *Engineering Preview* provides him with a stimulating review of engineering technology. A diligent student who gives deliberate attention to details and solves all the problems will find himself admirably equipped to enter a technological curriculum.

In the opinion of this reviewer, the first chapter on orientation should be read by all freshmen engineers to bring into focus the fields of engineering and their relationships to the basic sciences. The chapter on drawing is a gem as are all the figures in the whole volume. The reviewer disagrees with the exposition and figures relating to a unit solid angle on page 400.

D. L. HOLL

How To Solve It. A New Aspect Of Mathematical Method. By G. Polya. Princeton University Press, 1945. 15+204 pages. $2.50.

The interest of this book is pedagogical. First it should be noticed that Professor Polya explicity disavows the word "new" in the subtitle. As he observes in his preface, "heuristic . . . has a long past." Probably 2300 years would not be an excessive estimate. Then there is the statement on the jacket, "*A system of thinking which can help you solve any problem.*" Anyone who has been so rash as to write a book for what commercial publishers call "the trade," will know that an author is not responsible for the enthusiasms of his publishers, who must "make friends and influence people," or go out of business. Only those mathematicians who refrain from introspection and from observing their colleagues, might believe that a facility in solving problems in elementary mathematics will help them to "solve any problem." It may be suspected that Professor Polya did not write the copy for the jacket of his book.

What he did write, is an instructive exposition of the heuristic method applied to the solution of problems in elementary mathematics. Not to delude the reader into expecting more than can be offered, Professor Polya states (p. 158) that "Infallible rules of discovery leading to the solution of all possible mathematical problems would be more desirable than the philosopher's stone, vainly sought by all alchemists." And, quite bluntly: "The first rule of discovery is to have brains and good luck. The second rule of discovery is to sit tight and wait till you get a bright idea." Those of us who have little luck and less brains sometimes sit for decades. The fact seems to be, as Poincaré observed, it is the man, not the method, that solves a problem.

So far as instruction is concerned, the tactics of problem solving as expounded here are probably better known to teachers of secondary-school mathematics than they are to a majority of university professors. A generation ago, courses in the pedagogy of mathematics for prospective teachers in American secondary schools included substantially the subject-matter of this book. Such may still be the case; only those having direct contact with the teaching of teachers will know. If heuristic is no longer taught, *How To Solve It* may supply the deficiency. Every prospective teacher should read it. In particular, graduate students who are required to do some teaching, will find it invaluable if they have not already profited by observing one of their own teachers—if they were lucky enough to have one—who knew how to teach. "The traditional mathematics professor" (p. 181) who reads a paper before one of the Mathematical Societies, might also learn something from the book: "He writes a, he says b, he means c; but it should be d."

E. T. BELL

Sampling Statistics and Applications. By J. G. Smith and A. J. Duncan. New York, McGraw-Hill Book Co., 1945. 12+498 pages. $4.00.

This is the second of two volumes written under the joint title, *Fundamentals of the Theory of Statistics.* The previously published first volume, *Elementary Statistics and Applications*, has been reviewed in a recent issue of the MONTHLY. A 22-page introduction sets forth briefly those definitions and basic concepts of the first volume which are essential to the present one. Thus it can be read independently of the first, particularly by advanced students or research workers.

The body of the book has three main divisions: general theory of frequency curves, elementary theory of random sampling, and advanced sampling problems. The first of these begins with a discussion of probability and the symmetric binomial distribution along the same lines as in the first volume, proceeds to the asymmetric binomial and hypergeometric distribution and finally makes the transition to the various Pearson curves. The principal line of development is descriptive in nature with generous use of numerical examples and diagrams while the logical derivations are collected in a mathematical appendix immediately following. Then comes an excellent discussion of the Gram-Charlier type A series, cumulants and all, presented according to the same pedagogical pattern. Noteworthy is a short lucid comparison of the assumptions underlying these two approaches. There is a summary of conditions leading to normality and to non-normality and three outstanding non-normal distributions, t, χ^2, and F are defined and illustrated. A full chapter is devoted to numerical calculations for frequency curves, involving explanation of tables, plotting, computation of areas, curve fitting, and χ^2 test.

Part II presents first a comprehensive preview of sampling theory. This includes general remarks on the technique of random sampling and an unusually precise discussion of statistical inference, the testing of hypotheses, determination of confidence intervals, and maximum likelihood estimation of population parameters. Stratified and purposive sampling are mentioned briefly. Then these general aspects of sampling procedure are illustrated by a detailed study of the simplest case, samples from a discrete two-fold population. The Poisson distribution also gets treatment here. Next in order come samples from a normal population and these receive the most extensive discussion of the book. Samples of size two are used for the most part. This puts the usual multidimensional reasoning into the plane where a remarkably clear presentation is effected by means of elaborate tables and diagrams and a minimum of calculus. The distribution of the sample mean, variance, and t statistic are the end results. The N-dimensional case is also treated. In the testing of hypotheses regarding sample mean and variance a point is strongly made for the occasional use of unsymmetric regions of rejection. The distinction between the cases of known and estimated population variance is brought out with some unusual diagrams. The distribution of correlation coefficients, ordinary, partial, and multiple, are related to the normal and F distributions without proofs but with many illustrative problems.

The final division of the book opens with the multinomial distribution leading up to a clever and admittedly non-rigorous derivation of χ^2. Examples illustrate several different applications of the latter, particularly the testing of independence of classification. Quite novel is the joint study of mean and standard deviation; large tables illustrate the many possible areas of acceptance in joint significance tests. Also included is a treatment of sampling fluctuations in regression coefficients and higher order variances in any number of dimensions. Just about every question that can be raised in connection with pairs of samples is discussed: difference between percentages, means, variances, correlation coefficients, and regression statistics. Through these latter portions of the book proofs are generally omitted as being beyond its mathematical scope. However plenty of real-life examples are worked through to illustrate procedures. The chapter on analysis of variance is particularly instructive. Without developing highly specialized techniques the various fundamental procedures are presented through successive analyses of a set of student grades into more and more component variances. A short meaty concluding chapter, rich in references, introduces the reader to the difficulties of the problem of non-normality.

There is a subject and author index and an appendix with 12 mathematical tables including t, χ^2, F, and ratio of sample mean deviation to standard deviation. Scattered through the book are 113 figures, 56 tables, and footnote references to over 70 books and papers. The lack of exercises would seem to limit its use in the classroom. A number of irritating misprints and misstatements indicate some careless proofreading. However none of these, except possibly footnote 2 on page 117, should be permanently misleading.

The authors are economists, not professional mathematicians, and have bravely set about writing a book which was to be rational, comprehensive, even scholarly, yet be completely understandable to a reader having meagre acquaintance with the calculus. In pursuit of this last aim they sometimes become repetitious, sometimes confuse with excessive particulars, but more often they succeed in giving the reader a better working knowledge of the material than would a more concise and sophisticated presentation. Many of the artifices used to circumvent relatively difficult mathematics cannot help but clarify the subject, even to the initiated, by opening up new viewpoints. Some of these seem of general pedagogical importance. The book should help destroy the vacuum between purely descriptive texts on advanced statistical methods and formal treatises on the advanced mathematical theory.

J. L. VANDERSLICE

NEW BOOKS RECEIVED

Plane and Spherical Trigonometry. Second Edition. By H. A. Simmons. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1945. 11+387 pages. $2.25.

The River Mathematics. By A. Hooper. New York, Henry Holt and Co., 1945. 8+401 pages. $3.75.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 687. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement. Find the radius of the ball.

E 696. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Find four positive integers a, b, c, d such that a, b, c are in geometric progression, b, c, d in arithmetic progression, and $c+d=44$.

E 697. *Proposed by C. A. Murray, West Texas State College*

A certain geometry text raises the question whether the following procedure will inscribe a regular n-gon in a circle: AB being a diameter of the circle, construct an equilateral triangle ABC. Divide AB into n equal parts and let D be the second point of division from A. Draw CD, producing it to cut the circle at E. Is AE the side of a regular n-gon inscribed in the circle? For n equal to 3 or 4 the answer is readily affirmative. Does the procedure yield a regular pentagon for n equal to 5? If not, give a measure of the error.

E 698. *Proposed by J. M. Feld, Brooklyn College*

Let the complex numbers a and b represent two points on the Gauss plane. If $\rho=re^{i\alpha}$, (r and α real), show how one can construct the point corresponding to $z=(a+\rho b)/(1+\rho)$.

E 699. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 be the feet of the altitudes, the symmedians, and the cevians through the circumcenter, on the sides BC, CA, AB of a triangle ABC. (a) The lines B_1C_1, B_2C_2, B_3C_3 are concurrent in a point M_1; C_1A_1, C_2A_2, C_3A_3 in a point M_2; A_1B_1, A_2B_2, A_3B_3 in a point M_3. (b) Triangle $M_1M_2M_3$ is homological to, and circumscribes, triangle ABC.

E 700. *Proposed by Arnold Dresden, Swarthmore College*

If h and i are respectively positive and non-negative integers, then

(a)
$$\sum \frac{1}{p_0!p_1!\cdots p_i!} = \frac{(h+i-1)!}{i!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_0, p_1, \cdots, p_i$ which satisfy the conditions $p_0+p_1+\cdots+p_i=h$, $p_1+2p_2+\cdots+ip_i=i$;

(b) under the further restriction $i \geqq h-1$,

$$\sum \frac{1}{p_1!\cdots p_{i+1}!} = \frac{i!}{(i+1-h)!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_1, \cdots, p_{i+1}$ for which $p_1+\cdots+p_{i+1}=h$, $p_1+2p_2+\cdots+(i+1)p_{i+1}=i+1$.

SOLUTIONS

Self-reciprocal conics

E 659 [1945, 95]. *Proposed by R. A. Staal, University of Toronto*

Show that, if one conic is self-reciprocal with respect to another, then the two conics belong to a symmetrical set of four, each of which is self-reciprocal with respect to any of the other three. (However, not more than three of the four conics can be real.)

Solution by J. A. Jenkins, Loganton, Pa. The self-reciprocal property implies that the two polarities (say P and Q) associated with the conics are permutable. Thus the collineation obtained by performing them in turn is of period two, and so a harmonic homology (say H). Moreover, it is clear that the center and axis of this homology are pole and polar with respect to both P and Q. Each of these latter is seen to induce the same involution in the axis of the homology, since its points are invariant under their product. Choosing any pair of points in this involution we see that they form, with the center of the homology, the vertices of a triangle self-polar with respect to each of the given conics. Let us call the harmonic homologies, having as centers and axes the vertices and opposite sides of this triangle, H (as before) and K and L. The product correlations of these with P will be the polarity Q and two new polarities R and S. H, K, L, P, Q, R, S and the identical collineation form an abelian group of order eight. In particular, the conics associated with P, Q, R, and S are the required four conics. (Indeed we see that there is an infinity of such sets.) If the first two are real it is necessary that the pair of points chosen above be real. In this case, by examination of the corresponding involutions induced in the sides of the common self-polar triangle, we see that one of the associated conics will be real and the other not. This completes the proof.

A Tetrahedron of Constant Volume

E 667 [1945, 218]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let P, Q, R, P', Q', R' be arbitrary points on the respective edges BC, CA, AB, DA, DB, DC of a given tetrahedron $ABCD$. Prove that four planes, parallel to the faces BCD, CDA, DAB, ABC, drawn through the centroids of the respective tetrahedra $AQRP'$, $BRPQ'$, $CPQR'$, $DP'Q'R'$, form a tetrahedron of constant volume.

Solution by the Proposer. Let G, G', G'' be the mean centers (for equal multiples) of the sets of points $(A, B, C, D, P, Q, R, P', Q', R')$, (A, B, C, P, Q, R), (D, P', Q', R'). Now G'' is the centroid of tetrahedron $DP'Q'R'$, G' lies in the plane ABC, and G, G', G'' are collinear with $GG'/GG''=-2/3$. Similar statements may be made when any other vertex of the given tetrahedron is favored. We thus see that the final tetrahedron is inversely homothetic to the given tetrahedron, having G for homothetic center, and $-2/3$ for homothetic ratio. The volume of the final tetrahedron is then $8/27$ that of the given tetrahedron.

The analogous theorem and proof hold for the plane.

A Diophantine Equation

E 668 [1945, 218]. *Proposed by Walter Penney, Navy Department, Washington, D. C.*

Prove that the equation $x^2-3y^2=17$ has no solution in integers.

I. *Solution by E. P. Starke, Rutgers University.* Any integer x takes one of the forms $3n$, $3n\pm 1$. If these are substituted into the proposed equation, the results may be written as

$$3(3n^2 - y^2) = 17, \qquad 3(3n^2 \pm 2n - y^2) = 16,$$

respectively. These are evidently impossible in integers.

II. *Solution by W. F. Cheney, Jr., University of Connecticut.* The given equation demands that $x^2\equiv 17\equiv 2$, mod 3. But 2 is a quadratic non-residue, mod 3.

Also solved by D. W. Alling, Murray Barbour, D. H. Browne, H. N. Carleton, J. M. Danskin, Roy Dubisch, R. L. Duncan, Daniel Finkel, Irving Kaplansky, H. L. Lee, Benjamin Liebowitz, Irma Moses, W. J. Robinson, E. D. Schell, W. H. Thompson, and the proposer. Several solvers noted that 17 could be replaced by any integer of the form $2+3k$.

Editorial Note. The problem is also an immediate consequence of the theorem: *The diophantine equation $x^2-Cy^2=\pm H$, where C is positive and not a perfect square and H is positive and $>\sqrt{C}$, has no solution if no positive integer $K \not> \frac{1}{2}H$ can be found which makes $(K^2-C)/H$ integral.* By setting $K=1, 2, \cdots, 8$ in turn we find that $(K^2-3)/17$ is in no case an integer.

The above theorem is established on pages 482 and 483 of the second volume of Chrystal's *Algebra*. In arts. 15 through 19 of chapter XXXIII of this book Chrystal, using the theory of continued fractions, gives a complete discussion of integral solutions of quadratic forms of the type involved in this problem. For treatments not employing continued fractions see chapter XI (especially art. 6) of *Elementary Number Theory* by Uspensky and Heaslet, or chapter IV of *Modern Elementary Theory of Numbers* by Dickson.

The Centroid of n Points

E 669 [1945, 218]. *Proposed by J. H. Butchart, Grinnell College*

Let G be the centroid of n coplanar points P_i, Q any point in the same plane, and Δ_i the signed area of the triangle QGP_i. Show that

$$\Sigma\Delta_i = 0.$$

I. *Solution by Janet Ryder, Grinnell College.* Let rectangular coordinates of G, P_i, and Q be respectively $(\bar{x}, \bar{y})$, (x_i, y_i), and (x, y). Then $n\bar{x} = \Sigma x_i$, $n\bar{y} = \Sigma y_i$, and

$$\Sigma\Delta_i = \tfrac{1}{2}\Sigma \begin{vmatrix} x & y & 1 \\ \bar{x} & \bar{y} & 1 \\ x_i & y_i & 1 \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} x & y & 1 \\ \bar{x} & \bar{y} & 1 \\ \Sigma x_i & \Sigma y_i & n \end{vmatrix} = 0.$$

II. *Solution by C. E. Springer, University of Oklahoma.* If the centroid G is taken as origin of a set of n vectors V_i to the points P_i, then $\Sigma V_i = 0$. Denoting the vector GQ by V, the area Δ_i is one-half the scalar magnitude of the cross product of V and V_i. Now $\Sigma(V\times V_i) = V\times(\Sigma V_i) = 0$. Hence $\Sigma\Delta_i = 0$.

III. *Solution by L. M. Kelly, U. S. Coast Guard Academy.* It is well known that the algebraic sum of the perpendiculars dropped from any number of points P_i onto a line passing through the centroid G of these points is equal to zero. Taking the line as GQ, this algebraic sum times the common base GQ gives twice the algebraic sum of the areas Δ_i, and is zero.

That the algebraic sum of the perpendiculars from P_i onto GQ is zero may be seen at once by imagining unit masses placed at the points and then taking moments about GQ.

Also solved by J. M. Danskin, Sydney Glusman, Francis Hall, Irving Kaplansky, E. D. Shell (all like solution I), the proposer (like solution II), and S. T. Parker. The proposer noted that if the signed areas are represented by axial vectors, then the points P_i need not be coplanar. S. T. Parker solved, in a manner analogous to solution I, the more general problem: *Let G be the centroid of n coplanar points P_i, Q and R any two points in the same plane, Δ_i the signed area of the triangle QRP_i, and Δ the signed area of the triangle QRG. Then* $\Sigma\Delta_i = n\Delta$.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4181. *Proposed by P. D. Thomas, Lumberton, Miss.*

Lines are drawn from a point P on the circumcircle of an equilateral triangle parallel to the three sides thus determining six points, two on each side respectively. (1) Prove that the six points thus determined lie by threes on two straight lines. (2) If Q is the point of intersection of these two lines, find the locus of Q as P moves on the circumcircle.

4182. *Proposed by Cezar Coşniţa, Focşani, Roumania*

Show that the envelope of the conics circumscribing a given triangle and such that the angle between the asymptotes is constant is a curve of the fourth degree bitangent to the line at infinity at the circular points and having the vertices of the triangle for double points.

4183. *Proposed by P. M. Hummel, University of Alabama*

Let p and q be non-negative integers and x a variable. Define

$$f(x, p, q) = \sum_{i=0}^{p} (-1)^i {}_pC_i(x-i)^q,$$

where ${}_pC_i$ are binomial coefficients. Prove that $f(x, p, q)$ equals zero if $p>q$; equals $p!$ if $p=q$; and is a polynomial in x of degree $q-p$ if $p<q$.

4184. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what systems of numbers with the base less than 10000 are there the greatest number of squares of four digits of the form $aabb=(cc)^2$? Dedicated to E. P. Starke.

SOLUTIONS

Matrix Equation

4126 [1944, 352]. *Proposed by A. D. Wallace, University of Pennsylvania*

Let x, A, b denote respectively $(1, m)$, (m, n), $(1, n)$ matrices, an (i, j) matrix being one with i rows and j columns. If the matrix AA' is non-singular, show that the least square solution of $xA=b$ is the solution of an equation of the form $xA=b_0$. Determine b_0 and its geometric meaning.

Solution by the Proposer. Let $T(x)=xA$ so that T is a linear transformation taking E_m into E_n, euclidean spaces of dimensions m and n. Here we interpret a $(1, p)$ matrix as a point in E_p. Since AA' is non-singular it is readily seen that M, the transform of E_m under T, is of dimension m. While b is in E_n it will not in general lie in M. Let $\delta(x)=(xA-b)(xA-b)'$. Then $\delta(x)$ is the square of the distance from the point xA of M to the point b. It is also the sum of the squares of the deviations of $xA=b$. We assert that if $x_0AA'-bA'=0$ then $\delta(x_0)$ is an absolute minimum. For

$$\delta(x_0 + y) = \delta(x_0) + 2(x_0AA' - bA')y' + yAA'y', \ y \text{ any } (1, n) \text{ matrix.}$$

The second term vanishes and the third is $(yA)(yA)'$. This latter is a sum of squares and so non-negative. Further if $yA=0$ then $yAA'=0$ and thus $y=0$ since AA' has an inverse. We conclude that $\delta(x)$ is greater than $\delta(x_0)$ for any x different from x_0. If we multiply $x_0AA'-bA'=0$ on the right by $(AA')^{-1}$ we see that $x_0=bA'(AA')^{-1}$ and this latter is the least square solution of $xA=b$. The transform of x_0, $b_0=x_0A$, lies in M and is the nearest point of M to b. Now $x_0A=bA'(AA')^{-1}A=b_0$. Hence x_0 is a solution of $xA=bA'(AA')^{-1}A$. It is readily seen that the equation $xA=b_0$ has a unique solution.

Tetrahedron, Equal Powers

4131 [1944, 409]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $ABCD$ be a skew quadrangle; planes perpendicular at A to AC, at C to CB, at B to BD, at D to DA form a tetrahedron $A_1C_1B_1D_1$ with the centroid G_1. Show that the powers of A with respect to the sphere (G_1C) on G_1C as diameter, of C with respect to (G_1B), of B with respect to (G_1D), of D with respect to (G_1A) are equal.

Solution by Howard Eves, College of Puget Sound. We shall employ the following lemma—a theorem which is interesting in itself.

LEMMA. *Consider a tetrahedron T and a directed polygonal line p of four segments in space. If each segment of p is perpendicular to a face of T, is directed from the outside to the inside of this face, and is proportional to the area of this face, then p is a closed space polygon (skew-quadrangle). Conversely, if p is a skew-quadrangle, and each side of p is perpendicular to a face of T, then each side of p is proportional to the area of its corresponding face and, for one direction about p, is directed from the outside to the inside of this face.*

An indication of the proof of this lemma will suffice. Let m be any line in space and M a plane perpendicular to m. Let us project p orthogonally on m and T orthogonally on M. Since the sum of the projections on M of the signed areas of the faces of T is equal to zero, it follows that the sum of the projections on m of the signed segments of p is also zero. But m was taken arbitrarily, whence we see that p must be closed.

The converse is easily proved. Let us be given a line segment, p_1, perpendicu-

lar to face T_1 of T. Now there is one and only one skew-quadrangle $p_1p_2p_3p_4$ such that p_i is perpendicular to T_i. This, with the direct part of the lemma, establishes the converse.

Now, returning to the original problem, let G_a, G_c, G_b, G_d be the feet of the perpendicular from G_1 on planes $C_1B_1D_1$, $A_1B_1D_1$, $A_1C_1D_1$, $A_1C_1B_1$ respectively. Then

$$G_aG_1\cdot AC = \text{power of } A \text{ with respect to sphere } (G_1C),$$
$$G_cG_1\cdot CB = \text{power of } C \text{ with respect to sphere } (G_1B),$$
$$G_bG_1\cdot BD = \text{power of } B \text{ with respect to sphere } (G_1D),$$
$$G_dG_1\cdot DA = \text{power of } D \text{ with respect to sphere } (G_1A).$$

But, designating the altitudes of $A_1C_1B_1D_1$ from A_1, C_1, B_1, D_1 by h_a, h_c, h_b, h_d respectively, we have

$$G_aG_1:G_cG_1:G_bG_1:G_dG_1 = h_a:h_c:h_b:h_d.$$

And, from the lemma,

$$AC:CB:BD:DA = C_1B_1D_1:A_1B_1D_1:A_1C_1D_1:A_1C_1B_1.$$

Therefore

$$G_aG_1\cdot AC = G_cG_1\cdot CB = G_bG_1\cdot BD = G_dG_1\cdot DA = k \text{ (vol. of } A_1C_1B_1D_1),$$

where k is some constant of proportionality. This proves our theorem.

Notes: The analogous problem for a triangle in the plane is also true, but is much more easily proved.

The direct part of the lemma and its proof are readily extended to the case of any convex n-hedron T and the resulting skew n-gon p. The converse, however, is no longer true, as is seen by considering skew 6-gons associated with a cube.

Air Flight in Wind

4132. [1944, 475]. *Proposed by T. H. Matthews, McGill University*

If an aircraft travels at a constant airspeed, and traverses (with respect to the ground) a closed curve in a horizontal plane, the time taken is always less when there is no wind, than when there is any constant wind.

Solution by Gordon Pall, McGill University. Let V, W, v be respectively the airspeed of the craft when there is no wind, the wind speed with a constant direction, and the resultant speed of the craft in the forward direction along the closed curve C which it describes; and let τ be the angle from the wind direction to the tangent to C. If $\pi/2<|\tau|<\pi$ we must have $V>W$. In all cases

$$(1)\quad V^2 = W^2 + v^2 - 2Wv\cos\tau, \quad v = R + W\cos\tau, \quad R = \sqrt{V^2 - W^2\sin^2\tau}.$$

The negative sign before R must be dropped since $v \geqq 0$ and, if the negative sign is used, we would have $W \geqq V$. The total time T to traverse C is then

$$(2)\quad T=\int\frac{ds}{v}=\int\frac{R-W\cos\tau}{V^2-W^2}ds=\int\frac{Rds}{V^2-W^2},\quad\text{since}\quad\int\cos\tau ds=0.$$

Then

$$T=\int\frac{\sqrt{V^2-W^2\sin^2\tau}}{V^2-W^2}ds>\int\frac{\sqrt{V^2-W^2}}{V^2-W^2}ds=\int\frac{ds}{\sqrt{V^2-W^2}},$$

$$T>\int\frac{ds}{V}\cdot$$

This proves the desired result.

Solved also by Murray Barbour, A. M. Glicksman, T. A. Mossman, and G. W. Petrie.

Ring of Integers

4133 [1944, 475]. *Corrected. Proposed by A. L. Putnam, Yale University*

Let a, b, and c be integers with $b\neq 0$, and let d and f be the respective greatest common divisors of a and b and c and b. Then if

$$a\not\equiv\pm d\ (\text{mod } b)\text{ and } c\not\equiv\pm f(\text{ mod } b),$$

there is an infinite number of integers k for which the equation

$$ax+bxy+cy=k$$

has no solutions in integers.

I. *Solution by E. P. Starke, Rutgers University.* The given equation may be written

$$(bx+c)(by+a)=bk+ac,$$

and division throughout by df gives

$$(1)\qquad\left(\frac{b}{f}x+\frac{c}{f}\right)\left(\frac{b}{d}y+\frac{a}{d}\right)=\frac{b}{df}k+\frac{a}{d}\cdot\frac{c}{f}\cdot$$

If b is not a multiple of df, this equation has no solution in integers if k is any one of the infinity of integers prime to df, for then all terms are integers except bk/df. On the other hand if b is a multiple of df, b/df and ac/df are coprime because b/f and c/f are coprime, as are b/d and a/d. There are then (Dirichlet's theorem: See Dickson, *Modern Elementary Theory of Numbers*, 1939, pp. 291–305) infinitely many terms in the arithmetic progression, $ac/df+b/df$, $ac/df+2b/df$, $\cdots$, $ac/df+kb/df$, $\cdots$, which are primes. For each value of k corresponding to one of these primes, one of the factors in the left member of (1) must be ± 1. Hence one of the equations

$$bx+c=\pm f,\qquad by+a=\pm d$$

must be true. Since these are contradicted by the hypothesis, there is no integral solution for such values of k.

II. *Solution by Gordon Pall, McGill University.* We shall prove a more general result from which Putnam's follows:

The function $\phi = bxy+ax+cy$ (where a, b, c are integers, and $b \neq 0$) represents all but a finite number of integers if and only if either

$$(2) \qquad a \equiv \pm 1 \pmod{b}, \quad \text{or} \quad c \equiv \pm 1 \pmod{b},$$

or

$$(3) \qquad |b| = 6, \qquad a \equiv \pm 3, \qquad c \equiv \pm 2 \pmod{6},$$

or vice versa for a and c. When (2) *holds, ϕ represents all integers; when* (3) *holds, ϕ represents all integers save one, namely $-ac/6$.*

Proof. Consider the equation $bxy+ax+cy=n$, or

$$(4) \qquad (bx+c)(by+a) = bn + ac.$$

Replacing x by $x-h$ and y by $y-k$, we can add arbitrary multiples of b to a and c, while leaving $bn+ac$ unchanged. By altering the signs of x, y, and ϕ, we can secure $b>0$, $a \geqq 0$, $c \geqq 0$. These operations leave $\pm(bn+ac)$ unchanged, and reduce the problem to a form ϕ in which

$$(5) \qquad 0 \leqq a \leqq \tfrac{1}{2}b, \qquad 0 \leqq c \leqq \tfrac{1}{2}b.$$

If either a or c is 1, ϕ evidently represents all integers. If $a=0$ and $c>1$, then $\phi = y(bx+c)$ cannot represent any prime of the form $bn \pm 1$.

We can thus assume that $a \geqq c \geqq 2$, $b \geqq 5$. If $|x| \geqq 2$ and $|y| \geqq 2$, then $|bxy+ax+cy| \geqq 2a$ by (5). If x and y have the values 0, ± 1, the only values of ϕ which may equal 1 or 2 are $b-a-c$, a, c. Hence if ϕ is to represent 1 and 2, $b-a-c=1$, $c=2$, whence by (5) either

$$b = 5,\ a = c = 2; \quad \text{or} \quad b = 6,\ a = 3,\ c = 2.$$

In the first case ϕ does not represent 3. In the second case,

$$6xy + 3x + 2y + 1 = (3x+1)(2y+1),$$

where $3x+1$ represents $\pm 2^h$, and $2y+1$ is any odd number. Hence $6xy+3x+2y$ represents every integer except -1.

It follows that except when (2) or (3) holds, ϕ fails to represent some integer n such that $bn+ac \neq 0$. That is, in view of (4), if $bn+ac = \pm p_1 p_2 \cdots p_r$ as a product of primes, we cannot partition the product into two factors one of which is congruent to c, the other to a, mod b. The same property must hold for $\pm p_1 p_2 \cdots p_r p$, if p is any prime congruent to ± 1 mod b. Hence ϕ fails to represent infinitely many integers n.

We have used here the fact that *there exist infinitely many primes of the form* $nb \pm 1$. This was proved by *elementary* methods by several writers. (For refer-

ences, see Dickson's *History of the Theory of Numbers*, I, pp. 418–419.) It is of course a special case of Dirichlet's theorem on primes in an arithmetic progression, which has however not yet been proved by strictly elementary means.

Gordon Pall found the conditions for ϕ to represent all integers in 1931 (*A class of universal functions*, Bulletin of the American Mathematical Society, 38, 1932, 56–58), but overlooked the fact that $(3x+1)(2y+1)$ did not represent zero.

Solved also by the proposer as corrected.

Editorial Note. The above two solutions gave also proofs that the original statement with $d \not\equiv \pm f \pmod{b}$ is incorrect. Pall gave also a proof, using Dirichlet's theorem, independent of his above proof. The proposer's solution, also using this theorem, pointed out the following context in which the problem occurred:

Suppose R is a commutative ring with a unity and B is a proper ideal of R. In general the residue class $B+ac$ is larger than the set consisting of all products of elements from the two residue classes $B+a$ and $B+c$. It is natural to ask when equality will hold. In the special case of the ring of integers the ideal B has the form (b) where $b\neq 0$, and a necessary condition becomes: The equation

$$(a + by)(c + bx) = ac + kb$$

has integral solutions x and y for every integer k. As is shown in the solution of the problem, this requires solving the equation given in the problem.

Angle Trisection

4134 [1944, 475]. *Proposed by Hüseyin Demir, Columbia University*

Let $C_1^1C_2^1C_3^1$ be the inscribed triangle of a reference triangle $A_1A_2A_3$, and $C_1^2C_2^2C_3^2$ be that of $C_1^1C_2^1C_3^1$, and so on, obtaining a triangle $C_1^nC_2^nC_3^n$ after n steps. Denoting the angles of the nth triangle by C_i^n, prove that

1. $(C_i^n-\pi/3)(A_i-\pi/3)=(-1)^n2^{-n}$.

2. The limit of the direction of $C_2^nC_3^n$ as $n\to\infty$, is the direction of one of the trisectrices of the angle $(A_2A_3,\ C_2^1C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

Solution by Howard Eves, College of Puget Sound. 1. Designating the incenter of $A_1A_2A_3$ by I we have $C_2^1IC_3^1=2C_1^1$. Therefore $A_1+2C_1^1=\pi$. Similarly, $A_i+2C_i^1=\pi$, or

$$(C_i^1 - \pi/3)/(A_i - \pi/3) = -2^{-1}.$$

By the same process

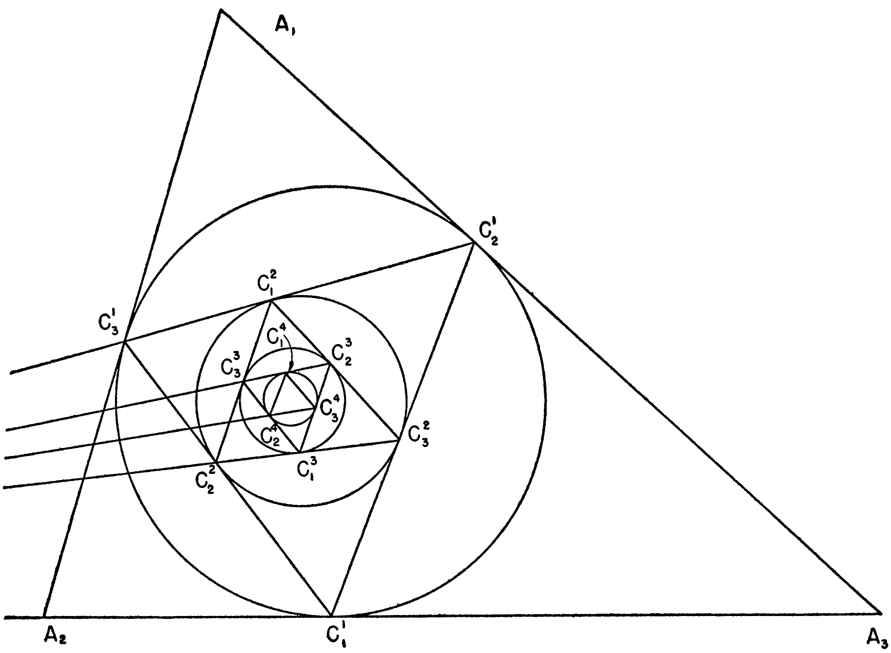
$$(C_i^j - \pi/3)/(C_i^{j-1} - \pi/3) = -2^{-1}, \qquad j = 2, \cdots, n.$$

By multiplication we then get

$$(C_i^n - \pi/3)/(A_i - \pi/3) = (-1)^n2^{-n}.$$

2. Now

$$\begin{aligned}\lim_{n\to\infty}\left[\text{angle }(A_2A_3,\ C_2^nC_3^n)\right] &= \sum_{n=1}^{\infty} \text{angle }(C_1^{2n-2}C_2^{2n-2},\ C_1^{2n}C_2^{2n}), \quad \text{where} \quad C_i^0 \equiv A_i\\ &= \sum_{1}^{\infty} (C_3^{2n-1} - C_1^{2n})\\ &= \sum_{1}^{\infty} \left[(C_3^{2n-1} - \pi/3) - (C_1^{2n} - \pi/3)\right]\\ &= \sum_{1}^{\infty} \left[- 2^{-2n+1}(A_3 - \pi/3) - 2^{-2n}(A_1 - \pi/3)\right],\end{aligned}$$



by the relations of part 1 above. The infinite sum on the right reduces to

$$\begin{aligned}\sum_{1}^{\infty} [\pi - A_1 - 2A_3]2^{-2n} = (A_2 - A_3)\sum_{1}^{\infty} 2^{-2n} = \tfrac{1}{3}(A_2 - A_3) &= \tfrac{2}{3}\left(\frac{A_2 - A_3}{2}\right);\\ &= (\tfrac{2}{3})\left[\tfrac{1}{2}(A_2 - \pi/3) - \tfrac{1}{2}(A_3 - \pi/3)\right]\\ &= (\tfrac{2}{3})\left[-(C_2^1 - \pi/3) + (C_3^1 - \pi/3)\right]\\ &= (\tfrac{2}{3})(C_3^1 - C_2^1) = (\tfrac{2}{3}) \text{ angle } (A_2A_3,\ C_2^1C_3^1).\end{aligned}$$

This establishes part 2. The method suggested here for asymptotically obtaining one of the angle trisectors of a given acute angle $(A_2A_3,\ C_2^1C_3^1)$ is apparent. It is

needless to say, however, that there are many better euclidean asymptotic constructions for trisecting an angle.

Solved also by the proposer.

Editorial Note. Below are some references to this MONTHLY regarding approximate methods of angle trisection with limits for the error:

1932, 478, *Angle Division*, article by E. C. Kennedy; 2972 [1925, 95]; 3114 [1925, 483]; 3522 [1933, 303]; 3563 [1934, 113]; The method of Pappus using conics 3490 [1932, 243].

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Assistant Professors H. G. Apostle and A. E. Basch of the University of Rochester and Rensselaer Polytechnic Institute, respectively, have been appointed to assistant professorships at Amherst College.

Associate Beulah M. Armstrong of the University of Illinois has been promoted to an assistant professorship.

Assistant Professor J. H. Butchart of Grinnell College, Grinnell, Iowa, has been appointed to a professorship at Arizona State Teachers College.

Dr. F. A. Butter, Jr., has been appointed acting assistant professor of mathematics at Stanford University.

Professor A. G. Clark and Assistant Professor M. L. Madison have returned to Colorado State College of Agricultural and Mechanical Arts.

Assistant Professor D. B. DeLury of the University of Toronto has been appointed to an associate professorship at Virginia Polytechnic Institute.

Assistant Professor R. P. Dilworth of the California Institute of Technology has been promoted to an associate professorship.

Dr. T. C. Doyle of Stanford University has been appointed to an assistant professorship at Dartmouth College.

Assistant Professor Samuel Eilenberg of the University of Michigan has been promoted to an associate professorship. He is, at present, visiting lecturer at Princeton University.

W. C. Griffith of DePauw University has been appointed to a professorship at Marion Institute, Marion, Alabama.

J. R. Hanna of the University of Wichita has been promoted to an assistant professorship.

Assistant Professor J. D. Hill of Michigan State College has been promoted to an associate professorship.

Assistant Professor K. D. Kelly of Fenn College, Cleveland, Ohio, has been promoted to an associate professorship.

Dr. Lois Kiefer of New Mexico State Teachers College has been appointed to an acting assistant professorship at the University of Tennessee.

Professor Lincoln LaPaz of Ohio State University has been appointed head of the department of mathematics at the University of New Mexico.

Associate Professor Eugene Lukacs of Berea College has been appointed to a professorship at Our Lady of Cincinnati College.

Associate Professor M. S. MacPhail of Acadia University, Wolfville, Nova Scotia, has been promoted to a professorship.

Dr. V. O. McBrien of the College of the Holy Cross has been promoted to an assistant professorship.

Professor Edith J. McKissock of Youngstown College, Youngstown, Ohio, has been appointed assistant dean of women at the University of Cincinnati.

Professor A. E. Meder, secretary of Rutgers University, has been appointed dean of administration.

Dr. N. S. Mendelsohn has been appointed lecturer at Queen's University, Kingston, Ontario, Canada.

Associate Professor R. R. Middlemiss of Washington University has been promoted to a professorship.

Dr. H. E. Nelson of Gustavus Adolphus College, St. Peter, Minnesota, has been appointed associate professor of mathematics and head of the department.

Associate Professor R. M. Pinkerton of the Agricultural and Mechanical College of Texas has been appointed professor of mathematics and acting head of the department.

Dr. A. R. Poole has been appointed to an assistant professorship at Montana State College.

Fred Robertson of Iowa State College has been promoted to an assistant professorship.

Dr. L. D. Rodabaugh of Oberlin College has been appointed statistician in the Bureau of the Census in Washington.

Dr. Peter Scherk of the University of Saskatchewan has been promoted to an assistant professorship.

Associate Professor Samuel Selby of the University of Akron has been appointed professor of mathematics and head of the department.

Dr. D. R. Shreve has been appointed to an associate professorship at the University of Tulsa.

Dr. W. C. Taylor of the University of Cincinnati has been promoted to an assistant professorship.

Assistant Professor C. C. Torrance of Case School of Applied Science has been promoted to an associate professorship.

Dr. B. R. Ullsvik of the State Teachers College in Eau Claire, Wisconsin, has been appointed to an associate professorship at Illinois State Normal University, Normal, Illinois.

Dr. Mary C. Vanhorn of Trinity College, Washington, D. C., has been promoted to an assistant professorship.

Dr. G. C. Vedova has been appointed professor of mathematics and head of the department at Pennsylvania Military College, Chester, Pennsylvania.

Assistant Professor G. L. Walker of the University of Delaware has been appointed to an assistant professorship at Temple University.

Dr. H. S. Wall of the Illinois Institute of Technology has been appointed to a professorship.

Dr. S. E. Warschawski has been appointed professor of mathematics and mechanics at the University of Minnesota.

The following appointments to instructorships are announced:
Everett Junior College, Everett, Washington: G. H. Van Arkel
North Carolina State College: G. C. Watson
Southern College of Optometry, Memphis, Tennessee: R. L. Coker
University of California: Dr. O. G. Owens
University of Chicago: Dr. Irving Kaplansky, Dr. William Karush
University of Illinois: Corinne Hattan
University of Minneaota: W. D. Munro

Professor Stefan Banach, according to a report from the Polish Press Agency died in Lvov at the age of fifty-three.

N. Durairajan, Executive Engineer in Mylapore, Madras, India, died July 15, 1945.

Dr. Willis Whited of Harrisburg, Pennsylvania, died April 28, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

UNIVERSITY PROGRAMS FOR EX-SERVICE PERSONNEL IN CANADA

F. S. NOWLAN, University of British Columbia

This paper is divided into two parts. The first is an abridgment of a talk that the author gave in June, in Montreal, before the Canadian Mathematical Congress. The second part represents an attempt to bring the Canadian picture up to date. Although educational conditions in the two countries are different, frequent reference is made to the American situation. It is believed that these comparisons are worthwhile.

Part I

The world situation has undergone great change since I was asked to undertake this survey. The shooting war in Europe is over and the next few months will see thousands of young men and women return to civilian life. Many of these left the high schools and universities to enter the armed services. For various reasons they wish to renew their schooling. They possibly realize that the brightest hope for a peaceful world lies in a new and broader outlook on life. To them, education offers the one opportunity to gain this outlook and to contribute to the building of an enlightened and vigorous Canada in a world, we hope, of finer ideals.

Numbers of these young people have witnessed the power of science. Their imaginations have been stimulated and they look forward to creative work in science or engineering. Some have seen and experienced the healing magic of medicine and they feel the urge to embark upon its study. There are other considerations, but whatever the urge, carefully conducted surveys have shown that thousands of young airmen, sailors, and soldiers propose to continue their studies and soon will be crowding our universities.

This influx will entail increased responsibilities for us. However, these young men and women have never failed us and they have the right to expect our best efforts in this emergency. Nevertheless, it is fair to add that we can not assume the entire responsibility in connection with the problems that arise. There are federal obligations, and federal aid should be given beyond the payment of gratuities, which, it is true, have been provided on a generous scale.

The terms of the Post Discharge Re-establishment Order-in-Council, P.C. 5210, provide for a maintenance grant on a graduated scale and the payment

of the usual fixed university fees,* exclusive of prescribed textbooks, to scholastically qualified and honorably discharged ex-service personnel for as many months as the individual has been in the Service. The monthly maintenance grant varies from $60 for an unmarried man or woman, with no dependents or disabilities, and $80 for a man and wife under similar conditions, to a hypothetical maximum of $197 in the case of a man with total disability (such as blindness), a wife and six children. An additional grant is made for each child. The allowance for each of the first two children is $12 a month, $10 is allowed for the third child, *etc.* In addition, $15 a month is allowed for a dependent parent.

The grants are designed to relieve the recipient from financial worry and so to enable him to devote his entire attention to his studies. They are on a somewhat more generous scale to the individual than the corresponding sums provided in the U.S.A. under the terms of Public Law 346, the G.I. Bill of Rights. The difference is accentuated by the fact that the cost of living in Canada is less than in the U.S.A. The American ex-service man receives a monthly maintenance grant of $50 or $75, depending upon his single or married status and, in addition, the payment of university fees, including textbooks and supplies, up to $500 a year. This sum may be increased in the case of universities which normally charge a higher tuition fee.

The conditions that govern Canadian grants to veterans are set forth in two pamphlets issued by the Department of Pensions and National Health. These are entitled *Back to Civil Life* and *Principles Governing Training for Rehabilitation*. For purpose of comparison, the educational provisions of the G.I. Bill of Rights can be found in this MONTHLY for October, 1944. Further information may be found in the numerous Bulletins on Higher Education and National Defense as issued by the American Council on Education, Washington, D. C.

The best available information indicates that 35,000, or more, ex-service personnel will seek admission or re-admission to Canadian universities within the next few years. What this implies is obvious when one considers that the total pre-war registration in our universities was approximately the same number. The surveys indicate that courses in engineering will be in greatest demand. Next in popularity is commerce and business, followed in order by arts, education, and medicine.

The magnitude of the administrative problem is appreciated when one realizes that the fees of a Canadian university student pay only about 40% of the institutional cost of his education. The question of buildings and equipment in universities that are already overcrowded and understaffed is extremely serious. It is even conceivable that unless some solution of this problem is reached, attendance at an institution such as the University of British Columbia, which is

* The regulations have been modified since this paper was given in Montreal. They now provide for an additional payment *to the university*, for each ex-service man, of $150 per university year, and a proportional sum if the student takes summer work. An additional sum of $5 weekly is granted a married student who supports two establishments. This provides for cases where a student's wife is not with him at the university, or where the student has dependent parents who do not live with him.

without dormitories, may be denied many veterans due to the difficulty in finding shelter.

A detailed discussion of Canadian university problems related to the training of veterans can be found in the Report of the National Conference of Canadian Universities on Post-War Problems, as adopted at their meeting at McMaster University in June, 1944. This Report is well worth reading. It is constructive, but non-controversial. In this latter respect it is in marked contrast to published statements from individual American educators in regard to the application of the G.I. Bill of Rights. However, these diverse points of view help in forming a well-balanced picture of the situation. I have particular reference to the fears expressed by President Hutchins of the University of Chicago* regarding the educational provisions of the G.I. Bill of Rights. Also, President Conant of Harvard University in his "President's Report" for 1943–44, expresses misgivings.

I feel that to some extent we should take issue with these gentlemen. They seem to fear a general lowering of standards. To quote President Hutchins: "A serious dilution of an already diluted educational system is in prospect." It occurs to me that the influx of new students may give us both the excuse and opportunity for raising standards. On the other hand, it would admittedly be unfair to expect higher standards of service men than we demand of civilian students, and so long as we tolerate play-boys and play-girls in our universities, I feel that we may dismiss our fears in regard to the veteran letting us down scholastically. So far, the experience both in Canada and the U.S.A. has been that the veteran is a serious-minded, hard worker, who attains better than average standing.

I recently sent a questionnaire to the various Canadian colleges and universities in the hope that I might get a more intimate picture of the problems which confront us as teachers of mathematics. The overall picture that I get is as follows:

1. There probably will not be as many ex-service men in university attendance as indicated by official surveys. Nevertheless, Toronto considers the possibility of 10,000 ex-service students within the next few years. Mt. Allison is preparing for a possible 50% increase in registration and the University of New Brunswick anticipates a similar increase. In general, the Western Universities hesitate to speculate upon the increase but this year the University of British Columbia had 2908 students during the regular session. It is expected that at least 3500 will register in the fall.

2. As a general rule, refresher courses will not be given and the feeling is that they are not required. Special summer courses will take care of the needs.

3. There is a keen realization of the importance of counselors, with academic and war experience, to advise and guide the service men. But, so far as possible, these men should be kept from feeling that they are receiving special attention.

4. Dean Beatty of the University of Toronto expressed the general feeling in

* *Cf.* The State of the University, October 19, 1944.

his reply: "I believe that as far as possible the instruction will be entrusted to the older and more experienced members of the staff, men who will appreciate what the boys have come through and who will understand where they are likely to find difficulties. The boys will need careful and sympathetic attention."

5. It is felt that the classes should be kept reasonably small. A number of educators favor classes of not more than 30. The largest figure reported comes from Toronto which sets 50 as a maximum.

6. It is felt that there is no likelihood, nor danger, of a lowering of standards and there would be opposition to any step in that direction. On the other hand, more emphasis will be placed upon the practical applications of mathematics.

The impression in the universities that have instructed groups of ex-service men during the past year is that in their studies these men more than keep up their end as compared with civilian students. Our U.B.C. group consisted mainly of airmen and at one time included six men with the D.F.C. Two heads of departments, who instructed these men, informed me that they worked harder than other students and, in fact, were at a higher level of mentality. Our mathematics instructors report that these students wish to work all the problems of the text whereas the average civilian student appears to feel in honor bound to work as few as possible.

These observations are in keeping with my experience last summer when I instructed in the V-12 program at Notre Dame University. The students were navy men and marines, about half of each. A number had seen action in the South Pacific, and I felt that they did the best work in the group. The men were of more than average ability. They were serious minded and among the hardest workers that I have met in my teaching experience.

It is my feeling that the average civilian student is so imbued with the notion of play that as a result we have formed an entirely false idea of how much can be required of him. The ex-service men are serious and have habits of work and application. Through the force of example they can do our universities a real service.

I recently made a study of freshman examination papers in mathematics, as given in our various colleges and universities. There appears to be a considerable divergence in standards and this, insofar as service men are concerned, has undesirable features. It means that a service man might fail in more than two subjects and so, according to the terms of the Act, lose his right to further instruction. Nevertheless, in knowledge and ability he might be superior to a student who had made a fair showing in a university with lower standards. Possibly something of this sort was in President Conant's mind when he made the Report to which we referred. The Congress might well take this matter under advisement.

I have brought to your attention some of the difficulties which face us in the post-war period and for which we have no ready-made solution. However, we should not forget, nor should we let others forget, that these difficulties are by-products of the war and as such it is fair to expect and demand increased

federal aid for their solution. Furthermore, the universities should be given priority in matters of new buildings and equipment. A country that has performed miracles of production will scarcely accept defeat when it comes to implementing plans for post-war reconstruction.

Finally, I would like to see this Congress go on record as recommending that the educational provisions of P.C. 5210 be extended to apply* in another generation to children of those men who have made the supreme sacrifice or who may have become totally incapacitated. Also, among the veterans are older men who are not personally interested in university attendance. It should be possible, under certain conditions, for such a person to transfer his right to an educational grant to a son or daughter. By such an arrangement, we could repay a portion of the debt that we owe these men, and we may be sure that an investment in education pays the best dividends.

Part II

This is a factual Report,—an attempt to picture the unprecedented situation in Canadian universities at the opening of the fall term, 1945–46. These universities, with few exceptions, have experienced a 40% to 90% increase in enrollment. This is due to the influx of ex-service men who are either entering upon or continuing their university education. This increased attendance has entailed some problems of a physical nature and others that relate to scholarship. Also, as is natural, different universities, as well as different departments within the same university, have varied in their anticipation of the problems as well as in their efforts to arrive at solutions.

In general, like conditions of overcrowding, insufficient staff, and inadequate student accommodations prevail in all Canadian universities. The overcrowding is most extreme in first and second year classes in arts and engineering and in courses in mathematics, chemistry, and physics. Since this has been written for a mathematical journal, the picture is restricted in its pedagogical aspects to conditions that affect instruction in that subject. Although reference is made to a number of Canadian universities, the University of New Brunswick, (U.N.B.), has been selected as typical of the eastern universities, and the University of British Columbia, (U.B.C.), to represent the West. At U.B.C., we have a 94% increase in enrollment, the present registration being 5,650, and overcrowding is extreme. Only those who have lived through a Times Square subway crush can appreciate conditions in the corridors of the Arts Building at U.B.C. at the intermission between classes. The subway, however, has the advantage of guards who propel laggards along. That is a device that to date we have overlooked.

In viewing the picture, we should remember that, traditionally, mathematics is a required subject of study for first year students in arts at Canadian universi-

* Resolutions favoring this, and the suggestion that follows, were adopted by the Congress and forwarded to the appropriate authorities.

ties. (Its study is said to have cultural value.) In addition, the subject is required in several years of engineering. It follows that, with the possible exception of English, more students take mathematics than any other subject in the curricula, and that, with the exception of science courses that involve laboratory work, mathematics is most affected by any sudden increase in registration.

University of New Brunswick. This is one of the smaller universities. There are 300 ex-service men in attendance this year, representing a 75% increase over last year's total registration. The department of mathematics gives 2,320 student-hours of instruction per week as compared with 1,180 last year. The mathematics staff has been increased from two to three professors. In addition, 9 senior students grade papers in mathematics, each for two hours a week. There are no special arrangements for helping students who may have difficulty with their work. Instructors meet first and second year classes in groups that range in numbers from 60 to 98. Renovation of existing buildings has provided the necessary classroom and laboratory space. A house to house survey of Fredericton found living quarters for all but 50 men. These have been provided with temporary quarters in the gymnasium and they do their studying in classrooms and the library. It is expected that 150 additional ex-service men will register and begin work in January.

Acadia University has found it necessary to limit civilian admissions.

Mt. Allison University has instituted a unit term system which has proved very helpful. The university provides assistance for veterans who need it, as well as regular tutorial periods for students in first and second year mathematics.

The University of Toronto has experienced a 50% increase in attendance and they expect a large additional enrollment in January. They will then take over 70,000 square feet of floor space from the Ajax defense plant which is located 18 miles from the University proper. This space will be utilized for housing and training some 1500 students in first year engineering who will begin their courses at that date.

Queens University expected, in early September, that a third of their students would be ex-service men.

The University of Western Ontario, as of October 2, had registered 800 freshmen, which is more than their total registration of arts students last year. The increase was due to ex-service men.

The University of Saskatchewan has approximately 1,200 veterans, with another large group expected in January.

The University of Alberta has been obliged to set a quota upon enrollment in its professional schools.

The University of British Columbia. The present plant of U.B.C. was opened in the fall of 1925 and was built to accommodate 1,500 students. In spite of a steady and rapid growth to last year's total of 2,908 students, there has been no extension of initial classroom space. It is true that last spring the provincial legislature voted $5,000,000 for university expansion, but federal restrictions and priorities have prevented any building. This was the situation when it was

realized last spring that the fall registration would be phenomenal. The problems that were faced were of two kinds, the physical (classrooms, equipment, and student accommodations), and the scholastic.

The Physical Problem. This problem entailed much labor in the way of negotiations and supervision. It owes its successful solution to painstaking work on the part of the administration and certain faculty members, combined with fine cooperation from the military authorities. As a consequence:

(a) The university obtained an army camp on Acadia Road, a half mile from the campus, as a gift from the Army.

(b) The Army loaned the university most of the buildings at the Point Grey Fortress, a coast artillery station 200 yards from the campus.

(c) The university purchased from the War Assets Corporation a heavy AA camp on Chancellor Boulevard, about two miles from the campus.

(d) The university obtained on indefinite loan from the R.C.A.F. three buildings at Jericho Beach, about four miles from the campus.

In addition to the foregoing, the university has obtained from the Army eleven buildings which were part of an anti-aircraft camp on Lulu Island. These buildings will be brought to the campus and used for living accommodations. The buildings will be ready by January, 1946, when, at least, 1,200 additional students (ex-service men) are expected. Furthermore, the Army and R.C.A.F. have provided, on loan, equipment for two kitchens as well as beds, blankets, and equipment for the dormitories.

The Scholastic Problem. Last year, the mathematics department at U.B.C. consisted of six full-time instructors and one part-time instructor. It has been increased this year by the return of a staff member who was on leave and by the appointment of five instructors, three young men and two women. In addition, there are seven student assistants (tutors) and four readers of class exercises.

Last year the mathematics department gave 6,230 student-hours of instruction per week. This year it gives 12,685 hours. The class sections are large, several ranging in number from 250 to 290. There are 1,730 first year students in arts, and they take their mathematics in eighteen sections. Of these, fourteen are between 64 and 85 in number, while four exceed 100. One section is around 260 in number. The instructor in charge of this section gives 3,010 student-hours of instruction a week.

No refresher courses are offered, but certain rooms are designated as study rooms for several hours each day, and are in charge of advanced students who give help if requested. Also, members of the Students' Council have offered to provide assistance as far as they are able. This will be provided without charge.

It has been indicated that this report is objective and, as such, personal opinions are out of place and are not expressed. It may, however, be observed that many of the ex-service men have been away from mathematics for periods ranging from two to seven, or more, years. It follows that personal attention is imperative if they are to have a reasonable chance of success in their work.

In closing, the writer apologizes to Presidents Conant and Hutchins.

THE AVAILABILITY OF METEOROLOGISTS AS INSTRUCTORS

The American Meteorological Society through its President, Professor C.-G. Rossby of the University of Chicago, wishes to call to the attention of American colleges that among the large number of carefully selected college students who were given professional academic training in meteorology and have served as professional meteorologists (weather officers) with the Army Air Forces and the Navy Aerological Service during the war, there are a number who before their entry into the meteorological field had done a considerable amount of graduate work in mathematics and who now might become available for teaching positions in the fields of mathematics and meteorology.

Most of our colleges are now offering nonprofessional courses in meteorology to meet the needs of students in geography, geology and in agricultural subjects. In the post-war period the interest in flying is likely to extend and thus a need will arise for nonprofessional courses in aeronautical meteorology, designed to meet the needs of liberal arts students who, as private flyers, might wish a certain amount of information about the atmosphere. The needs listed above are, however, in most cases so limited as to preclude the possibility of the employment of properly qualified professional meteorologists as full-time instructors and it has thus become standard practice to have these nonprofessional courses in meteorology taught by members of other departments. The return of many of the professionally trained service meteorologists offers an opportunity for our colleges to obtain the service of instructors who are capable of doing professional work in a mathematics department and at the same time are qualified as experienced and trained professional meteorologists to present stimulating nonprofessional courses in meteorology on a somewhat higher plane than may have been possible in the past.

In some cases the mathematicians who were brought into meteorology during the war acquired useful experience in certain phases of applied mathematics, particularly theoretical hydrodynamics, through research activities in theoretical meteorology.

The American Meteorological Society is anxious to aid in the development of a program which will permit the country to salvage some of the great forward strides made in meteorology during the war. As part of this program, the Society maintains an extensive file on the academic education, training and professional meteorological experience of the meteorologists who served with our Armed Forces during the war. The Society will be glad to assist any college or college department which might be interested in obtaining the services of properly qualified meteorologists, by providing such organizations with names of suitable candidates together with information concerning their academic background, experience and general suitability for academic work. Inquiries should be directed to the Executive Secretary, American Meteorological Society, 5727 University Avenue, Chicago 37, Illinois.

MEXICAN MATHEMATICS

RUFUS OLDENBURGER, Illinois Institute of Technology

The second National Mathematical Congress of Mexico in three years was held at Guadalajara in the State of Jalisco, May 28 to June 2, 1945. Over two hundred professional leaders attended including Señor don Jaime Torres Bodet, Secretary of Education in President Camacho's cabinet. Senor Torres gave one of the main addresses. The congress was dedicated to the late George D. Birkhoff whose lectures at the National University in 1942 and 1943 inspired much research activity on his theory of gravitation. The United States was represented by Professors Nelson Dunford, Solomon Lefschetz, Francis D. Murnaghan, Norbert Wiener, and Rufus Oldenburger. Professors Lefschetz and Wiener spoke on Birkhoff and his work.

Several papers were presented to the congress by the new group of young pure and applied mathematicians, including Professors Alberto Barajas and Roberto Vásquez of the Institute of Mathematics of the National University, and Professors Antonio Romero Juárez, Nabor Carrillo Flores and Carlos Graef Fernández of the Institute of Physics of this university. Of the men just named Professor Vásquez is the only one specializing in pure mathematics, his interest being in analysis. The formerly excessive teaching loads of these men have recently been reduced to as little as twelve hours a week, largely through the influence of Professor Ricardo Monges López, director of the faculty of sciences of the National University, whose kindness and fatherly encouragement mean a great deal to the scientists of Mexico.

The participants in the congress were entertained by the Governor of Jalisco at a theater performance of dancers, singers and instrumentalists, and at a banquet at the French Club. The University of Guadalajara took the congress to Lake Chapala for a day. In many ways the congress was honored by the government, receiving its financial support.

Although the active research mathematicians in Mexico can be counted on the fingers of two hands, the work of these men receives considerable support from people in other fields on a scale not attained in the United States. Without the attendance of these other men the meetings of the congress would have been very small. Mathematical productivity is being encouraged by the official journal, "Boletín de la Sociedad Matemática," of the Mexican Mathematical Society which has about one hundred and forty members. The president of this society is Dr. Alfonso Nápoles Gándara, director of the Institute of Mathematics of the National University.

At the present time, post-college courses in mathematics are given only by the National University in Mexico City. The encouragement of the government and private sources promises a rapid and extensive expansion of mathematical interest in Mexico, an interest stimulated by the commission of President Avila Camacho for promoting and coordinating scientific investigation.

MARINE CORPS CRITICIZES THE TEACHING OF MATHEMATICS

Lt. Col. Thomas B. Hughes, in charge of the Marine Corps training section, recently discussed the training program of the Marine Corps before the Connecticut School Board Association. His address, as reported by Albert R. Prince in the U. S. Education News, was critical of the instruction given to students in the United States in English, mathematics, foreign languages, and citizenship. Col. Hughes pointed out that too large a percentage of men coming to the Corps had been poor in mathematics, in English, and in citizenship training.

Much of the trouble encountered by the training section of the Marine Corps appeared to be caused by fundamental difficulties in the use of English. In too many cases, special training was necessary to bring trainees to the fourth grade level in their use of English. Moreover, "some who had had two years of college were so weak in English that they could not be approved for officer training. One such candidate made 60 mistakes in a one-page letter."

Col. Hughes recommended the adoption of a well rounded high school program with considerable emphasis upon communication rather than literature in the English courses, and with a minimum requirement of two years of mathematics. He also urged that more attention be given to the foreign languages, with new emphasis upon Russian, Spanish, and Portuguese.

THE MATHEMATICAL ASSOCIATION OF AMERICA

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MISSOURI
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NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
PHILADELPHIA
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SOUTHEASTERN
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

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The Chauvenet Prize

The Chauvenet Prize for a "noteworthy expository paper published in English during the five-year period by a member of the Association" was established in 1925 by a gift from Professor J. L. Coolidge, then president of the Association. Its frequency period was reduced to three years by an additional gift from Professor W. B. Ford during his presidency and by an anonymous gift in 1936.

As determined more recently by the Board of Governors, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The next two awards will be made in December 1947 and December 1950, covering the periods 1944-46 and 1947-49 respectively.

Seven awards have been made as follows:

1920-1924. G. A. Bliss, "Algebraic Functions and Their Divisors," *Annals of Mathematics.*

1925-1928. T. H. Hildebrandt, "The Borel Theorem and Its Generalizations," *Bulletin* of the American Mathematical Society.

1929-1931. G. H. Hardy, "An Introduction to the Theory of Numbers," *Bulletin* of the American Mathematical Society.

1932-1934. Dunham Jackson, "Series of Orthogonal Polynomials" and "Orthogonal Trigonometric Sums," *Annals of Mathematics;* "The Convergence of Fourier Series," *American Mathematical Monthly.*

1935-1937. G. T. Whyburn, "On the Structure of Continua," *Bulletin* of the American Mathematical Society.

1938-1940. Saunders Mac Lane, "Modular Fields," and "Some Recent Advances in Algebra," both in the *American Mathematical Monthly.*

1941-1943. R. H. Cameron, "Some Introductory Exercises in the Manipulation of Fourier Transforms," *National Mathematics Magazine.*

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ABLOW, C. M., A.M. (U.C.L.A.) *Lt., U.S.N.R. 14 Avenue 32, Venice, Calif.*

ACKLEY, Prof. H. M., A.M. (Olivet) Western Michigan Coll., Kalamazoo, Mich. *602 Axtell St.*

ADAMS, B. T., A.M. (Baylor) *Instr., Amarillo Army Air Field, Amarillo, Tex. 2172 Avenue I, Wichita Falls, Tex.*

ADAMS, Prof. C. R., Ph.D. (Harvard) Brown Univ., Providence, R. I. *60 Intervale Road*

ADAMS, Prof. E. P., Ph.D. (Harvard) Physics, Princeton Univ., Princeton, N.J.

ADAMS, G. J., Jr. Address unknown

ADAMS, Asst. Prof. LOUISE, A.M. (North Carolina) High Point Coll., High Point, N.C.

ADAMS, L. J., A.M. (Southern California) Head of Dept., Santa Monica City Coll., Santa Monica, Calif. *227–25th St.*

ADAMS, O. S., D.Sc. (Kenyon) Principal Mathematician, Retired, U. S. Coast and Geodetic Survey, Washington 25, D.C. *R.F.D. 4, Mt. Vernon, Ohio*

ADKINS, J. B., Ed.M. (Harvard) Teacher, Phillips Exeter Acad., Exeter, N.H. *Lt., U.S.N.R.*

ADKINS, Prof. L. K., M.S. (Chicago) Head of Dept., State Teachers Coll., La Crosse, Wis.

ADKISSON, Prof. V. W., Ph.D. (Pennsylvania) Chm. of Dept., Univ. of Arkansas, Fayetteville, Ark. *236 Buchanan St.*

ADLER, Asst. Prof. CLAIRE F., Ph.D. (New York Univ.) New York Univ., New York, N.Y. *189-21 Tioga Drive, St. Albans, N.Y.*

AGARD, Prof. H. L., Ph.D (Yale) Williams Coll., Williamstown, Mass. *Box 49*

AGNEW, Prof. R. P., Ph.D. (Cornell) Cornell Univ., Ithaca, N.Y. *112 White Hall*

AHL, Sister WINIFRED, A.B. (St. Joseph's Coll., Md.) Instr., St. Joseph's Coll., Emmitsburg, Md.

AHLFORS, Prof. L. V., Ph.D. (Helsingfors) Univ. of Helsingfors, Helsingfors, Finland. *Munksnäs, Finland*

AITCHISON, BEATRICE, Ph.D. (Johns Hopkins) Statistical analyst, Interstate Commerce Commission, Washington, D.C. *1929 S Street N.W., Washington 9, D.C.*

ALBERT, Prof. A. A., Ph.D. (Chicago) Univ. of Chicago, Chicago 37, Ill. *Asso. Director, Applied Math. Group, Northwestern Univ.*

ALBERT, Asst. Prof. G. E., Ph.D. (Wisconsin) Ohio State Univ., Columbus 10, Ohio. *1104 N. Oakland Ave., Indianapolis 1, Ind.*

ALBERT, Prof. O. W., Ph.D. (Washington) Univ. of Redlands, Redlands, Calif. *629 Buena Vista St.*

ALBERTI, FURIO, B.S. (Chicago) 1806 S. Sixth Ave., Maywood, Ill.

ALDEN, Asso. Prof. H. H., Ph.D. (Ohio State) Univ. of Wyoming, Laramie, Wyo. *Univ. Union Box 23*

ALEXANDER, Prof. C. K., Ph.D. (Calif. Inst. of Tech.) Head of Dept., Occidental Coll., Los Angeles 41, Calif.

ALEXANDER, Prof. J. W., Ph.D. (Princeton) Inst. for Advanced Study, Princeton, N.J. *29 Cleveland Lane*

ALFIERI, F. A. Cpl., U. S. Army. *9 Metropolitan Oval, New York 62, N.Y.*

ALLEN, Prof. E. B., Ph.D. (Rensselaer) Head of Dept., Math. and Astr., Rensselaer Poly. Inst., Troy, N.Y. *4 Sheldon Ave.*

ALLEN, EDWARD F., A.M. (Pennsylvania) Instr., Math. and Physics, Acad. of the New Church, Bryn Athyn, Pa.

ALLEN, Prof. ELBERT F., Ph.D. (Missouri) Oklahoma A. and M. Coll., Stillwater, Okla. *1409 College Ave.*

ALLEN, Asst. Prof. FLORENCE E., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *219 Lathrop St.*

ALLEN, HARRIET W., Ph.D. (Mass. Inst. of Tech.) Physicist, Air Reduction Co., Stamford, Conn.

ALLEN, Asso. Prof. JOSEPH, A.M. (Harvard) Retired, Coll. of the City of New York, New York, N.Y. *Seal Harbor, Maine*

ALLENDOERFER, Asso. Prof. C. B., Ph.D. (Princeton) Haverford Coll., Haverford, Pa. *Consultant, Weather Training Branch, Army Air Forces, Washington, D.C. 750 Rugby Road, Bryn Mawr, Pa.*

ALLIOT, Prof. EUGENE, Lic.ès Sc. (Paris) St. Mary's Seminary, Randolph, Vt.

ALMAN, J. E., A.M. (Claremont) *In Service*

AMUNDSON, Asst. Prof. N. R., Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis 14, Minn. *Dept. of Math. and Mech., Engg. Bldg.*

ANDERSEN, Prof. MAE R., Ph.D. (Chicago) Concordia Coll., Moorhead, Minn. *507 Tenth St. S.*

ANDERSON, A. H., M.E. (Marquette) Head of Science Dept., Whitefish Bay Schools, Whitefish Bay, Wis. *5143 N. Berkeley Blvd., Milwaukee 11, Wis.*

ANDERSON, Asst. Prof. E. W., Ph.D. (Iowa State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*

ANDERSON, P. H., Ph.D. (Illinois) Analyst, Office of Surplus Property, Dept. of Commerce, Washington 25, D.C. *1228 Blair Mill Road, Silver Spring, Md.*

ANDERSON, R. D., A.B. (Minnesota) Fellow, Univ. of Texas, Austin, Tex. *Lt., U.S.N.R.*

ANDERSON, Asst. Prof. R. LUCILE, Ph.D. (Bryn Mawr) Hunter Coll., 695 Park Ave., New York 21, N.Y.

ANDERSON, Prof. W. E., Ph.D. (Pennsylvania) Miami Univ., Oxford, Ohio. *112 E. Walnut St.*

ANDERTON, ETHEL L., Ph.D. (Yale) Teacher, High School, West Haven, Conn. *215 Park Terrace Ave.*

ANDREE, R. V., Ph.M. (Wisconsin) Grad. Teaching Asst., Univ. of Wisconsin, Madison, Wis. *612 Howard Place, Madison 3, Wis.*

ANN ELIZABETH, Sister, Ph.D. (Wisconsin) Prof., Registrar, St. Mary Coll., Xavier, Kans.

ANNEAR, Asst. Prof. P. R., M.S. (Case) Head of Dept., Math. and Astr., Baldwin-Wallace Coll., Berea, Ohio. *76 Rocky River Drive*

ANTONY, Rev. J. H., A.B. (Univ. of Dayton) Head of Dept., Purcell High School, Cincinnati, Ohio. *3901 Oak St., Apt. 7, Cincinnati 27, Ohio*

ARANY, Prof. DANIEL, Grad. (Univ. of Budapest) Emeritus, Math. in the royal school for Industry, Budapest, Hungary. *Korong-u 6, Budapest XIV, Hungary*

ARCHIBALD, Prof. R. C., Ph.D. (Strasbourg), Dr. (Univ. Padua), LL.D. (Mt. Allison) Emeritus, Brown Univ., Providence, R.I.

ARCHIBALD, Asso. Prof. R. G., Ph.D. (Chicago) Queens Coll., Flushing, N.Y.

ARENA, J. F., A.M. (Michigan) Grad. student, Univ. of Michigan, Ann Arbor, Mich. *518 E. William St.*

ARGUE, W. T., A.B. (Tulsa) Chief Computer, Seismograph Service Corp., Tulsa 1, Okla. *Box 1590*

ARMSTRONG, Asst. Prof. BEULAH M., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *364 Math. Bldg.*

ARNDT, Prof. W. F. C., Ph.D. (Göttingen) Univ. Coll. of the Orange Free State, Bloemfontein, Union of South Africa

ARNOLD, H. A., Ph.D. (Calif. Inst. of Tech.) Cruft Lab., Harvard Univ., Cambridge, Mass.

ARNOLD, H. C., A.M. (Ohio State), M.S. (Carnegie Inst. of Tech.) Tech. Director, Federal Enamel and Stamping Co., McKees Rocks, Pa. *76 Standish Blvd., Pittsburgh 16, Pa.*

ARNOLD, Asso. Prof. H. E., Ph.D. (Yale) Wesleyan Univ., Middletown, Conn.

ARNOLD, Asst. Prof. K. J., Ph.D. (Mass. Inst. of Tech.) Univ. of Wisconsin, Madison 6, Wis. *North Hall*

ARNOLD, Asso. Prof. W. C., M.S. (Chicago) DePauw Univ., Greencastle, Ind. *P.O. Box 466*

ARNOLDY, Sister M. NICHOLAS, Ph.D. (Catholic Univ.) Head of Dept., Marymount Coll., Salina, Kans.

AROIAN, L. A., Ph.D. (Michigan) Instr., Hunter Coll., New York 21, N.Y. *247 Wadsworth Ave., New York 33, N.Y.*

ARTIN, Prof. EMIL, Ph.D. (Leipzig) Indiana Univ., Bloomington, Ind. *715 S. Fess Ave.*

ASHBAUGH, LAURA M. (Mrs. F. R.), A.M. (Pennsylvania) Instr., Moravian Coll. for Women, Bethlehem, Pa. *115 Wall St.*

ASHCRAFT, Prof. T. B., Ph.D. (Johns Hopkins) Colby Coll., Waterville, Me. *34 Pleasant St.*

ASTRACHAN, Asso. Prof. MAX, Ph.D. (Brown) Chm. of Dept., Antioch Coll., Yellow Springs, Ohio

ATKIN, Asso. Prof. EDITH I., A.M. (Columbia) Emeritus, Illinois State Normal Univ., Normal, Ill. *815 S. Fell Ave.*

ATKINS, H. P., Jr., M.S. (Brown) Instr., Univ. of Rochester, River Campus, Rochester 3, N.Y.

ATKINSON, ROBERT, A.M. (Haverford) Chm., Math. and Sci. Depts., The Shipley School, Bryn Mawr, Pa.

AUCOIN, A. A., Ph.D. (Louisiana State) *Lt., U.S.N.R.*
AUDE, Prof. H. T. R., M.S. (Colgate) Colgate Univ., Hamilton, N.Y.
AURORA, SILVIO, A.B. (Columbia) Grad. student, Columbia Univ., New York, N.Y. *30-58 83rd St., Jackson Heights, N.Y.*
AVERS, H. G., A.B. (George Washington) Chief Mathematician, Div. of Geodesy, U. S. Coast and Geodetic Survey, Washington 25, D.C.
AYLOR, M. W., M.S. (Virginia) Instr., Univ. of Virginia, Charlottesville, Va. *Madison, Va.*
AYRE, Asso. Prof. H. G., Ph.D. (Peabody) Western Illinois State Teachers Coll., Macomb, Ill.
AYRES, Prof. FRANK, Jr., Ph.D. (Chicago) Dickinson Coll., Carlisle, Pa. *Army Univ. Center, APO 413, c/o Postmaster, New York, N.Y.*
AYRES, Asst. Prof. H. C., Ph.D. (California) U. S. Naval Acad., Annapolis, Md.
AYRES, Prof. W. L., Ph.D. (Pennsylvania) Head of Dept., Purdue Univ., Lafayette, Ind. *Dept. of Math.*

BABBITT, A. E., A.M. (Illinois) Vice-Pres. and Actuary, Lamar Life Ins. Co., Box 880, Jackson 107, Miss.
BABCOCK, Asst. Prof. L. E., M.S. (Illinois) Univ. of Richmond, Richmond, Va. *1411 Grove Ave., Apt. 15, Richmond 20, Va.*
BABCOCK, Dean R. W., Ph.D. (Wisconsin) School of Arts and Sci., Kansas State Coll., Manhattan, Kans.
BABCOCK, Asso. Prof. WEALTHY, Ph.D. (Kansas) Univ. of Kansas, Lawrence, Kans. *Route 6*
BACHMANN, J. G., B.S. (New York Univ.) Pres., Bachmann and Co., Inc., 8511 Sunset Blvd., Hollywood 46, Calif.
BACON, Prof. CLARA L., Ph.D. (Johns Hopkins) Emeritus, Goucher Coll., Baltimore, Md. *2316 N. Calvert St., Baltimore 18, Md.*
BACON, Asso. Prof. H. M., Ph.D. (Stanford) Stanford Univ., Stanford University, Calif. *Box 1144*
BAEZ, A. V., A.M. (Syracuse) Instr., Physics, Stanford Univ., Stanford University, Calif. *Dept. of Physics*
BAGBY, L. C., A.M. (Kansas) Engr., Conklin Industrial Service, Inc., Detroit 2, Mich. *62 Glendale Ave., Highland Park 3, Mich.*
BAIDAFF, Prof. B. I., Dr. (Buenos Aires) Analisis, Univ. Nac., Buenos Aires, Argentina. *Ave. de Mayo 560*
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BAKER, S. R., A.B. (Ursinus) Teacher, Hannah Penn Jr. High School, York, Pa. *Private, U. S. Army*
BAKST, AARON, Ph.D. (Columbia) 135-12 77th Ave., Flushing, N.Y.
BALDWIN, Prof. J. W., A.M. (Michigan) Wayne Univ., Detroit, Mich. *16191 Roselawn Ave., Detroit 21, Mich.*
BALL, Rev. H. W., A.M. (Boston Coll.) Missionary Priest, St. Helen's Mission, Linstead, Jamaica, B.W.I.
BALL, Asst. Prof. N. H., Ph.D. (Mass. Inst. of Tech.) U. S. Naval Acad., Annapolis, Md. *Capt., Coast Artillery*
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BALLARD, RUTH MASON (Mrs. F. K.), Ph.D. (Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill. *636 Wellington Ave., Chicago 14, Ill.*
BALLOU, Asst. Prof. D. H., Ph.D. (Harvard) Middlebury Coll., Middlebury, Vt. *27 Weybridge St.*
BALOF, C. A., M.S. (Iowa) Business Manager, Lincoln Coll., Lincoln, Ill. *623 N. Union St.*
BAMFORTH, Asso. Prof. F. R., Ph.D. (Chicago) Ohio State Univ., Columbus, Ohio. *Dept. of Math.*

BANCROFT, Asst. Prof. T. A., Ph.D (Iowa State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*

BANKS, Prof. G. B., Ph.D. (Niagara Univ.) Physical Sci., Niagara Univ., Niagara University, N.Y.

BARBOUR, Asso. Prof. J. M., Ph.D. (Cornell), Mus.D. (Toronto) Music, Michigan State Coll., East Lansing, Mich. *Music Dept.*

BARDELL, R. H., Ph.D. (Chicago) Acting Dir., Univ. of Wisconsin in Milwaukee, 623 W. State St., Milwaukee 3, Wis.

BAREIS, Asst. Prof. GRACE M., Ph.D. (Ohio State) Ohio State Univ., Columbus, Ohio. *164-13th Ave., Columbus 1, Ohio*

BARKER, C. B., Ph.D. (California) *O.S.R.D., Radiation Lab., Massachusetts Inst. of Tech., Cambridge 39, Mass. Room 22-267*

BARNARD, Asso. Prof. R. W., Ph.D. (Chicago) Univ. of Chicago, Chicago, Ill.

BARNES, Prof. G. F., Ph.D. (Indiana) Math. and Physics, Mississippi Coll., Clinton, Miss. *Box 182*

BARNES, J. C., B.S. (North Georgia Coll.) Head of Dept., North Georgia Coll., Dahlonega, Ga.

BARNETT, Prof. I. A., Ph.D. (Chicago) Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math.*

BARNETT, Asst. Prof. JOSEPH, Jr., A.M. (Columbia) Oklahoma A. and M. Coll., Stillwater, Okla. *520 Maple Ave., Apt. 5*

BARR, Prof. C. F., M.S. (Chicago) Univ. of Wyoming, Laramie, Wyo.

BARRAL-SOUTO, Prof. JOSÉ, Sc.D. Statistics, Faculty of Econ. Sciences, Univ. of Buenos Aires, Buenos Aires, Argentina. *Córdoba 1459*

BARROW, Prof. D. F., Ph.D. (Harvard) Univ. of Georgia, Athens, Ga. *260 Cherokee Ave.*

BARTEN, H. J., B.S. in M.E. (Johns Hopkins) Asst. Supervisor, Naval Gage Lab., Washington Navy Yard, Washington, D.C. *251-37th St. S.E., Washington 19, D.C.*

BARTKY, Prof. WALTER, Ph.D. (Chicago) Asso. Dean, Physical Sci., Univ. of Chicago, Chicago, Ill.

BARTON, Prof. HELEN, Ph.D. (Johns Hopkins) Head of Dept., Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *1027 Spring Garden St.*

BARTOO, Prof. G. C., A.M. (Michigan) Western Michigan Coll., Kalamazoo, Mich.

BARTRAM, P. R., A.B. (Hamilton) Clerk, Bethlehem Steel Co., Buffalo, N.Y. *385 Oakwood Ave., East Aurora, N.Y.*

BASOCO, Prof. M. A., Ph.D. (Calif. Inst. of Tech.) Univ. of Nebraska, Lincoln 8, Nebr.

BASS, T. J., Jr., A.M. (California) Instr., San Francisco Jr. Coll., San Francisco, Calif. *Lt. (j.g.), U.S.N.R.*

BASYE, R. E., Ph.D. (Texas) A. and M. Coll. of Texas, College Station, Tex. *Lt., U.S.N.R. 5529 Brooklyn Ave., Kansas City 4, Mo.*

BATCHELDER, Asso. Prof. P. M., Ph.D. (Harvard) Univ. of Texas, Austin, Tex. *910 West 22nd St., Austin 21, Tex.*

BATEMAN, Prof. HARRY, Ph.D. (Johns Hopkins) Math., Theoretical Physics, and Aeronautics, California Inst. of Tech., Pasadena 4, Calif.

BATEN, Prof. W. D., Ph.D. (Michigan) Michigan State Coll., East Lansing, Mich. *411 Marshall St.*

BATES, GRACE E., M.S. (Brown) Fellow, Univ. of Illinois, Urbana, Ill. *153 Math. Bldg.*

BATTIG, LEON, A.M. (Wisconsin) Instr., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *2535 Elizabeth St., Sheboygan, Wis.*

BATTIN, EDNA C., A.M. (Southern California) Head of Dept., Union High School, Colton, Calif. *1178 N. Eighth St.*

BATTIN, Asst. Prof. I. L., A.M. (Swarthmore) Drew Univ., Madison, N.J.

BAUER, L. M., A.M. (New Mexico) Teacher, Menaul School, Albuquerque, N.M. *P.O. Box 1295*

BAUMGART, J. K., A.M. (Michigan) *2nd Lt., Air Corps., 1441 S. 13th Ave., Maywood, Ill.*

BAUMGARTNER, R. A., A.M. (Illinois) Head of Dept., High School, Freeport, Ill. *Lt., U.S.N.R. Postgrad. School, U. S. Naval Acad., Annapolis, Md. 59 Maryland Ave., Annapolis, Md.*

BAUSER, A. V., A.M. (New York Univ.) Engr., Elliott Co., Ridgway, Pa. *105 N. Ferguson St., Shenandoah, Pa.*

BAUSERMAN, THOMAS, B.S. (W. Va. Inst. of Tech.) *Lt., CAC (AA)*

BAY, J. C. Librarian, The John Crerar Library, 86 E. Randolph St., Chicago, Ill.

BAYS, Dr. SÉVÉRIN. Prof. ord. math. pures, Univ. de Fribourg, Fribourg, Switzerland

BEACH, J. W., M.S. (Iowa State) Instr., Iowa State Coll., Ames, Iowa. *155 Campus Ave.*

BEAL, Asst. Prof. JUNA LUTZ, A.M. (Chicago) Acting Head of Dept., Butler Univ., Indianapolis 7, Ind. *3262 Broadway, Indianapolis 5, Ind.*

BEARD, Asst. Prof. HELEN P., Ph.D. (Mass. Inst. of Tech.) Newcomb Coll., New Orleans 18, La.
BEASLEY, Asst. Prof. S. LOUISE, Ed.M. (Peabody) Head of Dept., Drury Coll., Springfield, Mo.
BEATLEY, Asso. Prof. RALPH, A.M. (Columbia) Harvard Grad. School of Educ., Cambridge 38, Mass.
BEATTY, Asst. Prof. H. M., A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *200 Tibet Road*
BEATTY, Prof. SAMUEL, Ph.D. (Toronto) Dean, Faculty of Arts, Univ. of Toronto, Toronto, Ont., Canada. *Dept. of Math.*
BEATY, Prof. E. B., A.M. (California) Oregon State Coll., Corvallis, Ore.
BEAUMONT, Asst. Prof. R. A., Ph.D. (Illinois) Univ. of Washington, Seattle 5, Wash. *Dept. of Math.*
BEAVER, Prof. R. A., Ph.D. (Cornell) New York State Coll. for Teachers, Albany 3, N.Y.
BECKENBACH, Asso. Prof. E. F., Ph.D. (Rice) Univ. of California at Los Angeles, Los Angeles, Calif.
BECKER, H. W. Instr., Mare Island Navy Yard, Fire Control School, Mare Island, Calif. *1426 Amador St., Vallejo, Calif.*
BECKWITH, Prof. ETHELWYNN R., (Mrs. W. E.), Ph.D. (Radcliffe) Milwaukee-Downer Coll., Milwaukee 11, Wis.
BECKWITH, MABLE L., A.M. (Claremont) Supervising Prin., Brawley School Dist., Brawley, Calif. *Lt. (j.g.), U.S.N.R., Hotel Whitcomb, San Francisco, Calif.*
BECKWITH, Asso. Prof. W. S., A.M. (Harvard) Univ. of Georgia, Athens, Ga. *731 Cobb St.*
BEDWELL, T. H., A.B. (South Dakota) Instr., Physics, Univ. of South Dakota, Vermillion, S.D. *223 N. Yale St.*
BEEGLE, Prof. B. L., M.S. (Washington) Dean, Seattle Pacific Coll., 3307 Third Ave. W., Seattle 99, Wash.
BEELER, F. A., A.M. (Indiana) Hillsdale Coll., Hillsdale, Mich. *Lt., U.S.N.R., 43 Pierrepont St., Apt. B, Brooklyn 2, N.Y.*
BEEMAN, W. E., M.S. (N. Tex. State T.C.) Instr., North Texas State Teachers Coll., Denton, Tex. *Box 5613 T.C. Station*
BEENKEN, MAY M., Ph.D. (Chicago) Dir., Div. of Preprofessional Educ., State Teachers Coll., Oshkosh, Wis.
BEER, Asst. Prof. F. P., Ph.D. (Geneva) Physics, Williams Coll., Williamstown, Mass. *Thompson Phys. Lab.*
BEESLEY, Asso. Prof. E. M., Ph.D. (Brown) Acting Head of Dept., Math. and Mech., Univ. of Nevada, Reno, Nev.
BEGLE, Asst. Prof. E. G., Ph.D. (Princeton) Yale Univ., New Haven, Conn. *24 Wakefield St., Hamden, Conn.*
BEINERT, R. L., A.B. (Hobart) Asst., Cornell Univ., Ithaca, N.Y. *RT 3/c, U.S.N., 306 Woodward Ave., Buffalo 14, N.Y.*
BEITO, E. A., A.M. (Minnesota) *In Service*
BELDING, R. F., A.B. (Amherst) Instr., Vermont Acad., Saxtons River, Vt.
BELL, Asso. Prof. CLIFFORD, Ph.D. (California) Univ. of California at Los Angeles, Los Angeles 24, Calif. *10514 Rochester Ave.*
BELL, Prof. E. T., Ph.D. (Columbia) California Inst. of Tech., Pasadena 4, Calif.
BELL, R. F., M.S. (Michigan) Instr., Eastern Washington Coll. of Educ., Cheney, Wash. *Smithers, W.Va.*
BELL, Prof. TALMON, A.B. (Sterling) Sterling Coll., Sterling, Kans.
BELLAMY, B. C., B.S.C.E. (Wyoming) Civil Engineer, Laramie, Wyo. *Box 37, Lamont, Wyo.*
BELLARDO, J. E., M.S. (St. Bonaventure) Instr., Aquinas Coll., 69 Ransom Ave., Grand Rapids, Mich.
BEMMELS, Prof. W. D., Ph.D. (Colorado) Math. and Physics, Ottawa Univ., Ottawa, Kans.
BENDER, Asso. Prof. H. A., Ph.D. (Illinois) Rhode Island State Coll., Kingston, R.I.
BENEDICTA, Sister M. (BOYLE), M.S. (St. Bonaventure) Prof., Villa Maria Coll., Erie, Pa. *West Eighth St.*
BENNER, Asso. Prof. J. A., A.M. (Lafayette) Lafayette Coll., Easton, Pa. *2 E. Campus*
BENNETT, Prof. A. A., Ph.D. (Princeton) Brown Univ., Providence, R.I. *Major, A.U.S., Ordnance Dept., Ballistic Research Lab., Aberdeen Proving Ground, Md.*
BENNETT, Prof. THEODORE, Ph.D. (Illinois) Marietta Coll., Marietta, Ohio
BERGER, EDLA G., A.M. (Minnesota) Mathematician, Equitable Life Assur. Soc., New York, N. Y. *130 W. Twelfth St., New York 11, N.Y.*
BERGSTRESSER, C. A., A.M. (Lafayette), M.S. (Pennsylvania) Head of Dept., Jamaica High School, New York, N.Y. *112-66 175th Place, St. Albans, N.Y.*
BERKELEY, L. M., A.M. (Virginia) 415 West 118 St., New York, N.Y.

BERNARD ALFRED, Brother (WELCH), A.M. (Catholic Univ.) Head of Dept., Manhattan Coll., Spuyten Duyvil Parkway, New York 63, N.Y.
BERNSTEIN, Prof. B. A., Ph.D. (California) Univ. of California, Berkeley, Calif. *2785 Shasta Road, Berkeley 8, Calif.*
BERNSTEIN, DOROTHY L., Ph.D. (Brown) Instr., Univ. of Rochester, Rochester, N.Y.
BERNSTEIN, Prof. FELIX, Ph.D. (Göttingen) Emeritus, New York Univ., New York, N.Y. *949 Broadway, Room 1705, New York 10, N.Y.*
BERRY, Prof. A. C., Ph.D. (Harvard) Lawrence Coll., Appleton, Wis. *In Service*
Berry, E. M., Ph.D. (Iowa) Head of Dept., State Teachers Coll., Chadron, Nebr. *353 Chapin St.*
BERRY, T. E., B.S. (George Washington) Analyst, Social Security Board, Washington, D.C. *200 Rhode Island Ave. N.E., Washington 2, D.C.*
BERRY, Prof. W. J., A.M. (Harvard) Retired, Poly. Inst. of Brooklyn, Brooklyn, N.Y. *224 St. John's Place, Brooklyn 17, N.Y.*
BERT, Prof. O. F. H., A.M. (Geneva) Washington and Jefferson Coll., Washington, Pa. *28 N. Lincoln St.*
BERTRAND, Sister M. (WALTON), Ph.D. (Fordham) Teacher, Marywood Coll., Scranton, Pa.
BETTINGER, Asst. Prof. A. K., A.M. (Wisconsin) Creighton Univ., Omaha, Nebr.
BETZ, WILLIAM, A.M. (Rochester) Specialist in math., Rochester public schools; Univ. of Rochester, Rochester, N.Y. *652 Melville St.*
BEVERIDGE, Prof. H. R., Ph.D. (Illinois) Monmouth Coll., Monmouth, Ill. *1043 E. Detroit Ave.*
BEY, Asst. Prof. D. R., A.M. (Illinois) Illinois State Normal Univ., Normal, Ill. *402 W. Locust St.*
BIBB, Asso. Prof. S. F., M.S. (Chicago) Illinois Inst. of Tech., 3300 Federal St., Chicago, Ill.
BICKERSTAFF, Asst. Prof. T. A., A.M. (Mississippi) Registrar, Univ. of Mississippi, University, Miss. *Box 7*
BIESELE, Asst. Prof. F. C., Ph.D. (Texas) Univ. of Utah, Salt Lake City, Utah. *1876 S. Woodside St., Salt Lake City 7, Utah*
BIGELOW, W. W., A.B. (Beloit), B.S.C.E. (George Washington) Analyst, A. O. Smith Corp., Milwaukee, Wis. *Rockton, Ill.*
BIGGERSTAFF, J. S., M.S. (Washington) Instr., Rensselaer Poly. Inst., Troy, N.Y. *1520 Sage Ave.*
BILLIG, A. L., Ed.D. (Temple) Teacher, Science, High School, Allentown, Pa. *1328 Gordon St.*
BINGLEY, G. A., A.M. (Princeton) Tutor, St. John's Coll., Annapolis, Md.
BIRCHBY, Asst. Prof. W. N., A.M. (Colorado Coll.) California Inst. of Tech., Pasadena, Calif.
BIRCHENOUGH, Prof. HARRY, A.M. (Columbia) New York State Coll. for Teachers, Albany 3, N.Y.
BIRD, Asso. Prof. M. T., Ph.D. (Illinois) Utah State Agric. Coll., Logan, Utah
BIRKHOFF, Asso. Prof. GARRETT, A.B. (Harvard) Harvard Univ., Cambridge, Mass. *45 Fayerweather St.*
BISSINGER, B. H., Ph.D. (Cornell) Instr., Michigan State Coll., East Lansing, Mich. *962 Main St., Leominster, Mass.*
BLACK, Prof. A. H., Ph.D. (Cornell) Lebanon Valley Coll., Annville, Pa. *625 E. Main St.*
BLACK, Asso. Prof. FLORENCE L., Ph.D. (Kansas) Univ. of Kansas, Lawrence, Kans. *1300 Louisiana St.*
BLACK, Prof. H. L., Ph.D. (Illinois) Westminster Coll., New Wilmington, Pa. *239 New Castle St.*
BLACK, L. G., A.M. (Colorado) Instr., Purdue Univ., West Lafayette, Ind. *220 S. Salisbury St.*
BLACK, L. T., A.M. (Michigan) Instr., Long Beach City Coll., Long Beach, Calif. *Instr., Univ. of California at Los Angeles, Los Angeles, Calif. 1230 Hungerford St., Long Beach 5, Calif.*
BLACKALL, C. J., Ph.D. (Cornell) *2nd Lt., CAC. North Baltimore, Ohio*
BLACKWELL, Asso. Prof. R. C., Ph.D. (North Carolina) Furman Univ., Greenville, S.C. *322 University Ridge*
BLAIR, Prof. HAROLD, A.M. (Michigan) Western Michigan Coll., Kalamazoo, Mich. *1220 Academy St., Kalamazoo 49, Mich.*
BLAIR, Asso. Prof. LEORA, A.M. (Chicago) Northwestern State Coll., Natchitoches, La. *404 New Second St.*
BLAIR, Asso. Prof. R. V., Ph.D. (Peabody) Vanderbilt Univ., Nashville 4, Tenn.
BLAKE, ARCHIE, Ph.D. (Chicago) Ballistician, Ballistic Research Lab., Aberdeen Proving Ground, Md. *Box 86, Aberdeen, Md.*

BLAKE, R. G., A.B. (Florida) Instr., Univ. of Florida, Gainesville, Fla. *Route 2, Box 25*
BLAKEMAN, B. E., A.M. (Illinois) *Ensign, U.S.N.R.*
BLANCHE, E. E., Ph.D. (Illinois) Prin. Statistician, Foreign Econ. Admin., Washington, D.C. *Army Univ. Center in Italy. 9409 Montgomery Ave., North Chevy Chase, Md.*
BLAU, L. W., Ph.D. (Texas) Research consultant, Humble Oil & Refining Co., Houston 1, Tex. *2027 Colquitt Ave.*
BLEICK, Asst. Prof. W. E., Ph.D. (Johns Hopkins) U. S. Naval Acad., Annapolis, Md. *On leave, Lt. Comdr., U.S.N.R. 8 Taney Ave.*
BLINCOE, Prof. J. W., Ph.D. (Virginia) Randolph-Macon Coll., Ashland, Va.
BLISS, Prof. G. A., Ph.D. (Chicago) Emeritus, Univ. of Chicago, Chicago, Ill. *Flossmoor, Ill.*
BLOOM, G. M., A.M. (Northwestern) Teacher, Ballard Memorial School, Louisville, Ky. *Lt. Comdr., U.S.N. Naval Reserve Educ'l Center, 105 Market St., San Francisco, Calif.*
BLUMBERG, A. A., A B. (Texas) Instr., A. and M. Coll. of Texas, College Station, Tex. *Box 149 Faculty Exchange*
BLUMBERG, Prof. HENRY, Ph.D. (Göttingen) Ohio State Univ., Columbus, Ohio. *76 E. Blake Ave.*
BLUMBERG, Asst. Prof. J. O., Ph.D. (Pittsburgh) Univ. of Pittsburgh, Pittsburgh, Pa. *308 Foster St., Greensburg, Pa.*
BLUMENTHAL, Prof. L. M., Ph.D. (Johns Hopkins) Univ. of Missouri, Columbia, Mo. *Dept. of Math.*
BLYTH, C. R., B.A. (Queen's Univ.) Fellow, Queen's Univ., Kingston, Ont., Can. *R.R. 5, Guelph, Ont., Can.*
BOARDMAN, H. C., D.E. (S.D. St. Sch. of Mines) Dir. of Research, Chicago Bridge and Iron Co., Chicago, Ill. *10357 S. Hoyne Ave., Chicago 43, Ill.*
BOAS, R. P., Jr., Ph.D. (Harvard) Executive Editor, Mathematical Reviews. *21 Chauncy St., Cambridge 38, Mass.*
BOATMAN, Prof. A. O., A.M. (Indiana) Carthage Coll., Carthage, Ill.
BOCK, W. W., M.S. (Colorado) Medical student, Univ. of St. Louis, St. Louis, Mo. *3532 A Vista Ave., St. Louis 4, Mo.*
BODNAR, JOHN, M.S. in M.E. (Illinois Inst. of Tech.) Instr., Math. and Physics, Jr. Coll. of Connecticut, Bridgeport, Conn.; Test Engr., Chance Vought Aircraft, Stratford, Conn. *85 Summerfield Ave., Bridgeport, Conn.*
BOEDER, PAUL, Ph.D. (Göttingen) Dir., Bureau of Visual Science, American Optical Co., Southridge, Mass.
BOEHM, FRANK, Manager, Life Insurance, 80 Maiden Lane, New York 7, N.Y.
BOEHMER, Sister M. MIRABELLA, M.S. (Catholic Univ.) Instr., Math. and Physics, Alverno Teachers Coll., 1413 S. Layton Blvd., Milwaukee 4, Wis.
BOHNENBLUST, Prof. H. F., Ph.D. (Princeton) Indiana Univ., Bloomington, Ind. *Dept. of Math.*
BOLDYREFF, Asso. Prof. A. W., Ph.D. (Michigan) Univ. of Arizona, Tucson, Ariz. *Lt., Army Air Force. 2309 E. Fifth St., Tucson, Ariz.*
BOLKS, STANLEY, M.S. (Iowa State) Instr., Purdue Univ., West Lafayette, Ind. *R.R. 10, Lafayette, Ind.*
BOLTON, GRACE L., A.M. (Columbia) Instr., New Jersey Coll for Women, New Brunswick, N.J.
BOOKSTON, J. M., A.M. (Michigan) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *1st Lt., A.C.*
BORGMAN, Asso. Prof. W. M., Jr., Ph.D. (Chicago) Wayne Univ., Detroit, Mich.
BORNMANN, W. C., A.M. (Columbia) Instr., Georgia School of Tech., Atlanta, Ga.
BOROFSKY, Asst. Prof. SAMUEL, Ph.D. (Columbia) Brooklyn Coll., Brooklyn, N.Y. *1st Lt., A.C. 335 State St., c/o Gardner, Brooklyn 2, N.Y.*
BORTOLOTTI, Prof. ETTORE, Dottore in mat. (Bologna) Univ. of Bologna, Bologna, Italy. *Via Albertazzi 43*
BOSWELL, Prof. A. V., A.M. (Northwestern) Tennessee A. and I. State Coll., Nashville 8, Tenn.
BOTTS, T. A., Ph.D. (Virginia) 2800 Woodley Road, Washington 8, D.C.
BOURNE, S. G., A.M. (Johns Hopkins) Instr., Johns Hopkins Univ., Baltimore 18, Md. *Dept. of Math.*
BOUTELLE, Asst. Prof. H. D., B.S., Ch.E. (Worcester Poly. Inst.) Massachusetts State Coll., Amherst, Mass. *10 Allen St.*
BOWDEN, Prof. JOSEPH, Ph.D. (Yale) Emeritus, Adelphi Coll., Garden City, N.Y. *21 Carleton Pl., Baldwin, N.Y.*

BOWER, Asso. Prof. JULIA W., Ph.D. (Chicago) Connecticut Coll., New London, Conn. *Dept. of Math.*

BOWKER, Prof. J. G., Ed.M. (Harvard) Chm. of Dept., Middlebury Coll., Middlebury, Vt. *14 Adirondack View*

BOWLING, Asso. Prof. FLOYD, M.S. (Iowa) Lincoln Mem. Univ., Harrogate, Tenn. *Lt., U.S.N.R., Instr., U. S. Naval Pre-Flight School, St. Mary's College, Calif.*

BOYCE, Asso. Prof. FANNIE W., Ph.D. (Chicago) Wheaton Coll., Wheaton, Ill. *311 E. Wesley St.*

BOYCE, Prof. JESSIE W., A.M. (Minnesota) Chm. of Dept., State Teachers Coll., Wayne, Nebr. *518 Lincoln St.*

BOYCE, Asso. Prof. M. G., Ph.D. (Chicago) Western Reserve Univ., Cleveland 6, Ohio

BOYD, Dean P. P., Ph.D. (Cornell) Head of Dept., Math. and Astr., Univ. of Kentucky, Lexington, Ky.

BOYER, Asst. Prof. C. B., Ph.D. (Columbia) Brooklyn Coll., Bedford Ave. and Avenue H, Brooklyn 10, N.Y.

BOYER, L. E., Ed.D. (Pennsylvania State Coll.) State Teachers Coll., Millersville, Pa.

BOYLE, Sister M. RITA CECILE, M.S. (Michigan) Asst. Prof., Barry Coll., Miami 38, Fla.

BRACEWELL, Prof. K. H., Ph.D. (Indiana) Math. and Physics, Hamline Univ., St. Paul, Minn.

BRADFORD, Asst. Prof. W. H., M.S. (Louisiana) McNeese Jr. Coll. of L.S.U., Lake Charles, La. *Part-time Instr., Univ. of Texas, Austin, Tex. Box 1879 Univ. Sta.*

BRADLEY, Asst. Prof. A.D., Ph.D. (Columbia) Hunter Coll., New York, N.Y. *Lt., U.S.N.R. 66 Villard Ave., Hastings-on-Hudson, N.Y.*

BRADLEY, R. A., B.A. (Queen's Univ.) 378 Barrie St., Kingston, Ont., Can.

BRADSHAW, Prof. J. W., Ph.D. (Strassburg) Univ. of Michigan, Ann Arbor, Mich. *1304 Cambridge Road*

BRADY, W. G., B.S. (Washington and Jefferson) *Lt. (j.g.), U.S.N.R. P.O. Box 214, Washington, Pa.*

BRAMBLE, Prof. C. C., Ph.D. (Johns Hopkins) Math. and Mech., Postgraduate School, U.S. Naval Acad., Annapolis, Md. *Comdr., U.S.N.R. 145 Monticello Ave.*

BRAND, Prof. LOUIS, Ph.D. (Harvard) Chm. of Dept., Univ. of Cincinnati, Cincinnati 21, Ohio

BRANDEBERRY, Prof. J. B., Ph.D. (Michigan) Math. and Engg. Mech.; Acting Dean, Coll. of Engg., Univ. of Toledo, Toledo 6, Ohio

BRANDNER, Asst. Prof. F. A., A.M., M.S. (Chicago) Iowa State Coll., Ames, Iowa. *Dept. of Math.*

BRANSON, Asso. Prof. HERMAN, Ph.D. (Cincinnati) Physics, Howard Univ., Washington, D.C.

BRANSON, Dean J. W., M.S. (Purdue) Acting Pres., New Mexico Coll. of A. and M. A., State College, N.M.

BRAUER, Asso. Prof. A. T., Ph.D. (Berlin) Univ. of North Carolina, Chapel Hill, N.C. *401 McCauley St.*

BRAUER, Asso. Prof. RICHARD, Ph.D. (Berlin) Univ. of Toronto, Toronto 5, Ont., Can. *114 Balmoral Ave., Toronto, Ont., Can.*

BRAVERMAN, BENJAMIN, A.M. (Columbia) Chm. of Dept., Seward Park High School, New York, N.Y. *1309 Avenue L, Brooklyn, N.Y.*

BRAY, Prof. H. E., Ph.D. (Rice) Rice Inst., Houston, Tex.

BRENDEL, RUTH A., Ed.M. (Buffalo) Univ. of Buffalo, Buffalo, N.Y. *Ensign, U.S.N.R. Computation Lab., Cruft, Harvard Univ., Cambridge 38, Mass.*

BRENKE, Prof. W. C., Ph.D. (Harvard) Emeritus, Univ. of Nebraska, Lincoln, Nebr. *1250 S. 21st St.*

BRIANT, R. C., Ph.D. (Pittsburgh) Sr. Indus. Fellow, Mellon Inst., Pittsburgh, Pa. *403 Glasgow Road, Pittsburgh 21, Pa.*

BRIDGER, C. A., M.S. (Oregon State Coll.) Instr., Univ. of Utah, Salt Lake City 1, Utah. *336 Douglas St., Salt Lake City 2, Utah*

BRIGHT, H. F., M.S. (Rochester) Dir. of Guidance, San Angelo Coll., San Angelo, Tex.

BRINK, Prof. R. W., Ph.D. (Harvard) Chm. of Dept., Univ. of Minnesota, Minneapolis 14, Minn. *2243 Hoyt Ave., St. Paul 8, Minn.*

BRINKMANN, Asso. Prof. H. W., Ph.D. (Harvard) Swarthmore Coll., Swarthmore, Pa.

BRITTON, Asso. Prof. JACK, Ph.D. (Colorado) Univ. of Colorado, Boulder, Colo.

BRIXEY, Asso. Prof. J. C., Ph.D. (Chicago) Univ. of Oklahoma, Norman, Okla. *927 S. Pickard St.*

BROCK, PAUL, A.B. (Brooklyn) Grad. Student, Princeton Univ., Princeton, N.J. *Cpl., Signal Corps. 1421 East 34 St., Brooklyn, N.Y.*

BRODERICK, Prof. T. S., M.A. (Dublin) Trinity Coll., Dublin, Ireland
BRONSTEIN, SAMUEL, A.M. (New York Univ.) Teacher, High School, Hartford, Conn. *56 Adams St., Hartford 5, Conn.*
BROOKE, Prof. W. E., A.M. (Nebraska) Emeritus, Math. and Mech., Univ. of Minnesota, Minneapolis 14, Minn. *416 Walnut St. S.E.*
BROOKS, Asst. Prof. FOSTER, Ph.D. (Ohio State) Kent State Univ., Kent, Ohio. *O.S.R.D. c/o Mrs. Bronta Brooks, Sherrodsville, Ohio*
BROTHERS, Prof. W. H., Jr., Ph.D. (Michigan) Talladega Coll., Talladega, Ala.
BROWN, Asst. Prof. A.B., Ph.D. (Harvard) Queens Coll., Flushing, N.Y.
BROWN, Prof. B. H., Ph.D. (Harvard) Dartmouth Coll., Hanover, N.H. *7 Ripley Road*
BROWN, B. K., A.M. (Colorado) Instr., Colorado School of Mines, Golden, Colo. *Process Instr., Tennessee Eastman Corp., Oak Ridge, Tenn. 112 D E. Hunter Circle*
BROWN, Asso. Prof. C. H., Ph.D. (Kansas) Central Missouri State Teachers Coll., Warrensburg, Mo. *407 S. Maguire St.*
BROWN, Prof. E. C., A.M. (Maine) Worcester Poly. Inst., Worcester, Mass. *51 Grafton St., Shrewsbury, Mass.*
BROWN, H. K., Ph.D. (Michigan) Instr., Rensselaer Poly. Inst., Troy, N.Y. *703 Grand St.*
BROWN, Prof. H. S., M.S. (Lafayette) Emeritus, Hamilton Coll., Clinton, N.Y. *162 Proctor Blvd., Utica, N.Y.*
BROWN, Prof. LILLIAN O., A.M. (Columbia) Hood Coll., Frederick, Md.
BROWN, Asst. Prof. M. C., A.M. (Kentucky) Univ. of Kentucky, Lexington, Ky. *448 Clifton Ave.*
BROWN, Asso. Prof. MYRTLE C., A.M. (Texas) North Texas State Teachers Coll., Denton, Tex. *1415 W. Oak St.*
BROWN, Prof. O. E., Ph.D. (Chicago) Case School of Appl. Sci., Cleveland, Ohio. *1842 Taylor Road, East Cleveland, Ohio*
BROWN, R. D., Jr. Engr. in charge, Patent Dept., Philco Corp., Philadelphia, Pa. *Blue Bell, Montgomery Co., Pa.*
BROWN, Prof. R. E., M.S. (Mass. Inst. of Tech.) Mech. Engg., Rhode Island State Coll., Kingston, R.I. *Box 211*
BROWN, Prof. T. H., Ph.D. (Yale) Business Statistics, Harvard Grad. School of Bus. Admin., Boston, Mass. *25 Meadow Way, Cambridge, Mass.*
BROWN, W. B., Ph.D. (Ohio State) Instr., Fenn Coll., Cleveland, Ohio. Physicist, Nat'l Advisory Comm. for Aeronautics, Cleveland, Ohio. 13983 Clifton Blvd., Lakewood 7, Ohio
BROWNE, D. H. 10 Indian Orchard Pl., Buffalo 10, N.Y.
BROWNE, Prof. E. T., Ph.D. (Chicago) Univ. of North Carolina, Chapel Hill, N.C. *Army Univ. Center in England*
BROWNE, P. L., A.M. (Michigan State Coll.) *RT 3/c*
BROYDE, SAMUEL, B.S. (Chicago), J.D. (De Paul) Instr., Northwestern Univ., Evanston, Ill. *1319 Touhy Ave., Chicago, Ill.*
BRUCE, Prof. C. W., Ph.D. (Virginia) Wesleyan Coll., Macon, Ga.
BRUCE, Prof. R. E., Ph.D. (Boston Univ.) Emeritus, Boston Univ., Boston, Mass.
BRUCK, Asst. Prof. R. H., Ph.D. (Toronto) Univ. of Wisconsin, Madison 6, Wis. *206 North Hall*
BRUNK, H. D., Ph.D. (Rice) *Ensign, U.S.N.R., Staff, Commander Carrier. 251 East St., Healdsburg, Calif.*
BRYAN, Prof. N. R., Ph.D. (Columbia) Univ. of Maine, Orono, Me. *4 University Place*
BRYSON, A. M., A.M. (Pittsburgh) Instr., Univ. of Pittsburgh, Pittsburgh, Pa. *McCutcheon Lane, Wilkinsburg, Pa.*
BUCHANAN, Asso. Prof. ANN S., A.M. (Louisiana State) Southwestern Louisiana Inst., Lafayette, La. *1106 Lee Ave.*
BUCHANAN, Prof. DANIEL, Ph.D. (Chicago) Dean of Faculty of Arts and Sci., Univ. of British Columbia, Vancouver, B.C., Can.
BUCHANAN, Prof. H. E., Ph.D. (Chicago) Tulane Univ., New Orleans 15, La.
BUCHANAN, Dean SCOTT, Ph.D. (Harvard) St. John's Coll., Annapolis, Md.
BUCK, R. C., A.M. (Cincinnati) Teaching Fellow, Harvard Univ., Cambridge 38, Mass. *24 Chauncy St.*
BUELL, E. L., Ph.D. (Mass. Inst. of Tech.) Instr., Northwestern Univ., Evanston, Ill. *Physicist, N.D.R.C., Northwestern Tech. Inst., Evanston, Ill.*
BUIKSTRA, B. H., M.S. (Kansas State Coll.) Instr., Kansas State Coll., Manhattan, Kans. *Lt., U.S.N.R., 435 West 119 St., New York 27, N.Y.*
BUKER, W. E., A.M. (Ohio State) Teacher, Public school, Pittsburgh, Pa. *Teacher in war training program at Univ. of Pittsburgh. 3833 Oswego St.*
BULLARD, Prof. J. A., Ph.D. (Clark) Chm. of Dept., Math. and Mech., Univ. of Vermont, Burlington, Vt. *110 Summit St.*

BULLITT, W. M., B.S. (Princeton) Bullitt and Middleton, Kentucky Home Life Bldg., Louisville 2, Ky.
BULLOCK, Asso. Prof. R. C., Ph.D. (Chicago) North Carolina State Coll., Raleigh, N.C. *Box 5548 State College Sta.*
BUMER, Prof. C. T., Ph.D. (Ohio State) Chm. of Dept., Kenyon Coll., Gambier, Ohio
BUNYAN, Asst. Prof. L. H., Ph.D. (Wisconsin) Rutgers Univ., New Brunswick, N.J.
BURDICK, O. Z., M.S. (Illinois) Lawyer, 713 S. Courtland St., Hart, Mich.
BURGESS, R. W., Ph.D. (Cornell) Chief Statistician, Western Elec. Co., 195 Broadway, New York, N.Y.
BURINGTON, Asso. Prof. R. S., Ph.D. (Ohio State) Case School of Appl. Sci., Cleveland, Ohio. *Head mathematician, Bureau of Ordnance, Navy Dept., Washington, D.C. 5200 N. Carlin Spring Road, Arlington, Va.*
BURK, Asso. Prof. J. D., B.A. (Toronto) Univ. of Toronto, Toronto 5, Ont., Can.
BURKE, Rev. J. G., A.M. (Mt. St. Mary) Mount St. Mary Coll., Emmitsburg, Md.
BURKE, Sister LEONARDA, Ph.D. (Catholic Univ.) Prof., Regis Coll., Weston 93, Mass.
BURKETT, Asso. Prof. F. J. H., Ph.D. (New York Univ.) Union Coll., Schenectady, N.Y. *Army Univ. Center*
BURNAM, Prof. J. E., A.M. (Texas) Hardin-Simmons Univ., Abilene, Tex. *1141 Grape St.*
BURNS, Asst. Prof. H. E., Ph.D. (Northwestern Univ.) Indiana Univ. Exten., 3901 Indianapolis Blvd., East Chicago, Ind.
BURNS, Sister MARY ROBERDETTE, A.M. (Illinois) Asso. Prof., Clarke Coll., Dubuque, Iowa
BURNS, Rev. W. F., M.S. (Boston Coll.) Asso. Prof., Coll. of the Holy Cross, Worcester, Mass.
BURR, Asso. Prof. I. W., Ph.D. (Michigan) Purdue Univ., West Lafayette, Ind.
BURT, MARY M. (Mrs. L. N.), A.M. (Wisconsin) 800-24th Ave. N., St. Petersburg, Fla.
BURWELL, W. R., Ph.D. (Oxford) Chm. of the Board, Brush Devel. Co., 3311 Perkins Ave., Cleveland 14, Ohio.
BUSEMANN, Asst. Prof. HERBERT, Ph.D. (Göttingen) Smith Coll., Northampton, Mass.
BUSH, Prof. L. E., Ph.D. (Ohio State) Chm. of Dept., Coll. of St. Thomas, St. Paul 1, Minn.
BUSHEY, Asso. Prof. JEWELL HUGHES (Mrs. J. H.), Ph.D. (Chicago) Hunter Coll., New York, N.Y. *501 West 113 St., New York 25, N.Y.*
BUSHEY, Asso. Prof. J. H., Ph.D. (Michigan) Hunter Coll., 695 Park Ave., New York 21, N.Y.
BUSSEY, Prof. W. H., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn. *1421 E. River Road*
BUTCHART, Prof. J. H., Ph.D. (Illinois) Arizona State Coll., Flagstaff, Ariz.
BUTLER, Asso. Prof. C. H., Ph.D. (Missouri) Western Michigan Coll., Kalamazoo, Mich.
BUTLER, L. G., A.M. (Oregon) Instr., State Coll. of Washington, Pullman, Wash. *1904 B Street*
BUTTER, F. A., Jr., Ph.D. (Stanford) Acting Asst. Prof., Stanford Univ., Stanford University, Calif. *Box 1250*
BUTTERFIELD, Prof. A. D., A.M. (Columbia) Emeritus; Dir., Veteran's Educ., Univ. of Vermont, Burlington, Vt. *479 Main St.*
BUXTON, Asso. Prof. C. L., M.S. (Case School) Physics, Clarkson Coll. of Tech., Potsdam, N.Y. *91 Market St.*
BYRD, P. F., M.S. (Chicago) *1st Lt., U.S. Army Air Corps, Weather Detachment, TAAF, Tuskegee, Ala.*
BYRNE, Prof. W. E., Ph.D. (Rensselaer) Virginia Military Inst., Lexington, Va. *Box 836*

CAIN, Asso. Prof. W. H., A.M. (Columbia) Western Michigan Coll., Kalamazoo, Mich. *1402 Hillcrest Ave., Kalamazoo 39, Mich.*
CAIRNS, Asst. Prof. S. S., Ph.D. (Harvard) Queens Coll., Flushing, N.Y. *148-39 Kalmia Ave.*
CAIRNS, Prof. W. D., Ph.D. (Göttingen) Emeritus, Oberlin Coll., Oberlin, Ohio. *1675 Kenneth Way, Pasadena 3, Calif.*
CALCAGNO, H. E. Patria 715, Montevideo, Uruguay
CALKINS, Asst. Prof. ELEANOR, A.M. (Michigan) Coll. of William and Mary, Williamsburg, Va. *121 Chandler Ct.*
CALKINS, Prof. HELEN, Ph.D. (Cornell) Head of Dept., Pennsylvania Coll. for Women, Pittsburgh 6, Pa.
CALLAGHAN, Sister M. PATRICIA, A.M. (St. Louis Univ.) Prof., Fontbonne Coll., Wydown and Big Bend Blvd., St. Louis, Mo.
CALLAHAN, Asso. Prof. ETHEL B., Ph.D. (Columbia) Hartwick Coll., Oneonta, N.Y. *122 Chestnut St.*
CALLAWAY, Asso. Prof. IRIS, A.M. (Peabody) Univ. of Georgia, Athens, Ga. *258 Hill St.*

CALLAWAY, Mrs. THEODOSIA T., A.M. (Columbia) Prof., Stephens Coll., Columbia, Mo. *1515 Ross St.*
CALVERT, Asst. Prof. R. L., A.M. (Illinois) Utah State Agric. Coll., Logan, Utah. *Lt. (j.g.), U.S.N.R. 122 W. 15th St., Hutchinson, Kans.*
CAMERON, Asso. Prof. E. A., Ph.D. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *Lt., U.S.N.R.*
CAMERON, Prof. R. H., Ph.D. (Cornell) Univ. of Minnesota, Minneapolis, Minn. *819 Superior St.*
CAMP, Prof. B. H., Ph.D. (Yale) Wesleyan Univ., Middletown, Conn. *110 Mt. Vernon St.*
CAMP, Prof. C. C., Ph.D. (Cornell) Univ. of Nebraska, Lincoln, Nebr. *Dept. of Math.*
CAMP, Prof. E. J., Ph.D. (Chicago) Macalester Coll., St. Paul 5, Minn.
CAMPAIGNE, H. H., Ph.D. (Northwestern) Instr., Univ. of Minnesota, Minneapolis, Minn. *Lt. Comdr., U.S.N.R., 2502 N. Fourth St., Arlington, Va.*
CAMPBELL, D. F., Ph.D. (Harvard) Consulting Actuary, 160 N. LaSalle St., Chicago, Ill.
CAMPBELL, Asso. Prof. FRANCES L., Ph.D. (Michigan) George Pepperdine Coll., 1121 W. 79th St., Los Angeles, Calif.
CAMPBELL, G. A., Ph.D. (Harvard) Retired, American Tel. and Tel. Co., New York, N.Y. *129 Bellevue Ave., Upper Montclair, N.J.*
CAMPBELL, J. D., A.M. (Illinois) Instr., Rensselaer Poly. Inst., Troy, N.Y. *Dept. of Math.*
CAMPBELL, JESSIE R., A.B. (Syracuse) Instr., High School, Hollywood, Calif. *12 The Strand, Hermosa Beach, Calif.*
CAMPBELL, Prof. J. W., Ph.D. (Chicago) Univ. of Alberta, Edmonton South, Alta., Can. *Dept. of Math.*
CAMPBELL, Asst. Prof. W. B., A.M. (Cornell) Philadelphia Textile Inst., Philadelphia, Pa. *6848 N. Seventh St., Philadelphia 26, Pa.*
CANDY, Prof. A. L., Ph.D. (Nebraska) Emeritus, Univ. of Nebraska, Lincoln, Nebr. *1003 H Street*
CANNING, JOSEPH, A.B. (Intermountain) Roundhouse Foreman, Northern Pacific R.R., Garrison, Mont.
CAPARÓ, Prof. J. A., Ph.D. (Notre Dame) Elec. Engg., Univ. of Notre Dame, Notre Dame, Ind. *1024 Leeper Blvd., South Bend, Ind.*
CAPESIUS, Rev. JOHN, A.M. (Alabama) Dean, St. Bernard Coll., St. Bernard, Ala.
CARAÇA, Prof. B. DE J. (Ci. Ec. e Fin.) Inst. Superior de Cien., Econ. e Finan., Univ. Tecnica de Lisboa, Lisbon, Portugal. *Rua Almeida e Sousa 63, 1 andar*
CAREY, Asso. Prof. E. F. A., M.S. (California) Emeritus, Univ. of Montana, Missoula, Mont.
CARIS, Asso. Prof. P. A., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia 4, Pa. *717 Shadeland Ave., Drexel Hill, Pa.*
CARIS, Asst. Prof. V. B., A.M. (Defiance Coll.) Ohio State Univ., Columbus, Ohio. *403 E. Oakland Ave.*
CARLEN, MILDRED E., M.S. (Brown) Instr., Registrar, Grad. School, Brown Univ., Providence 12, R.I.
CARLSON, Prof. C. S., M.S. (Iowa) Head of Dept., St. Olaf Coll., Northfield, Minn.
CARLSON, Prof. J. A., M.S. (Washington) Head of Dept., Whitworth Coll., Spokane, Wash.
CARLSON, Asst. Prof. ELIZABETH, Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *3024-14th Ave. S., Minneapolis 7, Minn.*
CARLTON, Asst. Prof. L. VIRGINIA, A.M. (Tulane) Wesleyan Coll., Macon, Ga. *Box 504 Wesleyan Coll.*
CARMAN, M. G., Ph.D. (Illinois) Head of Dept., State Teachers Coll., Murray, Ky. *Box 63 College Sta.*
CARMICHAEL, Prof. F. L., A.M. (Princeton) Statistics, Univ. of Denver, School of Commerce, Glenarm Place at 20th St., Denver, Colo.
CARMICHAEL, Prof. R. D., Ph.D. (Princeton) Dean of Grad. School, Univ. of Illinois, Urbana, Ill. *207 W. Washington Blvd.*
CARNAHAN, Asst. Prof. W. H., A.M. (Indiana) Purdue Univ., Lafayette, Ind. *708 Vine St., West Lafayette, Ind.*
CARPENTER, Prof. D. R., A.M. (Princeton) Roanoke Coll., Salem, Va.
CARPENTER, F. M., A.M. (Illinois) Fellow, Univ. of Iowa, Iowa City, Iowa. *910 S. Dodge St.*
CARPENTER, Asso. Prof. P. N., M.S. (Univ. of Washington) Grove City Coll., Grove City, Pa. *214 E. Pine St.*
CARR, Prof. F. E., Ph.D. (Chicago) Math. and Astr., Oberlin Coll., Oberlin, Ohio. *284 Forest St.*
CARROLL, C. L., Jr., A.M. (North Carolina) *Ensign, U.S.N.R.; Instr., Navigation, U.S. Navy Pre-Flight School, Chapel Hill, N.C. 698 Gimghoul Road*
CARROLL, Asso. Prof. I. S., A.M. (Columbia) Syracuse Univ., Syracuse, N.Y. *511 Comstock Ave.*

CARSCALLEN, Asso. Prof. G. E., A.M. (Illinois) Wabash Coll., Crawfordsville, Ind. *112 N. Barr St.*
CARSON, Prof. T. C., A.M. (Duke) State Teachers Coll., Johnson City, Tenn.
CARTER, C. C. Attorney, Bluffs, Ill.
CARVER, Prof. W. B., Ph.D. (Johns Hopkins) Cornell Univ., Ithaca, N.Y. *White Hall*
CASE, Rev. J. E., Ph.D. (Chicago) Head of Dept., St. Louis Univ., St. Louis, Mo.
CATON, W. B., Ph.D. (Yale) State Teachers Coll., Duluth, Minn.
CAVALLI, S. F., D.Sc. (Univ. of Milan, Italy) Consulting Engr., Philco Corp., Philadelphia, Pa. *271 S. 58th St., Philadelphia 39, Pa.*
CAWLEY, Asso. Prof. JOHN, M.S. (Lafayette) Lafayette Coll., Easton, Pa. *621 Coleman St.*
CEBULA, Rev. RICHARD, M.S. (Michigan) Instr., St. Martin's Coll., Lacey, Wash. *On leave. Univ. of Michigan, Ann Arbor, Mich. 326 N. Ingalls St., Ann Arbor, Mich.*
CEDERBERG, Prof. W. E., Ph.D. (Wisconsin) Augustana Coll., Rock Island, Ill. *2542-22½ Ave.*
CELAURO, Asst. Prof. F. L., A.M. (New York Univ.) Lehigh Univ., Bethlehem, Pa. *Dept. of Math.*
CELL, Asso. Prof. J. W., Ph.D. (Illinois) North Carolina State Coll., Raleigh, N.C. *Dept. of Math.*
CERINO, RAPHAEL, A.B. (Brooklyn) *RT 2/c, U.S.N.R., 2898 Pitkin Ave., Brooklyn, N.Y.*
CHACALOS, L. H. Student, 108 Ward St., Orange, N.J.
CHANEY, J. G., A.M. (Texas) *In Service*
CHANLER, Asst. Prof. JOSEPHINE H., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *364 Math. Bldg.*
CHAROSH, MANNIS, M.S. (New York Univ.) Teacher, New Utrecht High School, Brooklyn, N.Y. *8305-19th Ave., Brooklyn 14, N.Y.*
CHASE, L. R. Teacher, Rogers High School, Newport, R.I. *Boulevard Terrace*
CHELLEVOLD, Asst. Prof. J. O., A.M. (Northwestern) Wartburg Coll., Waverly, Iowa. *Lt. Comdr., U.S.N.R., Midshipmen's School, New York, N.Y. 419 West 119 St., New York 27, N.Y.*
CHEN, WAY MING, Ph.D. (California) 21 Mosswood Road, Berkeley 4, Calif.
CHENEY, Prof. W. F., Jr., Ph.D. (Mass. Inst. of Tech.) Univ. of Connecticut, Storrs, Conn. *P. O. Box 26*
CHESNA, JOHN. Physicist, Eastman Kodak Co., Rochester, N.Y. *12 Otilla St., Rochester 5, N.Y.*
CHESSIN, P. L. Student, Univ. of Wisconsin, Madison, Wis. *316 West 93 St., New York 25, N.Y.*
CHIN, LOUISE H., A.M. (California) Asso., Univ. of California, Berkeley, Calif. *2008 Cedar St., Berkeley 7, Calif.*
CHITTENDEN, Prof. E. W., Ph.D. (Chicago) Univ. of Iowa, Iowa City, Iowa. *221 Physics Bldg.*
CHRISTIAN, H. S., Jr. Student, Illinois Inst. of Tech., Chicago, Ill. *2936 Diversey Ave.*
CHRISTIE, D. E., Ph.D. (Princeton) Instr., Bowdoin Coll., Brunswick, Me. *36 Boody St.*
CHRISTMAN, LAURA E., A.M. (Wisconsin) Teacher, Senn High School, Chicago, Ill. *1217 Elmdale Ave.*
CHURCH, Asst. Prof. RANDOLPH, Ph.D. (Yale) Postgrad. School, U.S. Naval Acad., Annapolis, Md. *Lt. Comdr., U.S.N.R. 316 N. Glen Ave.*
CHURCHILL, Prof. R. V., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *924 S. Forest Ave.*
CIVIN, PAUL, Ph.D. (Duke) Instr., Univ. of Buffalo, Buffalo, N.Y. *32 Avery Ave., Buffalo 16, N.Y.*
CLARK, Prof. A. G., A.M. (Colorado) Head of Dept., Colorado State Coll. of A. and M. A., Fort Collins, Colo.
CLARK, Asso. Prof. B. G., Ph.D. (Illinois) Univ. of Alabama, University, Ala.
CLARK, C. R., A.M. (Michigan) Teacher, McKinley High School, Washington, D.C. *2707 Adams Mill Road N.W.*
CLARK, K. L., Ph.B. (Wisconsin) Mathematician, A. O. Smith Corp., Milwaukee, Wis. *2251 N. Summit Ave.*
CLARKE, Prof. E. H., Ph.D. (Chicago) Hiram Coll., Hiram, Ohio
CLARKE, Mrs. L. BEATRICE, A.M. (Michigan) Instr., Florida A. and M. Coll., Tallahassee, Fla.
CLARKE, Sister M. BORGIA, A.M. (Catholic Univ.) Prof., Webster Coll., Webster Groves, Mo.
CLARKE, WALTER B. Senior Partner, W. B. Clarke & Co., Nurserymen, San José, Calif. *848 N. First St.*

CLAUDIAN, Prof. VIRGIL, Grad. (Univ. of Bucharest) Liceul Mihai Viteazul, Bucharest 4, Roumania
CLAWSON, Prof. J. W., M.A. (New Brunswick) Ursinus Coll., Collegeville, Pa. *6 Glenwood Ave.*
CLELAND, Prof W. E., Ph.D. (Princeton) Geneva Coll., Beaver Falls, Pa.
CLEMENT, Asst. Prof. MARY D., Ph.D. (Chicago) Wells Coll., Aurora, N.Y.
CLEMENTS, Prof. G. R., Ph.D. (Harvard) U.S. Naval Acad., Annapolis, Md. *7 Thompson St.*
CLIFFORD, Asst. Prof. A. H., Ph.D. (Calif. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge, Mass. *Lt. (j.g.), U.S.N.R.*
CLIFFORD, P. C., A.M. (Columbia) Instr., State Teachers Coll., Upper Montclair, N.J.
CLINTON, FLORENTINA M. (Mrs. J. C.), A.M. (Ohio State) Chillicothe High School, Chillicothe, Ohio. *172 Vine St.*
CLUTZ, Prof. F. H., Ph.D. (Johns Hopkins) Emeritus, Gettysburg Coll., Gettysburg, Pa. *159 Broadway*
COBLE, Prof. A. B., Ph.D. (Johns Hopkins) Univ. of Illinois, Urbana, Ill. *702 W. Washington Blvd.*
COE, Asst. Prof. C. J., Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *2022 Hill St.*
COFFIN, Prof. L. M., A.M. (Michigan) Emeritus, Coe Coll., Cedar Rapids, Iowa. *633 S. Bixel St., Los Angeles 14, Calif.*
COHEN, Prof. ABRAHAM, Ph.D. (Johns Hopkins) Emeritus, Johns Hopkins Univ., Baltimore 18, Md.
COHEN, A. C., Jr., Ph.D. (Michigan) Address unknown
COHEN, Prof. TERESA, Ph.D. (Johns Hopkins) Pennsylvania State Coll., State College, Pa. *315 S. Atherton St.*
COKER, R. L., A.M. (Alabama) Instr., Math. and Physics, Southern Coll. of Optometry, Memphis, Tenn. *1877 Cowden Ave.*
COLE, Asst. Prof. NANCY, Ph.D. (Radcliffe) Connecticut Coll., New London, Conn.
COLE, Asst. Prof. R. H., Ph.D. (Wisconsin) Univ. of Western Ontario, London, Ont., Can.
COLEMAN, Asst. Prof. E. P., M.S. (Iowa) Univ. of Omaha, Omaha, Nebr. *Major, Coast Artillery Corps; U.S. Military Acad., West Point, N.Y. Dept. of Math.*
COLEMAN, J. B., Ph.D. (California) 340 West 55 St., New York 19, N.Y.
COLEMAN, RUTH T., A.M. (Columbia) Teacher, River Edge School, River Edge, N.J. *326 River Road, Bogota, N.J.*
COLEMAN, W. B., A.M. (Harvard) Univ. of Texas, Austin 12, Tex. *Room 9, Waggener Hall*
COLLIER, Prof. MYRTIE, Ph.D. (Strasbourg) Chm. of Dept., Immaculate Heart Coll., Los Angeles, Calif. *225 Thurston Ave., Los Angeles 24, Calif.*
COLSON, H. D., A.B. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn. *Ensign, U.S.N.R. Wadena, Minn.*
COMBELLACK, Asst. Prof. W. J., Ph.D. (Boston Univ.) Northeastern Univ., 360 Huntington Ave., Boston 15, Mass.
COMER, LOUISE M., M.S. (Brown) Instr., Barnard Coll., 607 West 119 St., New York 27, N.Y. *Milbank Hall*
COMFORT, E. G. H., Ph.D. (Brown) Instr., Univ. of Arkansas, Fayetteville, Ark. *Major, Air Corps. 812 Prairie Ave., Wilmette, Ill.*
COMSTOCK, Prof. C. E., A.M. (Knox) Emeritus, Bradley Poly. Inst., Peoria, Ill. *203 Fredonia Ave., Peoria 5, Ill.*
CONGDON, Prof. A. R., Ph.D. (Columbia) Retired, Univ. of Nebraska, Lincoln, Nebr. *359 N. 33rd St.*
CONGER, L. H., Jr., A.M. (Harvard) *Lt., Army Air Force. 560 W. Webster Ave., Muskegon, Mich.*
CONKWRIGHT, Asso. Prof. N. B., Ph.D. (Illinois) Univ. of Iowa, Iowa City, Iowa. *209-B Physics Bldg.*
CONLAN, Sister MARIA LOYOLA, A.M. (Fordham) Asso. Prof., Coll. of Mount St. Vincent, New York 63, N.Y.
CONSTABLE, MARY LOUISE, A.M. (Pennsylvania) Teacher, Philadelphia High School for Girls, Philadelphia, Pa. *The Whittier, 140 N. 15th St., Philadelphia 2, Pa.*
CONWELL, G. M., Ph.D. (Princeton) Master in Math., St. Paul's School, Concord, N.H. *307 Pleasant St.*
CONWELL, Prof. H. H., Ph.D. (Wisconsin) Dean, Beloit Coll., Beloit, Wis.
COOK, Asso. Prof. A. J., Ph.D. (Chicago) Univ. of Alberta, Edmonton, Alta., Can.
COOK, Sister ROSE MARGARET, M.S. (Notre Dame) Prof., Loretto Heights Coll., Loretto, Colo.
COOKE, Asst. Prof. J. V., Ph.D. (Peabody) North Texas State Teachers Coll., Denton, Tex. *Capt., Air Corps*

Cooley, Asso. Prof. H. R., Ph.D. (New York Univ.) New York Univ., 100 Washington Sq. E., New York 3, N.Y.
Cooley, Prof. J. A., Ph.D. (Illinois) Univ. of Tennessee, Knoxville, Tenn.
Coolidge, Prof. J. L., Ph.D. (Bonn) Emeritus, Harvard Univ., Cambridge 38, Mass. *27 Fayerweather St.*
Cooper, Elizabeth M., Ph.D. (Illinois) Chm. of Dept., Hunter Coll. High School, New York, N.Y. *201 East 71 St., New York 21, N.Y.*
Cope, Asso. Prof. T. F., Ph.D. (Chicago) Chm. of Dept., Queens Coll., Flushing, N.Y. *33-69 167th St.*
Copeland, Prof. A. H., Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *616 Oswego St.*
Copeland, Prof. Lennie P., Ph.D. (Pennsylvania) Wellesley Coll., Wellesley, Mass. *14 Waban St.*
Copp, P. T., A.M. (Ohio State) Industrial Engr., Carnegie Illinois Steel Works, Gary, Ind. *Box 64, Porter, Ind.*
Corbin, Prof. C. E., A.M. (Northwestern) Emeritus, Coll. of the Pacific, Stockton, Calif. *P.O. Box 115, Carmel, Calif.*
Cordrey, W. A., Ph.D. (Peabody) Address unknown
Corliss, J. J., Ph.D. (Michigan) Chm. of Dept., De Paul Univ., 64 E. Lake St., Chicago, Ill.
Corona, Sister Maria, Ph.D. (Fordham) Dean, Coll. of Mount St. Joseph, Mount St. Joseph, Ohio
Corral y Alemán, J. I., Ing. de Mines. Dir. de Montes y Mines, Republic of Cuba, Havana, Cuba. *Calle B #514, entre 21 y 23, Vedado*
Cosby, Byron, A.M. (Missouri) Owner, Teacher Placement Bureau, Columbia, Mo. *One Ridgeley Road*
Cothran, Prof. J. C., Ph.D. (Cornell) Head of Dept. of Chem., State Teachers Coll., Duluth, Minn. *301½ Irvine Place, Elmira, N.Y.*
Coulter, W. H., Licentiate of instruction (Peabody) Retired, Railway mail clerk. *134 N. Summit Ave., Decatur, Ill.*
Courant, Prof. Richard, Ph.D. (Göttingen) Chm. of Dept., New York Univ., New York, N.Y. *142 Calton Road, New Rochelle, N.Y.*
Court, Prof. N. A., D.Sc. (Ghent, Belgium) Univ. of Oklahoma, Norman, Okla.
Cowan, R. W., Ph.D. (California) Instr., Univ. of Alabama, University, Ala. *338 Thomas St., Tuscaloosa, Ala.*
Cowles, Prof. W. H. H., A.M. (Columbia) Head of Dept., Pratt Inst., Brooklyn, N.Y. *132 Joralemon St., Brooklyn 2, N.Y.*
Cox, Asst. Prof. H. M., A.M. (Duke) Dir., Bureau of Instructional Research, Univ. of Nebraska, Lincoln 8, Nebr. *B-3 Admin. Bldg.*
Cox, P. C., A.M. (New Mexico) *1st Lt., A.C., Separation Center, Fort McPherson, Ga. 1540 Athens Ave. S.W., Atlanta, Ga.*
Coxeter, Asso. Prof. H. S. M., Ph.D. (Cambridge) Univ. of Toronto, Toronto, Ont., Can. *24 Strathearn Blvd., Toronto 10, Ont., Can.*
Craft, Plummer, B.S. (Mississippi State) *In Service*
Craig, Prof. A. T., Ph.D. (Iowa) Univ. of Iowa, Iowa City, Iowa. *Dept. of Math.*
Craig, Prof. C. C., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *3020 Angell Hall*
Craig, Prof. H. V., Ph.D. (Wisconsin) Math. and Astr., Univ. of Texas, Austin, Tex.
Crain, K. W., M.S. (Iowa) Instr., Purdue Univ., West Lafayette, Ind. *238 Lincoln St.*
Cramer, Asst. Prof. G. F., Ph.D. (Missouri) Tulane Univ., New Orleans, La. *Dept. of Math.*
Cramer, Paul, A.M. (Illinois) Head of Dept., Physics, Huron Coll., Huron, S.D. *1178 Illinois Ave. S.W.*
Cramlet, Asso. Prof. C. M., Ph.D. (Washington) Univ. of Washington, Seattle, Wash. *Physics Hall*
Crane, Asso. Prof. Rufus, A.M. (Ohio State) Ohio Wesleyan Univ., Delaware, Ohio. *269 W. William St.*
Crane, R. E., B.S. (Harvard) Amer. Tel. & Tel. Co., 195 Broadway, New York, N.Y. *120 Early St., Morristown, N.J.*
Crathorne, Prof. A. R., Ph.D. (Göttingen) Emeritus, Univ. of Illinois, Urbana, Ill. *802 Pennsylvania Ave.*
Crawford, Asst. Prof. W. S. H., A.M. (Minnesota) Mount Allison Univ., Sackville, N.B., Can. *Box 174*
Crispin, J. W., Jr., B.S. (Michigan) Grad. Student, Univ. of Michigan, Ann Arbor, Mich. *363 Richton Ave., Detroit 3, Mich.*

CROMWELL, J. W., Jr., A.M. (Dartmouth) Certified Pub. Accountant, Washington, D.C. *1815-13th St., N.W.*
CROSS, J. H., A.M. (Texas) 2015-15th St., Lubbock, Tex.
CROW, E. L., Ph.D. (Wisconsin) Mathematician, Bur. of Ordnance, Navy Dept., Washington, D.C. *119 Fifth St. N.E., Washington 2, D.C.*
CRUDELI, Prof. UMBERTO, Dr. in mat. (Rome) Mathl. physics, Univ. of Naples, Naples, Italy
CRULL, Prof. H. E., Ph.D. (Illinois) Math. and Astr., Park Coll., Parkville, Mo. *Lt. (j.g.), U.S.N.R.*
CULMER, Prof. ORPHA ANN, A.M. (Michigan) Head of Dept., State Teachers Coll., Florence, Ala. *College Station*
CUNNINGHAM, A.B., Ph.D. (West Virginia) Instr., Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
CURFMAN, Prof. L. E., M.S. (Colorado) Kansas State Teachers Coll., Pittsburg, Kans. *406 W. Adams St.*
CURRIE, Asst. Prof. J. C., A.M. (Mississippi) Northeast Jr. Coll. of L.S.U., Monroe, La.
CURRIER, Asst. Prof. A. E., Ph.D. (Harvard) U.S. Naval Acad., Annapolis, Md. *Dept. of Math.*
CURRY, Prof. H. B., Ph.D. (Göttingen) Pennsylvania State Coll., State College, Pa. *Principal mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.*
CURTIS, Prof. H. B., Ph.D. (Cornell) Lake Forest Coll., Lake Forest, Ill. *11 College Campus*
CURTISS, Prof. D. R., Ph.D. (Harvard) Emeritus, Northwestern Univ., Evanston, Ill. *632 Buena Vista St., Redlands, Calif.*
CURTISS, Asst. Prof. J. H., Ph.D. (Harvard) Cornell Univ., Ithaca, N.Y. *Lt., U.S.N.R. Code 345 Bureau of Ships, Washington, D.C.*
CUTLER, Asst. Prof. E. H., Ph.D. (Harvard) Lehigh Univ., Bethlehem, Pa.
CUTTING, L. H., A.M. (Missouri) Teacher, High School, Kansas City, Mo. *406 E. 43rd St., Kansas City 4, Mo.*

DADOURIAN, Prof. H. M., Ph.D. (Yale) Math. and Nat. Philos., Trinity Coll., Hartford, Conn. *125 Vernon St.*
DALAL, R. D. Stock Broker, retired, London, England
DALY, J. F., Ph.D. (Princeton) *Lt. Comdr., U.S.N.R. Bureau of Ships, Navy Dept., Washington, D.C. 4213-18th St. N.E., Washington 18, D.C.*
DAMSGARD, Asst. Prof. L. C., A.B. (Union Coll., Nebr.) Math. and Astr., Pasadena Jr. Coll., Pasadena, Calif. *2000 Loma Vista St., Pasadena 7, Calif.*
DANCER, Prof. WAYNE, Ph.D. (Michigan) Univ. of Toledo, Toledo 6, Ohio. *2233 Dundas Road*
DANCEY, Prof. L. S., A.M. (Illinois) Carroll Coll., Waukesha, Wis. *125 N. Charles St.*
DANIELLS, Asst. Prof. MARIAN E., A.M. (Iowa State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
DANSKIN, J. M., Jr., A.B. (U.C.L.A.) *RT 1/c, U.S.N.R.*
DANZL, Rev. ARTHUR, A.M. (Columbia) Prof., St. John's Univ., Collegeville, Minn.
DAOUST, J. H., A.M. (Minnesota) *Capt., A.C., Mary Esther, Fla.*
DAPPERT, J. W., C.E. (Valparaiso Univ.) Civil Engr., Taylorville, Ill. *505 Gandy Ave.*
D'ARCO, PAUL, B.S. (Chicago) Instr., De Paul Univ., Chicago, Ill. *1116 W. Polk St.*
DARKOW, Asso. Prof. MARGUERITE D., Ph.D. (Chicago) Hunter Coll., 695 Park Ave., New York, N.Y. *16 East 82 St., New York 28, N.Y.*
DARLING, F. W., A.B. (Cornell) Mathematician, U.S. Coast and Geodetic Survey, Washington, D.C. *45 Kenilworth Ave., Garrett Park, Md.*
DARRAUGH, J. E., A.M. (Brooklyn) Clerk, Consolidated Edison Co., 4 Irving Pl., New York, N.Y. *2nd Lt., Army Air Forces. 2917 Glenwood Road, Brooklyn, N.Y.*
D'ATRI, A. J., C.E. (Brooklyn Poly. Inst.) Civil Engr., Dept. of Public Works, New York, N.Y. *1821 Madison Pl., Brooklyn 29, N.Y.*
DAUGHERTY, J. D., A.M. (Pennsylvania) Head of Dept., Eastside High School, Paterson, N.J.
DAUM, J. A., Ph.D. (Nebraska) *Capt., A.U.S., Signal Corps. 337 Sylvania Ave., Avon-by-the-Sea, N.J.*
DAUS, Prof. P. H., Ph.D. (California) Univ. of California at Los Angeles, Los Angeles 24, Calif.
DAVIS, Asst. Prof. A. W., Ph.D. (Iowa State) Iowa State Coll., Ames, Iowa. *Theoret. and Appl. Mech. Dept.*
DAVIS, Prof. D. R., Ph.D. (Chicago) State Teachers Coll., Montclair, N.J. *Army Univ. Center, England and France*
DAVIS, Asso. Prof. H. A., Ph.D. (Cornell) West Virginia Univ., Morgantown, W.Va. *307 Duquesne Ave.*
DAVIS, Dean J. B., M.S. (Northwestern) Itasca Jr. Coll., Coleraine, Minn.

DAVIS, Asst. Prof. JAMES EDGAR, A.M. (Ohio State) Univ. of Illinois, 808 S. Wood St., Chicago 12, Ill.
DAVIS, Prof. JAMES ELMER, A.M. (Wisconsin) Drexel Inst. of Tech., 32nd and Chestnut St., Philadelphia 4, Pa.
DAVIS, MARGARET R., Ed.D. (Columbia) 31 S. Main St., Ashley, Pa.
DAVIS, Asst. Prof. U. P., A.M. (Florida) Univ. of Florida, Gainesville, Fla. *1635 W. Mechanic St.*
DAVIS, Prof. W. M., Ph.D. (Chicago) Cornell Coll., Mount Vernon, Iowa. *Army Univ. Center in England. 616 Seventh Ave. N., Mount Vernon, Iowa*
DAVISON, Asso. Prof. RACHEL, A.M. (Oberlin) Houghton Coll., Houghton, N.Y.
DEAN, ALICE C., A.M. (Rice) Fellow in Math., Acting Librarian, Rice Inst., Houston 1, Tex. *P.O. Box 1892*
DEAN, Mrs. MILDRED W., Ph.D. (Johns Hopkins) Instr., Queens Coll., Flushing, N.Y. *254-09 West End Drive, Great Neck, N.Y.*
DEARBORN, Prof. D. C., Ph.D. (Duke) Registrar, Catawba Coll., Salisbury, N.C.
DECHERD, Asst. Prof. MARY E., A.M. (Texas) Retired, Univ. of Texas, Austin, Tex. *2313 Nueces St., Austin 21, Tex.*
DE CICCO, Asst. Prof. JOHN, Ph.D. (Columbia) Illinois Inst. of Tech., 3300 Federal St., Chicago, Ill. *6340 Blackstone Ave.*
DECK, Prof. L. J., A.M. (Pennsylvania) Muhlenberg Coll., Allentown, Pa.
DECKER, Prof. F. F., Ph.D. (Syracuse) Syracuse Univ., Syracuse, N.Y. *312 Marshall St.*
DECLEENE, Rev. L. A. V., Ph.D. (Catholic Univ.) Prior, Philadelphia Priory. *920 Christian St., Philadelphia 47, Pa.*
DECOU, Prof. E. E., M.S. (Chicago) Emeritus, Univ. of Oregon, Eugene, Ore. *929 Hilyard St.*
DEDERICK, L. S., Ph.D. (Harvard) Asso. Director, Ballistic Research Lab., Aberdeen Proving Ground, Md.
DE LA GARZA, ELEUTERIO. Box 304, Brownsville, Tex.
DE LA SALLE (SEILER), Brother LOUIS, Ph.D. (Catholic Univ.) Prof., Dean of Studies, St. Mary's Coll., Winona, Minn.
DEMING, Prof. R. M., M.S. (Iowa State) Registrar, Upper Iowa Univ., Fayette, Iowa
DEMOSS, M. L., M.S. (Kansas St. T.C., Pittsburg) Instr., General Motors Inst., Flint, Mich. *Lt. (j.g.), U.S.N.R. 15326 S. Atkinson St., Gardena, Calif.*
DENBOW, Asst. Prof. CARL, Ph.D. (Chicago) Ohio Univ., Athens, Ohio. *Lt. (j.g.), U.S.N.R.*
DENNIS, Asso. Prof. F. L., Ph.D. (Illinois) Ursinus Coll., Collegeville, Pa. *930 Main St.*
DENNISON, C. H., B.S. (Mass. Inst. of Tech.) Chemist, Archer Rubber Co., Milford, Mass. *12 Crescent St., Natick, Mass.*
DENTON, W. W., Ph.D. (Illinois) 4211 Third Ave., Detroit 1, Mich.
DE REGT, M. P., B.S. (Webb Inst.) Supervisor, Hull Tech. Section, Todd-Pacific Shipyards, Tacoma, Wash. *Route 6, Box 241-B*
DIAMOND, A. H., Ph.D. (California) *Lt., U.S. Army, S.A.A.C.C., San Antonio, Tex.*
DICKSON, Prof. L. E., Ph.D. (Chicago) Emeritus, Univ. of Chicago, Chicago, Ill.
DILLINGHAM, Prof. ALEXANDER, A.M. (Harvard) U.S. Naval Acad., Annapolis, Md.
DILWORTH, Asso. Prof. R. P., Ph.D. (Calif. Inst. of Tech.) California Inst. of Tech., Pasadena, Calif. *204 Astrophysics Bldg.*
DIMICK, Prof. C. E., A.M. (Pennsylvania) Retired, U.S. Coast Guard Acad., New London, Conn. *Gales Ferry, Conn.*
DIMSDALE, BERNARD, Ph.D. (Minnesota) *In Service*
DINES, Prof. L. L., Ph.D. (Chicago) Emeritus, Carnegie Inst. of Tech., Pittsburgh, Pa. *322 N. 24th St., Quincy, Ill.*
DINKINES, Adj. Prof. FLORA, A.M. (Michigan) Univ. of South Carolina, Columbia, S.C. *On leave. Grad. Student, Univ. of Chicago. Room 201 Blackstone Hall, 5748 S. Blackstone Ave., Chicago, Ill.*
DIX, Prof. L. E., M.S. (Norwich Univ.) Norwich Univ., Northfield, Vt. *5 Spring St.*
DOBBIN, Sister MARIOLA, A.M. (Wisconsin) Rosary Coll., River Forest, Ill.
DOBSON, W. P., M.A. (Toronto) Dir. of Research, Hydro Electric Power Commission of Ontario, 620 University Ave., Toronto 2, Ont., Can.
DODSON, N. E., A.B. (Berea Coll.) Grad. Asst., Univ. of Alabama, University, Ala. *Box 2194*
DOERFLER, HILARY, A.M. (St. John's Univ., Minn.) Head of Dept., St. Gregory's Coll., Shawnee, Okla.
DOERMANN, F. W., Ph.D. (Vienna) Retired.
DOLL, W. R., A.B. (Columbia Coll.) 709 Kennedy Bldg., Tulsa 3, Okla.
DOMINIC, Brother, A.B. (St. Mary's Coll., Calif.) Instr., St. Mary's Coll., St. Mary's College, Calif.

DONALDSON, J. D., A.M. (Pittsburgh) Consolidated Vultee Aircraft Co., San Diego, Calif. *2027 Drescher St., San Diego 10, Calif.*
DONCHIAN, P. S., A.B. (Yale) Pres. and Treas., Samuel Donchian Rug Co., Hartford, Conn. *85 Gillett St.*
DONER, Prof. R. D., Ph.D. (Illinois) Head of Dept., Alabama Poly. Inst., Auburn, Ala. *477 E. Samford Ave.*
DORROH, Asst. Prof. J. L., Ph.D. (Texas) Louisiana State Univ., University Sta., Baton Rouge, La. *Dept. of Math.*
DORWART, Asso. Prof. H. L., Ph.D. (Yale) Washington and Jefferson Coll., Washington, Pa. *R.D. 6*
DOSTAL, Asst. Prof. B. F., A.M. (Indiana) Univ. of Florida, Gainesville, Fla. *106 Peabody Bldg.*
DOTTERER, Prof. J. E., A.M. (Illinois) *Teaching in war training program, Indiana Central Coll., Indianapolis 3, Ind.*
DOUGHERTY, LUCY T., A.M. (Kansas) Sec'y, Fairfax Hills, Inc., Kansas City, Kans. *Gould Hotel, Kansas City 14, Kans.*
DOUGLAS, Asst. Prof. JESSE, Ph.D. (Columbia) Brooklyn Coll., Brooklyn 10, N.Y. *2610 Glenwood Road*
DOUGLASS, Prof. R. D., Ph.D. (Mass. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge, Mass. *18 Oak Ave., Belmont, Mass.*
DOWNING, F. A. Asst. Director of Traffic, North Carolina Utilities Com., Raleigh, N.C. *212 Taylor St.*
DOWNING, Prof. H. H., Ph.D. (Chicago) Math. and Astr., Univ. of Kentucky, Lexington, Ky. *138 State St., Lexington 36, Ky.*
DOWNING, R. H., Ph.D. (West Virginia) Analyst, Engineering Dept., Fleetwings, Inc., Bristol, Pa.
DOYLE, Asst. Prof. T. C., Ph.D. (Princeton) Dartmouth Coll., Hanover, N.H. *Graduate Club*
DOYLE, Asso. Prof. W. C., Ph.D. (St. Louis Univ.) Rockhurst Coll., Kansas City 4, Mo.
DOYLE, Rev. W. G., M.S (Catholic Univ.) *In Service*
DRAPER, Prof. OLIVE M., A.M. (Michigan) Taylor Univ., Upland, Ind.
DRESDEN, Prof. ARNOLD, Ph.D. (Chicago) Swarthmore Coll., Swarthmore, Pa. *606 Elm Ave.*
DRESSEL, Asst. Prof. F. G., Ph.D. (Duke) Duke Univ., Durham, N.C. *309 Francis St.*
DRESSLER, B. B., A.M. (Illinois) Instr., Alabama Poly. Inst., Auburn, Ala. *327 E. Samford Ave.*
DREW, Asst. Prof. J. W., A.M. (Cornell) Virginia Union Univ., Richmond, Va.
DRIBIN, D. M., Ph.D. (Chicago) *M/Sgt., Signal Corps. 4320 Fourth St. N., Arlington, Va.*
DRIVER, D. D., A.M. (Nebraska) Registrar, Hesston Coll., Hesston, Kans.
DUBÉ, Prof. L. H., Ph.D. (Gregorian Univ., Rome) Ottawa Univ., Ottawa, Ont., Can.
DUERKSEN, J. A., A.B. (Bethel) Mathematician, U.S. Coast and Geodetic Survey, Washington, D.C. *3134 Monroe St. N.E., Washington 18, D.C.*
DUFFIE, J. A. H., B.S. (Univ. of New Brunswick) Grad. student, Chemistry-Physics, Catholic Univ. of America, Washington 17, D.C. *Box 794 Catholic Univ.*
DUFFNER, R. T., G.E. (Colo. School of Mines) Grad. student, Colorado School of Mines, Golden, Colo. *Lt. (j.g.), U.S.N.R. c/o VRF4, NAS, Brooklyn, N.Y.*
DUNCAN, D. C., Ph.D. (California) Instr., Los Angeles City Coll., 851 N. Vermont Ave., Los Angeles, Calif.
DUNCAN, R. L., A.B. (Tulsa) Landman, Carter Oil Co., Tulsa, Okla. *Box 801*
DUNFORD, Prof. NELSON, Ph.D. (Brown) Yale Univ., New Haven, Conn. *374 Fountain St.*
DUNKEL, Prof. OTTO, Ph.D. (Harvard) Emeritus, Washington Univ., St. Louis 5, Mo.
DUNLAP, Asso. Prof. L. T., A.M. (Pennsylvania State) Pennsylvania State Coll., State College, Pa.
DUNLAP, P. R., A.B. (Taylor Univ.) Kalkaska, Mich.
DUPASQUIER, Prof. L. G., Ph.D. (Zurich) Univ. of Neuchâtel, Neuchâtel, Switzerland
DURAND, JANET C., A.M. (Pennsylvania) Instr., Vassar Coll., Poughkeepsie, N.Y. *1 LaGrange Ave.*
DURELL, FLETCHER, Ph.D. (Princeton) Emeritus, Head of Dept., Lawrenceville School, Lawrenceville, N.J. *Belleplain, Woodbine R.F.D., N.J.*
DUREN, Asso. Prof. W. L., Jr., Ph.D. (Chicago) Tulane Univ., New Orleans 15, La. *3323 Joseph St.*
DURFEE, Dean WALTER H., Ph.D. (Cornell) Hobart Coll., Geneva, N.Y. *Coxe Hall*
DURFEE, WILLIAM H., Ph.D. (Cornell) *N.D.R.C. Northwestern Univ., Evanston, Ill. Room 293 Northwestern Tech. Inst.*
DURHAM, R. L., B.S. (Trinity Coll.) President, Southern Sem. and Jr. Coll., Buena Vista, Va.

DWYER, Asso. Prof. P. S., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *640 Oxford Road*
DYE, Asso. Prof. L. A., Ph.D. (Cornell) The Citadel, Charleston, S.C.

EAGLE, E. L., A.B. (Miami) Training Unit, Hq. Air Tech. Service Comm., Wright Field. *R.R. 1, Martinsville, Ohio*
EARHART, F. C., A.M. (Wisconsin) High School, Hopkinton, Iowa
EARL, Prof. J. M., Ph.D. (Minnesota) Municipal Univ. of Omaha, Omaha, Nebr. *528 S. 53rd St.*
EASTHAM, J. N., Ph.D. (Catholic Univ.) Head of Dept., Nazareth Coll., Rochester, N.Y. *74 Eastland Ave., Rochester 7, N.Y.*
EATON, J. E., Ph.D. (Yale) Instr., Queens Coll., Flushing, N.Y. *Staff Member, Radiation Lab., Massachusetts Inst. of Tech. 26 Concord Ave., Apt. 508, Cambridge 38, Mass.*
EAVES, Asso. Prof. E. D., Ph.D. (Texas) Univ. of Tennessee, Knoxville 16, Tenn. *Univ. Sta.*
EBERHART, Prof. PAUL, Ph.D. (Brown) Head of Dept., Math. and Engg., Washburn Univ., Topeka, Kans. *2068 Lane St.*
EDINGTON, Prof. W. E., Ph.D. (Illinois) Head of Dept., Math. and Astr., DePauw Univ., Greencastle, Ind.
EDISON, T. M., B.S. (Mass. Inst. of Tech.) Pres., Calibron Products, Inc., 51 Lakeside Ave., West Orange, N.J.
EDMONSON, Prof. NAT, Jr., Ph.D. (Rice) A. and M. Coll. of Texas, College Station, Tex. *Box 121 Faculty Exchange*
EDWARD JOSEPH, Sister, A.M., M.S. (Notre Dame) Instr., St. Mary's Coll., Notre Dame, Holy Cross, Ind.
EDWARDS, Prof. P. D., Ph.D. (Indiana) Ball State Teachers Coll., Muncie, Ind. *Army Univ. Center No. 1. 25 Warwick Road, Muncie, Ind.*
EGGERS, Asst. Prof. H. C. T., Ph.D. (Michigan) Univ. of Minnesota, Minneapolis, Minn. *2100-31st Ave. S.*
EGGERT, O. E. Postal clerk, U.S. Railway Mail Service, Philadelphia, Pa. *1916 Forest Ave., Morton, Pa.*
EIDE, Asso. Prof. MARGARET C. (Mrs. R. B.), A.M. (Wisconsin) State Teachers Coll., River Falls, Wis. *308 S. Second St.*
EIESLAND, Prof. J. A., Ph.D. (Johns Hopkins) Emeritus, West Virginia Univ., Morgantown, W.Va.
EILENBERG, Asso. Prof. SAMUEL, Ph.D. (Warsaw) Univ. of Michigan, Ann Arbor, Mich.
EISELE, CAROLYN, A.M. (Columbia) Instr., Hunter Coll., New York, N.Y. *257 West 86 St., New York 24, N.Y.*
EISENHART, Prof. L. P., Ph.D. (Johns Hopkins) Dean of Grad. School, Princeton Univ., Princeton, N.J. *73 Nassau St.*
EKMAN, Prof. W. E., A.M. (South Dakota) Math. and Astr., Univ. of South Dakota, Vermillion, S.D. *104 S. Yale St.*
ELDER, Asst. Prof. J. D., A.M. (Princeton) Physics, Wabash Coll., Crawfordsville, Ind.
ELLIOTT, Prof. W. W., Ph.D. (Cornell) Duke Univ., Durham, N.C. *Box 533*
ELLIS, J. R., M.S. (New Mexico) Teacher, Booker T. Washington High School, Tulsa, Okla. *1110 E. Pine St., Tulsa 6, Okla.*
ELSTON, J. S., A.B. (Cornell) Asst. Actuary, Travelers Ins. Co., Hartford, Conn.
EMMONS, Prof. C. W., A.M. (Illinois) Simpson Coll., Indianola, Iowa. *1009 North B St.*
EMMONS, Asso. Prof. H. W., D.S. (Harvard) Mech. Engg., Grad. School of Engineering, Harvard Univ., Cambridge, Mass. *307 Pierce Hall*
ENGSTROM, Asso. Prof. H. T., Ph.D. (Yale) Yale Univ., New Haven, Conn. *Comdr., U.S. N.R., Navy Dept., Washington, D.C. 357 N. Glebe Road, Arlington, Va.*
ENRIQUES, Prof. FEDERIGO. Univ. of Rome, Rome, Italy. *Via Sardegna 50*
EPSTEIN, BENJAMIN, Ph.D. (Illinois) Mathematician, Quality Control Dept., Westinghouse Elec. and Mfg. Co., East Pittsburgh, Pa. *633 Rivermont Drive, Pittsburgh 7, Pa.*
ERICKSON, R. W., Ph.D. (Minnesota) Instr., Hibbing Jr. Coll., Hibbing, Minn. *Capt., U.S. Army, A.G.D. O'Reilly General Hospital, Springfield, Mo.*
ERICKSON, W. S., Ph.D. (Wisconsin) Engr., Bell Aircraft Corp., 2050 Elmwood Ave., Buffalo 7, N.Y. *Engg. Div.*
ERIKSON, Asso. Prof. C. M., Ph.D. (Michigan) Michigan State Norman Coll., Ypsilanti, Mich. *101 Wallace Blvd.*
ERKILETIAN, D. H., Jr., A.M. (Illinois) Instr., Univ. of Missouri, School of Mines & Met., Rolla, Mo. *P.O. Box 434*
ERNSBERGER, IVA B., A.M. (Nebraska) Instr., Fullerton Jr. Coll., Fullerton, Calif.
ERNSDORFF, Rev. L. E., M.S. (Univ. of Notre Dame) Instr., Loras Coll., Dubuque, Iowa
EROKAN, ISLAM, M.C.E. (Cornell) *In Service*

ERRERA, Prof. ALFRED. L'Univ. libre de Bruxelles, Belgium. *1039 Chaussee de Waterloo, Uccle 3, Belgium*
ERWIN, GRACE, A.B. (Nebraska) Gilead, Nebr.
ERWIN, M. C. Teacher, Industrial Arts Dept., High School, Reynolds, Ind.
ESPOSITO, J. P., M.S. (De Paul) Teacher, Crane Tech. High School, Chicago, Ill. *1640 N. Parkside Ave., Chicago 39, Ill.*
ESTY, Prof. T. C., A.M. (Amherst) Emeritus, Amherst Coll., Amherst, Mass. *Spofford, N.H.*
ETTINGER, W. J., B.S. in M.E. (Lewis Inst.) Designing Engr., Edison Genl. Elec. Appliance Co., 5600 W. Taylor St., Chicago, Ill.
ETTLINGER, Prof. H. J., Ph.D. (Harvard) Univ. of Texas, Austin, Tex. *3110 Harris Park Ave.*
EVANS, Prof. G. C., Ph.D. (Harvard) Univ. of California, Berkeley 4, Calif.
EVANS, G. W., A.B. (Harvard) Retired. 16 Beverly Road, Swampscott, Mass.
EVANS, Prof. H. B., Ph.D. (Pennsylvania) Emeritus, Univ. of Pennsylvania, Philadelphia, Pa. *88 Merbrook Lane, Merion Station, Pa.*
EVANS, Prof. H. P., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison 6, Wis. *North Hall*
EVENSEN, E. J. Actuarial Section, Metropolitan Life Ins. Co., Pacific Coast Head Office. *T/4, U.S. Army.*
EVERETT, Prof. H. S., Ph.D. (Chicago) Extension Prof., Univ. of Chicago, Chicago 37, Ill.
EVERETT, J. P., Ph.D. (Columbia) Chm. of Dept., Western Michigan Coll., Kalamazoo, Mich. *907 W. South St.*
EVERETT, Asso. Prof. J. R., A.M. (Wisconsin) Colorado School of Mines, Golden, Colo. *1700 Washington St.*
EVES, Asst. Prof. H. W., A.M. (Harvard) Coll. of Puget Sound, Tacoma 6, Wash.
EWING, Asso. Prof. G. M., Ph.D. (Missouri) Univ. of Missouri, Columbia, Mo. *Engineering Bldg.*

FAGERSTROM, Asst. Prof. W. H., Ph.D. (Columbia) Coll. of the City of New York, Convent Ave. and 139 St., New York 31, N.Y. *706 Riverside Drive, New York 31, N.Y.*
FAIR, Asst. Prof. L. A., A.M. (Peabody) Morehead State Teachers Coll., Morehead, Ky. *468 Second St.*
FALVEY, FRANCES E., A.M. (Southern Methodist) Instr., Educ., Hollins Coll., Hollins, Va.
FARNELL, A.B., Ph.D. (California) *Instr., U.S. Military Acad., West Point, N.Y. Dept. of Math. Capt.*
FATTU, NICHOLAS, Ph.D. (Minnesota) Instr., Research Statistician, Univ. of Minnesota, Minneapolis, Minn. *Educ. Planning Office, Bldg. 300 U.S. Naval Trng. Center, Great Lakes, Ill. 204 W. Cedar St., Chippewa Falls, Wis.*
FAULKNER, F. D., M.S. (Kansas State Coll.) Jr. Physicist, Johns Hopkins Univ., Silver Spring, Md. *Applied Physics Lab.*
FAWCETT, Prof. H. P., Ph.D. (Columbia) Ohio State Univ., Columbus, Ohio. *54 W. Beechwold Blvd., Columbus 2, Ohio*
FAY, E. A., A.M. (Harvard) *In Service*
FEDERICO, P. J., A.M. (George Washington) Principal Examiner, U.S. Patent Office, Washington, D.C. *3634 Jocelyn St. N.W.*
FEEMSTER, Prof. H. C., A.M. (Nebraska) Retired, Math. and Astr., York Coll., York, Nebr.
FEHR, Asso. Prof. H. F., Ph.D. (Columbia) State Teachers Coll., Upper Montclair, N.J. *544 Highland Ave.*
FELD, J. M., Ph.D. (Columbia) Instr., Queens Coll., Flushing, N.Y.
FELDER, VIRGINIA I., M.S. (Tulane) Head of Dept., Sunflower Junior Coll., Moorhead, Miss. *Box 95*
FELLER, Prof. WILLIAM, Ph.D. (Göttingen) Cornell Univ., Ithaca, N.Y. *512 Highland Road*
FELTGES, EDNA M., A.M. (Wisconsin) Chm. of Dept., Wilson Jr. Coll., Chicago, Ill. *2552 E. 76th St., Chicago 49, Ill.*
FERGUSON, HARRY, B.S. (Boston Univ.) Instr., Tufts Coll., Medford, Mass. *83 Brattle St., Apt. 25, Cambridge 38, Mass.*
FERGUSON, W. E., A.M. (Missouri) Instr., Univ. of Missouri, Columbia, Mo. *Ensign, U.S. N.R. Butterfield, Mo.*
FERNS, Prof. H. H., Ph.D. (Toronto) Univ. of Saskatchewan, Saskatoon, Sask., Can. *Dept. of Math.*
FERRY, F. C., Ph.D. (Clark) Pres., Emeritus, Hamilton Coll., Clinton, N.Y. *324 Hart St., New Britain, Conn.*
FETTERER, Rev. R. A., M.S. (Catholic Univ.) Instr., St. Francis Seminary, Milwaukee, Wis. *3257 S. Lake Drive, Milwaukee 7, Wis.*
FICKEN, Prof. F. A., Ph.D. (Princeton) Univ. of Tennessee, Knoxville 16, Tenn.
FIELD, Prof. FLOYD, A.M. (Harvard) Dean of Men, Georgia School of Tech., Atlanta, Ga. *2865 Tupelo St. S.E.*

FIELD, Prof. PETER, Ph.D. (Cornell) Univ. of Michigan, Ann Arbor, Mich. *904 Olivia Ave.*
FIELD, S. E., A.M. (Michigan) Head of Dept., Wright High School, Ironwood, Mich. *635 E. Cloverland Drive*
FIELDS, Asst. Prof. W. L., A.M. (Indiana) Louisville Municipal Coll., Louisville, Ky. *2239 W. Chestnut St.*
FINAN, Asso. Prof. E. J., Ph.D. (Ohio State) Catholic Univ. of America, Washington, D.C. *604 Girard St. N.E., Washington 17, D.C.*
FINCH, J. V., A.M. (Wisconsin) *Capt., U.S. Army Air Corps, 6104 S. Woodlawn Ave., Chicago 37, Ill.*
FINDLAY, Prof. WILLIAM, Ph.D. (Chicago) McMaster Univ., Hamilton, Ont., Can. *32 South Oval*
FINE, N. J., A.M. (Pennsylvania) 1312 Pine St., Philadelphia, Pa.
FINKEL, Prof. B. F., Ph.D. (Pennsylvania) Emeritus, Drury Coll., Springfield, Mo. *1227 Clay St.*
FINKEL, DANIEL, B.S. (C.C.N.Y.) Instr., Amherst Coll., Amherst, Mass. *27 Woodside Ave.*
FIRESTONE, C. D., B.S. (New Mexico) Part-time Instr., Cornell Univ., Ithaca, N.Y. *R.T. 3/C, U.S. Navy. White Hall, Cornell Univ., Ithaca, N.Y.*
FIRST, DOUGLAS. Shipwright, New York Navy Yard, Brooklyn, N.Y. *415 Avenue C.*
FISCHER, Asso. Prof. C. H., Ph.D. (Iowa) Univ. of Michigan, Ann Arbor, Mich. *3016 Angell Hall*
FISCHER, Asst. Prof. I. C., M.S. (Marquette) Engg., Exten. Div., Univ. of Minnesota, Minneapolis, Minn. *4252 Columbus Ave., Minneapolis 7, Minn.*
FISHER, ANSELM, A.M. (New York Univ.) Instr., Macalester Coll., St. Paul, Minn. *41 Clarence Ave. S.E., Minneapolis 14, Minn.*
FITE, Prof. W. B., Ph.D. (Cornell) Emeritus, Columbia Univ., New York, N.Y. *411 West 114 St., New York 25, N.Y.*
FITTERER, Prof. J. C., C.E. (Colorado) Head of Dept., Colorado School of Mines, Golden, Colo. *1620 Maple St.*
FITZGERALD, E. L., B.S. (California) Student of Theology, Alma Coll., Alma, Calif.
FITZPATRICK, J. D., A.M. (Creighton) Supervisor of Cost Control, Allis-Chalmers Mfg. Co., West Allis, Wis. *Lt., U.S.N.R., 2102 W. Wisconsin Ave., Apt. 6, Milwaukee 3, Wis.*
FLAGG, Asst. Prof. ELINOR B., M.S. (Illinois) Illinois State Normal Univ., Normal, Ill. *29 Payne Place*
FLAHERTY, Asst. Prof. W. C., A.B. (Georgetown) Georgetown Univ., Washington, D.C.
FLANDERS, Prof. R. L., M.C.E. (Cornell) Civ. Eng., Oklahoma A. and M. Coll., Stillwater, Okla.
FLEISHER, Asso. Prof. EDWARD, Ph.D. (New York Univ.) Brooklyn Coll., Bedford Ave. and Ave. H, Brooklyn, N.Y. *20 Plaza St., Brooklyn 17, N.Y.*
FLEMING, J. A., D.Sc. (Cincinnati) Dir., Dept. of Terrestrial Magnetism, Carnegie Institution of Washington, 5241 Broad Branch Road N.W., Washington 15, D.C.
FLEXNER, Asso. Prof. W. W., Ph.D. (Princeton) Cornell Univ., Ithaca, N.Y. *White Hall*
FLOGSTAD, IDA, M.S. (Iowa State) State Teachers Coll., Superior, Wis. *1009 N. 18th St.*
FOARD, C. W., Ph.D. (Iowa) Physicist, Eastman Kodak Co., Rochester 4, N.Y. *686 Marshall Road, Rochester 11, N.Y.*
FOBES, Asst. Prof. M. P., A.M. (Harvard) Coll. of Wooster, Wooster, Ohio
FOCKE, Prof. T. M., Ph.D. (Göttingen) Dean Emeritus, Case School of Appl. Sci., Cleveland, Ohio. *2517 Wellington Road, Cleveland 18, Ohio*
FOLLEY, Asso. Prof. K. W., Ph.D. (Toronto) Wayne Univ., Detroit, Mich. *19230 Gainsborough St., Detroit 23, Mich.*
FORAKER, Prof. F. A., M.S. (Ohio Northern) Univ. of Pittsburgh, Pittsburgh, Pa. *1313 Macon Ave.*
FORD, CLARENCE, A.M. (Kentucky) Teacher, Male High School, Louisville, Ky. *1410 St. Anthony Place, Louisville 4, Ky.*
FORD, Prof. L. R., Ph.D. (Harvard) Chm. of Dept., Illinois Inst. of Tech., Technology Center, Chicago 16, Ill.
FORD, Prof. W. B., Ph.D. (Harvard) Emeritus, Univ. of Michigan, Ann Arbor, Mich. *Hayt Corners, Seneca Co., N.Y.*
FOREMAN, W. C., A.M. (Kansas) *Lt., U.S.N.R. Route 4, Hannibal, Mo.*
FORMAN, WILLIAM, A.M. (Brooklyn Coll.) Instr., Brooklyn Coll. Evening Session, Brooklyn, N.Y. *P.F.C., Signal Corps. 494 Powell St., Brooklyn 12, N.Y.*
FORRAY, M. J., M.S. (New York Univ.) Instr., Poly. Inst. of Brooklyn, Brooklyn, N.Y. *1329 E. Seventh St., Brooklyn 30, N.Y.*
FORSYTH, Prof. C. H., Ph.D. (Michigan) Dartmouth Coll., Hanover, N.H.
FORT, Prof. TOMLINSON, Ph.D. (Harvard) Head of Dept., Univ. of Georgia, Athens, Ga.

FOSTER, Prof. R. M., B.S. (Harvard) Poly. Inst. of Brooklyn, Brooklyn, N.Y. *136 Stanmore Place, Westfield, N.J.*
FOUST, Prof. J. W., Ph.D. (Michigan) Central Michigan Coll. of Educ., Mt. Pleasant, Mich.,
FOX, Asso. Prof. A. H., Ph.D. (Yale) Union Coll., Schenectady, N.Y. *1101 Millington Road*
FOX, Sister CATHERINE DOROTHEA, Ph.D. (Boston Coll.) Pres., Trinity Coll., Washington 17, D.C.
FRAME, Prof. J. S., Ph.D. (Harvard) Head of Dept., Michigan State Coll., East Lansing, Mich. *821 E. Grand River Ave.*
FRANCIS, S. A., A.M. (California) Chm. of Dept., Math. and Engg., San Mateo Jr. Coll., San Mateo, Calif. *181 Iris Way, Palo Alto, Calif.*
FRANK, F. T., A.B. (Stanford) Asst. to Mgr., Ins. Dept., Parrott and Co., San Francisco, Calif. *708 Ashbury St., San Francisco 17, Calif.*
FRANKEL, E. T., B.S. (C.C.N.Y.) Budget Analyst, Federation of Social Agencies, 519 Smithfield St., Pittsburgh 22, Pa.
FRANKENBUSH, BERTHA E., A.M. (Tulane) Retired, High School Teacher, New Orleans, La. *1217 Jefferson Ave.*
FRANKLIN, Prof. PHILIP, Ph.D. (Princeton) Massachusetts Inst. of Tech., Cambridge, Mass. *312 Pleasant St., Belmont, Mass.*
FRASER, W. A., Ph.D. (Iowa State) Research Engr., The Perkin-Elmer Corp., Glenbrook, Conn.
FREAS, ELIZABETH, A.M. (Louisiana State) Asst., Louisiana State Univ., University Sta., Baton Rouge, La. *Dept. of Math.*
FREDERICKSON, ELVY L., A.B. (Willamette) Computer, Boeing Aircraft Co., Seattle, Wash. *903 Bellevue Place, Seattle 2, Wash.*
FREESE, Asst. Prof. FRANCES, A.M. (Radcliffe) Dean of Women, Mount Union Coll., Alliance, Ohio
FRICK, Prof. C. H., Ph.D. (North Carolina) Mary Washington Coll., Fredericksburg, Va. *Lt. (S), U.S.N.R. 1 Reid Court*
FRIED, HANS, Ph.D. (Vienna) Lecturer, Swarthmore Coll., Swarthmore, Pa. *305 E. Essex Ave., Lansdowne, Pa.*
FRINK, Prof. ORRIN, Jr., Ph.D. (Columbia) Pennsylvania State Coll., State College, Pa. *706 Sunset Road*
FRY, THORNTON C., Ph.D. (Wisconsin) Dir., Switching Research, Bell Telephone Labs., 463 West St., New York 14, N.Y.
FRY, W. J., M.S. (Pennsylvania State) Asst. Physicist, Naval Research Lab., Washington, D.C.
FUDGE, HELEN G., Ph.D. (Pennsylvania) Teacher, Holmes Jr. High School, Philadelphia, Pa. *Rosemont, Pa.*
FULLER, Prof. GORDON, Ph.D. (Michigan) Alabama Poly. Inst., Auburn, Ala. *346 Payne St.*
FULLER, GRACE A., A.B. (Fresno State Coll.) Head of Dept., Union High School, Madera, Calif.
FULLER, Asso. Prof. K. G., A.M. (Nebraska) Teachers Coll. of Connecticut, New Britain, Conn. *Major, A.U.S., U.S. Military Acad., West Point, N.Y. Dept. of Math.*
FULMER, Asso. Prof. H. K., A.M. (Columbia) Georgia School of Tech., Atlanta, Ga.
FUNKHOUSER, H. G., Ph.D. (Columbia) Instr., Phillips Exeter Acad., Exeter, N.H. *Cilley Hall*
FURMAN, ALBERT, M.S. (New Hampshire) *Capt., Inf. 175th Engr (GS) Regt. 221 N. Smith St., Aurora, Ill.*

GABA, Prof. M. G., Ph.D. (Chicago) Univ. of Nebraska, Lincoln, Nebr.
GABRIELLE MARIE, Sister, M.S. (Catholic Univ.) Instr., Dunbarton Coll., Washington, D.C. *2935 Upton St., Washington 8, D.C.*
GADDIS, ALETHA C., A.M. (Columbia) Instr., Coe Coll., Cedar Rapids, Iowa. *1021 W. 23rd St., Cedar Falls, Iowa*
GADSKE, R. E., Ph.D. (Northwestern) *In Service*
GAFFNEY, M. P., B.S. (Harvard) 525 Ash St., Winnetka, Ill.
GAGE, Prof. W. H., M.A. (Univ. of B.C.) Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*
GAGER, W. A., Ph.D. (Peabody) Acting Asso. Prof., Univ. of Florida, Gainesville, Fla. *Peabody 13*
GAINES, Prof. R. E., A.M. (Furman) Univ. of Richmond, Richmond, Va.
GALBRAITH, Asst. Prof. M. G., M.S. (Rutgers) Rutgers Univ., New Brunswick, N.J. *121 Magnolia St.*
GALE, Prof. A. S., Ph.D. (Yale) Emeritus, Univ. of Rochester, Rochester, N.Y. *93 Bellevue Drive, Rochester 7, N.Y.*

GARABEDIAN, Prof. C. A., Ph.D. (Harvard) Wheaton Coll., Norton, Mass.
GARDNER, R. W., A.M. (Boston Univ.), D.D. (Olivet Coll.) *In Service*
GARNER, Asso. Prof. L. L., A.M. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *507 North St.*
GARRETT, E. T., A.B. (Cornell Coll.) *In Service*
GARRETT, Prof. J. A., A.M. (Peabody) Arkansas A. and M. Coll., Monticello, Ark.
GARRETT, Prof. W. H., A.M. (Illinois Coll.) Vice-President, Baker Univ., Baldwin, Kans.
GARRISON, Asso. Prof. L. M., A.M. (Missouri), Ed.M. (Peabody) Louisiana Poly. Inst., Ruston, La. *Box 67 Tech. Sta.*
GARVIN, Sister MARY CLEOPHAS, Ph.D. (St. Louis Univ.) Head of Dept. of Science, Notre Dame Coll., Cleveland 21, Ohio
GASKELL, R. E., Ph.D. (Michigan) Research Asso., Brown Univ., Providence, R.I. *15 Everett Ave., Providence 6, R.I.*
GASS, Asso. Prof. C. B., A.M. (Nebraska) Nebraska Wesleyan Univ., Lincoln, Nebr. *Pvt., Engr. Corps. S.E.D., P.O. Box 180, Santa Fe, N.M.*
GATEWOOD, B. E., Ph.D. (Wisconsin) Acting Chief Structural Engr., McDonnell Aircraft Corp., St. Louis, Mo. *9038 McNulty Drive, St. Louis 21, Mo.*
GAULT, Prof. A. E., M.S. (Chicago) Dean, Arts and Sci., Bradley Poly. Inst., Peoria 5, Ill.
GAUTHIER, Prof. ABEL, A.M. (Columbia) Univ. of Montreal, 2900 Blvd. Mont-Royal, Montreal, P.Q., Can.
GAVER, Prof. W. H., A.B. (Randolph-Macon) Newberry Coll., Newberry, S.C. *Box 345*
GAY, Prof. H. J., A.M. (Clark) Worcester Poly. Inst., Worcester, Mass. *7 Belvidere Ave., Worcester 5, Mass.*
GAYLORD, Asst. Prof. LESLIE J., M.S. (Chicago) Agnes Scott Coll., Decatur, Ga.
GEHMAN, Prof. H. M., Ph.D. (Pennsylvania) Univ. of Buffalo, Buffalo 14, N.Y. *Army Univ. Center in England. 163 Winspear Ave., Buffalo 15, N.Y.*
GELBART, Asst. Prof. ABE, Ph.D. (Mass. Inst. of Tech.) Syracuse Univ., Syracuse 10, N.Y.
GENTRY, Asso. Prof. F. C., Ph.D. (Illinois) Louisiana Poly. Inst., Ruston, La. *200 E. Arizona St.*
GENTZLER, W. E., A.M. (Columbia) Bursar, Columbia Univ., New York 27, N.Y.
GEORGES, J. S., Ph.D. (Chicago) Chm. of Dept., Wright Jr. Coll., Chicago, Ill. *4515 N. Kildare Ave.*
GERE, B. H., Ph.D. (Mass. Inst. of Tech.) *Lt., U.S.N.R. Instr., Postgrad. School, U.S. Naval Acad., Annapolis, Md.*
GERGEN, Prof. J. J., Ph.D. (Rice) Duke Univ., Durham, N.C. *P.O. Box 4771 Duke Sta.*
GERST, Rev. F. J., Ph.D. (Johns Hopkins) Prof., Loyola Univ., 6525 Sheridan Road, Chicago 26, Ill.
GERST, IRVING, A.M. (Columbia) Tech. Consultant, Army Service Forces. *1055 Clarkson Ave., Brooklyn, N.Y.*
GETCHELL, B. C., Ph.D. (Michigan) Research Analyst, U.S. War Dept. *903 N. Wayne St., Apt. 305, Arlington, Va.*
GIBBENS, Asst. Prof. GLADYS, Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn. *122 Folwell Hall*
GIBBONS, Sister MARY SERAPHIM, A.M. (Minnesota) Coll. of St. Catherine, St. Paul 1, Minn.
GIESE, J. H., Ph.D. (Princeton) Aerodynamicist, Bell Aircraft Corp., Buffalo, N.Y. *8650 Rivershore Drive, Niagara Falls, N.Y.*
GILL, Prof. B. P., Ph.D. (Columbia) Coll. of the City of New York, New York, N.Y. *493 Warwick Ave., West Englewood, N.J.*
GILLESPIE, Prof. WILLIAM, Ph.D. (Chicago) Emeritus, Princeton Univ., Princeton, N.J. *28 Haslet Ave.*
GILLETTE, E. F., A.B. (Hamilton) Master, The Taft School, Watertown, Conn.
GILLIS, M. E., A.M. (Chicago) Instr., Univ. of Tennessee, Knoxville, Tenn.
GILMAN, Asso. Prof. R. E., Ph.D. (Princeton) Brown Univ., Providence, R.I. *44 E. Manning St.*
GINGRICH, Prof. C. H., Ph.D. (Chicago) Math. and Astr., Carleton Coll., Northfield, Minn.
GINNINGS, R. M., M.S. (Chicago) Emeritus, Head of Dept., Western Illinois State Teachers Coll., Macomb, Ill. *314 N. Ward St.*
GINSBURG, A. M., A.M. (Columbia) Asst., Bronx Voc. High School, New York, N.Y. *1563 Metropolitan Ave., New York 62, N.Y.*
GINSBURG, Prof. JEKUTHIEL, A.M. (Columbia) Head of Dept., Yeshiva Coll., New York, N.Y. *610 West 139 St., New York 31, N.Y.*
GIVENS, Asst. Prof. J. W., Ph.D. (Princeton) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
GLAZIER, Asst. Prof. HARRIET E., A.M. (Chicago) Emeritus, Univ. of California at Los Angeles, Los Angeles, Calif. *1307 Lucile Ave., Los Angeles 26, Calif.*
GLEASON, A. M., B.S. (Yale) *Lt., U.S.N.R. 2017 Eye St. N.W., Washington 6, D.C.*

GLENN, W. H., A.M. (U.C.L.A.) Asst. Dir., Service and Develop. Lab., A. O. Smith Corp., Los Angeles, Calif. *1425 Beech St., South Pasadena, Calif.*
GLOVER, Prof. B. C., A.M. (Chicago) Otterbein Coll., Westerville, Ohio. *220 Hiawatha Ave.*
GODDERZ, H. W., B.S. (Northern St. T.C.) Coll. of St. Thomas, St. Paul 1, Minn.
GODFREY, E. L., A.M. (Indiana) Township High School, Arlington Heights, Ill.
GOINS, MARY A., A.M. (Michigan) 86 Gillett St., Hartford, Conn.
GOLD, Prof. J. S., A.M. (Bucknell) Math. and Astr., Bucknell Univ., Lewisburg, Pa. *306 S. Third St.*
GOLDBERG, MICHAEL, A.M. (George Washington) Principal Engr., Bureau of Ordnance, Navy Dept., Washington, D.C. *5823 Potomac Ave. N.W., Washington 16, D.C.*
GOLDSTINE, H. H., Ph.D. (Chicago) Instr., Univ. of Michigan, Ann Arbor, Mich. *Capt., Ordnance Dept. Moore School of Elec. Engineering, 200 S. 33rd St., Philadelphia 4, Pa.*
GOLOMB, Asst. Prof. MICHAEL, Ph.D. (Berlin) Purdue Univ., Lafayette, Ind. *Mathematician, N.D.R.C., Franklin Inst., Philadelphia 3, Pa.*
GONZÁLEZ, Prof. M. O., D.P.M.S. (Havana) Univ. of Havana, Havana, Cuba. *Escuela de Ciencias*
GOOD, J. M., M.S. (Brown) *Lt., U.S.N.R. c/o H. R. Spencer, Box 193, R.F.D. 1, Westwood, N.J.*
GOOD, Asst. Prof. R. A., Ph.D. (Wisconsin) Univ. of Maryland, College Park, Md. *Dept. of Math.*
GOODE, C. J., A.B. (Fordham) *In Service*
GOODMAN, A. W., A.M. (Cincinnati) Stress Analyst, Republic Aviation Corp., Farmingdale, N.Y. *3 Williams St., Rockville Centre, N.Y.*
GOODPASTURE, R. A., B.S. (Colorado State Coll.) Asst. Engr., U.S. Bureau of Reclamation, Custom House, Denver, Colo. *Capt., 157th Inf.*
GOODRICH, M. T., A.M. (Clark Univ.) Head of Dept., Keene Teachers Coll., Keene, N.H. *36 Wyman Way*
GORDON, Asso. Prof. W. O., A.M. (Pennsylvania State) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
GORE, G. D., Ph.D. (Chicago) Chm. of Dept., Math. and Engg. Sci., Roosevelt Coll. of Chicago, Chicago 4, Ill. *1707 Burlington St., Downers Grove, Ill.*
GORMAN, J. R., A.M. (U.C.L.A.) Instr., Compton Jr. Coll., Compton, Calif. *Lt., U.S.N.R. 328 S. Hamel Road, Apt. #1, Los Angeles 36, Calif.*
GORRELL, Prof. G. W., A.M. (Ohio State) Emeritus, Univ. of Denver, Denver, Colo. *Box 1243, Estes Park, Colo.*
GOTTSCHALK, W. H., Ph.D. (Virginia) Instr., Univ. of Pennsylvania, Philadelphia 4, Pa. *College Hall, Box 6*
GOUGH, Prof. E. S. J., B.A. (Montreal) Jacques Cartier Normal School, 1211 Sherbrooke St. E., Montreal 24, P.Q., Can. *4415 Coronation Ave., Montreal 28, P.Q., Can.*
GOULD, ALICE B., A.B. (Bryn Mawr) 35 Congress St., Boston, Mass.
GOUWENS, Asso. Prof. CORNELIUS, Ph.D. (Chicago) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
GOVE, H. E., M.S. (Washington Univ.) Union Elec. Co. of Missouri, 315 N. 12th Blvd., St. Louis 1, Mo. *Engg. and Const. Dept.*
GRABBE, Rev. HYACINTH, A.M. (Catholic Univ.) Prof., St. Joseph's Coll. and Milit. Acad., Hays, Kans.
GRABLE, Asst. Prof. E. S., A.M. (Washington and Jefferson) Univ. of Richmond, Richmond, Va. *Box 104*
GRAD, ARTHUR, A.M. (Columbia) Mathematician, U.S. Coast and Geodetic Survey, Washington, D.C. *3216 Walbridge Place N.W., Washington 10, D.C.*
GRAESSER, Prof. R. F., Ph.D. (Illinois) Univ. of Arizona, Tucson, Ariz. *1648 E. Fifth St.*
GRAFF, F. G., A.M. (Pittsburgh) Instr., Amherst Coll., Amherst, Mass. *97 Lincoln Ave.*
GRAHAM, Asso. Prof. MARIA D., A.M. (Columbia) Retired, East Carolina Teachers Coll., Greenville, N.C. *Warrenton, N.C.*
GRAHAM, Asso. Dean P. H., A.M. (Virginia) Chm. of Dept., New York Univ., Washington Square Coll., Washington Square E., New York, N.Y.
GRANT, ALICE A., A.M. (Brown) Apt. 241, 215 College St., Toronto, Ont., Can.
GRANT, Asst. Prof. H. S., Ph.D. (Pennsylvania) Rutgers Univ., New Brunswick, N.J. *108 S. Third Ave., Highland Park, N.J.*
GRAVATT, Prof. T. E., M.S. (Rutgers) Pennsylvania State Coll., State College, Pa. *344 E. College Ave.*
GRAVES, Asst. Prof. C. H., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa. *Fuel Rationing Division, O.P.A., Washington, D.C. 700 N. Wayne St., Arlington, Va.*
GRAVES, Asso. Prof. G. H., Ph.D. (Columbia) Purdue Univ., West Lafayette, Ind. *227 S. Grant St.*
GRAVES, Prof. L. M., Ph.D. (Chicago) Univ. of Chicago, Chicago 37, Ill.
GRAVES, Asst. Prof. W. L., A.M. (Pennsylvania) Drury Coll., Springfield, Mo. *In Service*

GRAY, MARION C., Ph.D. (Bryn Mawr) Member Tech. Staff, Bell Telephone Labs., 463 West St., New York 14, N.Y.
GRAY, MARY W. (Mrs. A. B.), A.M. (Connecticut Coll.) Instr., Barnard School for Girls, 554 Ft. Washington Ave., New York, N.Y. *87-40 Elmhurst Ave., Elmhurst, N.Y.*
GREEN, Asst. Prof. J. W., Ph.D. (California) Univ. of California at Los Angeles, 405 Hilgard Ave., Los Angeles 24, Calif. *Dept. of Math.*
GREEN, L. J., Ph.D. (Chicago) *Mathematician, N.A.C.A. 11717 Sellers Ave., Cleveland 8, Ohio*
GREENE, LAURA Z., M.S. (Chicago) Instr., Washburn Univ., Topeka, Kans. *1516 Boswell Ave.*
GREENLEAF, Prof. H. E. H., Ph.D. (Indiana) DePauw Univ., Greencastle, Ind. *1024 S. College Ave.*
GREENWOOD, J. A., Ph.D. (Missouri) *Lt., U.S.N.R., Armament Test, N.A.S., Patuxent River, Md.*
GREENWOOD, R. E., Jr., Ph.D (Princeton) Instr., Univ. of Texas, Austin, Tex. *Lt., U.S.N.R. 1374 Harvard St. N.W., Washington 9, D.C.*
GREER, EDISON, M.S. (Kansas State Coll.) Instr., Univ. of Kansas, Lawrence, Kans. *1301 New York St.*
GREGORY, Asso. Prof. C. D., A.M. (Yale) Coll. of William and Mary, Williamsburg, Va.
GREGORY, Brother JOSEPHUS, A.M. (Loyola Univ.) Instr., St. Mary's Coll., Winona, Minn.
GREVILLE, T. N. E., Ph.D. (Michigan) Actuarial Mathematician, Bureau of the Census, Washington, D.C. *1714-34th St. N.W., Washington 7, D.C.*
GRIFFIN, Prof. F. L., Ph.D. (Chicago) Reed Coll., Portland 2, Ore.
GRIFFIN, Asst. Prof. HARRIET M., Ph.D. (New York Univ.) Brooklyn Coll., Brooklyn, N.Y. *3609 Farragut Road, Brooklyn 10, N.Y.*
GRIFFIN, J. I., Ph.D. (Columbia) Instr., Econ., Coll. of the City of New York, New York, N.Y. *115 Henry St., Brooklyn 2, N.Y.*
GRIFFITH, Prof. W. C., A.M. (Oregon) Marion Inst., Marion, Ala.
GROSSMAN, GEORGE, A.M. (Columbia) Teacher, Manhattan High School of Aviation Trades, New York, N.Y. *3425 Gates Place, New York 67, N.Y.*
GROVE, Asst. Prof. C. C., Ph.D. (Johns Hopkins) Coll. of the City of New York, New York 10, N.Y. *143 Milburn Ave., Baldwin, N.Y.*
GROVE, Prof. V. G., Ph.D. (Chicago) Michigan State Coll., East Lansing, Mich. *438 Rosewood Ave.*
GROVES, MARJORIE J., A.M. (Chicago) Teacher, George Rogers Clark School, Whiting, Ind. *1542 Warwick St.*
GUARD, Asst. Prof. H. T., M.S. (Colorado) Colorado State Coll. of A. and M. A., Fort Collins, Colo. *Capt., F. A. U.S. Military Acad., West Point, N.Y. Dept. of Math.*
GUELPA, L. B., Jr., A.M. (New York Univ.) *Lt., U.S.N.R. Educational Officer in Math. and Sci., U.S. Merchant Marine Acad., Kings Point, N.Y. 10 Welwyn Road, Great Neck, N.Y.*
GUNDER, Asso. Prof. D. F., Ph.D. (Wisconsin) Civil Engg., Colorado State Coll. of A. and M. A., Fort Collins, Colo. *Sr. Engr., Rocket Research Div., Aberdeen Proving Ground, Md. 619 S. Sherwood St., Fort Collins, Colo.*
GUNDERSON, N. G., A.M. (Cornell) Teaching Asst., Cornell Univ., Ithaca, N.Y. *White Hall*
GYSLAND, Mrs. MARIAN S., A.M. (California) 1158 S. Clayton St., Denver, Colo.

HAAG, V. H., A.M. (Duke) Instr., Hershey Jr. Coll., Hershey, Pa. *Box 13*
HACKER, Asst. Prof. S. G., Ph.D. (Princeton) State Coll. of Washington, Pullman, Wash. *Lt. (j.g.), U.S.N.R. Research and Develop. Div., Bureau of Ordnance, Navy Dept., Washington, D.C. 4303 N. Pershing Drive, Arlington, Va.*
HADLEY, J. R., B.S. (Ohio State) 743 Oakdale Ave., Apt. B, Chicago 14, Ill.
HADLEY, Prof. LAURENCE, Ph.D. (Michigan) Purdue Univ., West Lafayette, Ind. *Box 482*
HADLEY, LENA C. (Mrs. A. J.), A.M. (Missouri) Instr., Kansas State Coll., Lawrence, Kans. *1300 Louisiana St.*
HADLOCK, Prof. E. H., Ph.D. (Cornell) Hastings Coll., Hastings, Nebr.
HAGEN, Asso. Prof. BEATRICE L., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa. *126 E. Nittany Ave.*
HAILPERIN, THEODORE, Ph.D. (Cornell) Mathematician, Aberdeen Proving Ground, Md. *233 Bloomsbury Ave., Havre de Grace, Md.*
HAIR, C. L., A.M. (Duke) *Colonel*, Head of Dept., The Citadel, Charleston, S.C.
HALL, Asst. Prof. D. W., Ph.D. (Virginia) Univ. of Maryland, College Park, Md. *Box 162*
HALL, Prof. H. L., A.M. (Indiana) Northwestern State Coll., Alva, Okla. *923 Seventh St.*
HALL, N. A., Ph.D. (Calif. Inst. of Tech.) Mech. Engr., United Aircraft Corp., East Hartford 8, Conn. *Research Div.*

HALLER, Asst. Prof. MARY E., Ph.D. (Washington) Univ. of Washington, Seattle 5, Wash. *Dept. of Math.*
HALLERBERG, A. E., A.M. (Illinois) Instr., Illinois Coll., Jacksonville, Ill. *RT 2/C, U.S.N. 845 S. East St., Jacksonville, Ill.*
HALLETT, W. N., Ph.D. (Pennsylvania) Instr., State Teachers Coll., Frostburg, Md. *1st Lt., A.G.D. Armed Forces Induction Sta., Indianapolis 2, Ind. 4140 Carrollton Ave., Indianapolis 5, Ind.*
HALMOS, Asst. Prof. P. R., Ph.D. (Illinois) Syracuse Univ., Syracuse 10, N.Y. *Dept. of Math.*
HAMILTON, Asst. Prof. H. J., Ph.D. (Brown) Pomona Coll., Claremont, Calif. *735 Yale St.*
HAMILTON, O. H., Ph.D. (Texas) *In Service*
HAMILTON, W. M., A.M. (Michigan) Retired. 1512 Granger Ave., Ann Arbor, Mich.
HAMMER, P. C., Ph.D. (Ohio State) Dir., Quality Control, Lockheed Aircraft Corp., Burbank, Calif.
HAMMING, R. W., Ph.D. (Illinois) P.O. Box 1663, Santa Fe, N.M.
HAMMOND, Prof. E. S., Ph.D. (Princeton) Bowdoin Coll., Brunswick, Me. *9 Thompson St.*
HAMMOND, Asst. Prof. J. R., A.M. (Harvard) U.S. Naval Acad., Annapolis, Md. *Dept. of Math.*
HANCOCK, CLARA L., A.M. (Iowa) Instr., Junior Coll., Virginia, Minn. *Box 706*
HAND, Sister MIRIAM, A.B. (Immaculate Heart) Instr., Immaculate Heart Coll., 2021 N. Western Ave., Hollywood, Calif.
HANHAUSER, M. A., A.B. (St. Bonaventure) Instr., St. Bonaventure Coll., St. Bonaventure, N.Y. *St. Bonaventure Monastery*
HANNA, Asst. Prof. J. R., M.S. (Kans. St. T. C., Emporia) Univ. of Wichita, Wichita 6, Kans.
HANSMAN, Asst. Prof. MARGARET M., Ph.D. (Illinois) Colorado Coll., Colorado Springs, Colo. *2038 Armstrong Ave.*
HANSON, E. H., Ph.D. (Ohio State) Chm. of Dept., North Texas State Teachers Coll., Denton, Tex. *Lt. Comdr., U.S.N.R. 3864 Porter St. N.W., Washington 16, D.C.*
HARDIN, Dean J. A., A.M. (Chicago) Head of Dept., Centenary Coll., Shreveport 16, La.
HARDING, HOWARD, B.M.E. (Michigan) Rochester Gas and Elec. Corp., Rochester, N.Y. *29 Kingston St.*
HARDMAN, W. R., A.B. (Indiana) Instr., Purdue Univ., West Lafayette, Ind. *103 Waldron St.*
HARDY, Prof. G. H., M.A. (Trinity, Cambridge) Trinity Coll., Cambridge Univ., Cambridge, England
HARKIN, D. C., Ph.D. (Chicago) Instr., Brooklyn Coll., Brooklyn, N.Y. *Research Div., Bureau of Ordnance, Navy Dept., Washington, D.C. 3000 Crest Ave., Cheverly, Md.*
HARMAN, H. H., M.S. (Chicago) Supervisor, Div. of Research and Statistics, Dept. of Public Welfare, Springfield, Ill. *855 S. Park Ave.*
HARMON, ELIZABETH HARRIS (Mrs. J. P.), M.S. (Washington Univ.) 4800 Forbes St., Pittsburgh 13, Pa.
HARP, E. L., Jr., A.M. (New Mexico) Head of Dept., High School, Roswell, N.M. *805 W. Mathews St.*
HARPER, Asst. Prof. F. S., Ph.D. (Iowa) Univ. of Nebraska, Lincoln, Nebr. *307-D M.A. Hall*
HARRELL, E. G., Ph.D. (Iowa) Head of Dept., State Teachers Coll., Platteville, Wis.
HARRINGTON, C. E., M.S. (Buffalo) Engr., W. H. Smith, Inc., Springville, N.Y. *52 Winter St., Buffalo 13, N.Y.*
HARRINGTON, W. J., Ph.D. (Cornell) Instr., Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
HARRIS, Asso. Prof. ISABEL, A.M. (Columbia) Westhampton Coll., Univ. of Richmond, Richmond, Va.
HARRIS, MINNIE W. C. (Mrs. D. P.), A.M. (Missouri) Part-time Instr., Meredith Coll., Raleigh, N.C. *1203 Filmore St.*
HARRISON, R. A., Ph.D. (Cornell) Master, St. Mark's School, Southboro, Mass.
HARSHBARGER, Asso. Prof. FRANCES, Ph.D. (Illinois) Kent State Univ., Kent, Ohio
HART, BERTHA I., A.M. (Cornell) Chief Computer, Ballistic Research Lab., Aberdeen Proving Ground, Md.
HART, Prof. W. L., Ph.D. (Chicago) Univ. of Minnesota, Minneapolis, Minn. *100 Folwell Hall*
HART, Asso. Prof. W. W., A.B. (Chicago) Retired, Univ. of Wisconsin. *139 Tudor Place, Kenilworth, Ill.*
HARTIG, Prof. H. E., Ph.D. (Minnesota) Asst. Director, Communication Engg., Univ. of Minnesota, Minneapolis, Minn. *3940 Lake Curve, Robbinsdale 12, Minn.*

HARTLEY, Asst. Prof. M. C., Ph.D. (Illinois) Educ., Univ. of Illinois, Urbana, Ill. *1203 W. Illinois St.*
HARTNELL, GEORGE. Retired, Magnetic Observ., U.S. Coast and Geodetic Survey. *Wyoming, N.Y.*
HARTUNG, Asso. Prof. M. L., Ph.D. (Wisconsin) Teaching of Math., Univ. of Chicago, Chicago 37, Ill.
HARTZLER, Prof. H. H., Ph.D. (Rutgers) Bluffton Coll., Bluffton, Ohio
HARVEY, Asso. Prof. G. G., Ph.D. (Washington Univ.) Physics, Massachusetts Inst. of Tech., Cambridge 39, Mass.
HARVEY, W. R., M.S. (Michigan) Memb. Tech. Staff, Bell Telephone Labs., New York, N.Y.
HARWOOD, Asst. Prof. MAY N., A.M. (Syracuse) Syracuse Univ., Syracuse, N.Y. *803 Comstock Ave.*
HASKINS, ELIZABETH M., M.S. (Mass. Inst. of Tech.) *2nd Lt., W.A.C., 27 Vernon St., Worcester 4, Mass.*
HASSLER, Prof. J. O., Ph.D. (Chicago) Math. and Astr., Univ. of Oklahoma, Norman, Okla. *425 Lahoma Ave.*
HASTINGS, CECIL, Jr., B.S. (Florida) Computer, Franklin Inst., New York, N.Y. *Shelton Hotel 349, Lexington Ave. and 49 St.*
HATCH, Asso. Prof. D. A., A.M. (Columbia) Lafayette Coll., Easton, Pa. *705 High St.*
HATCHER, Prof. T. W., Ph.D. (Cornell) Head of Dept., Virginia Poly. Inst., Blacksburg, Va.
HATFIELD, Prof. CHARLES, A.M. (Tennessee) Georgetown Coll., Georgetown, Ky. *301 Clayton Ave.*
HATFIELD, CHARLES, Jr., Ph.D. (Cornell) *Ensign, S.C., U.S.N.R., 301 Clayton Ave., Georgetown, Ky.*
HATTAN, CORINNE R., Ph.D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill. *1008 W. Stoughton*
HAUSMANN, Rev. B. A., Ph.D. (Yale) Chm. of Dept., Univ. of Detroit, McNichols Road at Livernois, Detroit 21, Mich.
HAVILAND, Prof. E. K., Ph.D. (Harvard; Johns Hopkins) Lincoln Univ., Chester Co., Pa. *Box 382, Oxford, Pa.*
HAWTHORNE, FRANK, B.S. in Educ. (Penn. St. T.C., Edinboro) *Univ. of California Div. of War Research, U.S. Navy Radio and Sound Lab., San Diego 52, Calif. 1 Stonelea Place, New Rochelle, N.Y.*
HAY, Asst. Prof. G. E., Ph.D. (Toronto) Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*
HAYES, J. J., B.S. (Utah) Instr., Univ. of Utah, Salt Lake City, Utah
HAYNES, NOLA L. A. (Mrs. E. S.), Ph.D. (Missouri) 1408 Rosemary Lane, Columbia, Mo.
HAYWARD, Mrs. LEOTA C., M.S. (Colorado State Coll.) Asst. Prof., Acting Head of Dept., Colorado State Coll. of A. and M.A., Fort Collins, Colo. *Dept. of Math.*
HAZARD, Prof. C. T., A.M. (Indiana) Purdue Univ., West Lafayette, Ind. *344 N. Western Ave.*
HAZELTINE, Prof. B. A., A.M. (Columbia) Middlebury Coll., Middlebury, Vt.
HAZELTINE, L. A., M.E. (Stevens), A.M. (Columbia) 15 Tower Drive, Maplewood, N.J.
HAZLETT, Asso. Prof. OLIVE C., Ph.D. (Chicago) Univ. of Illinois, Urbana, Ill.
HAZLEWOOD, E. A., Ph.D. (Cornell) *Capt., A.C. U.S. Military Acad., West Point, N.Y. Dept. of Math.*
H'DOUBLER, F. T., Ph.D. (Wisconsin), M.D. (Harvard) Surgeon, Springfield, Mo. *Medical Arts Bldg.*
HEARN, Rev. J. R., A.M. (Woodstock) Old St. Joseph's Church, 321 Willing's Alley, Philadelphia, Pa.
HEASLET, M.A., Ph.D. (Stanford) Head, Dept. of Theoretical Aerodynamics, N.A.C.A., Moffett Field, Calif. *Box 402, Los Altos, Calif.*
HEATH, R. V. Member, New York Stock Exchange, New York, N.Y. *40 Wall St.*
HEBEL, Asso. Prof. I. L., M.S. (Colorado) Colorado School of Mines, Golden, Colo.
HEDBERG, Asso. Prof. E. A., Ph.D. (Missouri) Baylor Univ., Waco, Tex. *Visiting Asso. Prof., Elec. Communications, Harbor Bldg. School, Massachusetts Inst. of Tech., Cambridge, Mass. 105 Harvard St., Newtonville, Mass.*
HEDLUND, Prof. G. A., Ph.D. (Harvard) Univ. of Virginia, Charlottesville, Va. *1834 Fendall Ave.*
HEFNER, Prof. R. A., Ph.D. (Chicago) Georgia School of Tech., Atlanta, Ga. *Faculty Exchange Box 2102*
HEILMAN, Prof. C. E., A.M. (Duke) Math. and Physics, Elizabethtown Coll., Elizabethtown, Pa. *352 N. Tenth St., Lebanon, Pa.*
HEIMANN, Prof. E. E., A.M. (Texas) East Central State Coll., Ada, Okla. *523 S. Highland St.*
HEINEMAN, Asso. Prof. E. R., A.M. (Wisconsin) Texas Tech. Coll., Lubbock, Tex.

HEINS, Asst. Prof. M. H., Ph.D. (Harvard) Illinois Inst. of Tech., 3300 Federal St., Chicago, Ill. *Mathematician, Office Chief of Ordnance. 4829A S. 28th St., Fairlington Project, Arlington, Va.*
HEINZMAN, Asso. Prof. W. P., A.M. (Illinois) Acting Head of Dept., New Mexico Coll. of A. and M.A., State College, N.M. *Box 144*
HELLINGER, E. D., Ph.D. (Göttingen) Lecturer, Northwestern Univ., Evanston, Ill. *2215 Maple Ave.*
HELLMICH, Asso. Prof. E. W., Ph.D. (Columbia) Northern Illinois State Teachers Coll., DeKalb, Ill.
HELMS, C. B., B.S. in Educ. (Temple) Pennsylvania Military Coll., Chester, Pa.
HELSEL, R. G., Ph.D. (Ohio State) Instr., Ohio State Univ., Columbus 10, Ohio. *Dept. of Math.*
HELTON, Asso. Prof. F. F., A.M. (Missouri) Head of Dept., Central Coll., Fayette, Mo. *Morrison Observatory*
HEMENWAY, Asso. Prof. L. D., A.M. (Harvard) Math. and Physics, Simmons Coll., 300 The Fenway, Boston, Mass.
HENDERSON, Prof. ARCHIBALD, Ph.D. (North Carolina; Chicago) Head of Dept., Univ. of North Carolina, Chapel Hill, N.C. *721 E. Franklin St.*
HENDRIX, Asst. Prof. GERTRUDE, M.S., A.M. (Illinois) Eastern Illinois State Teachers Coll., Charleston, Ill. *1425 Fourth St.*
HENNEL, Prof. CORA B., Ph.D. (Indiana) Indiana Univ., Bloomington, Ind. *410 S. Park Ave.*
HENRIQUES, Asst. Prof. ANNA S. (Mrs. D. E.), Ph.D. (Chicago) Univ. of Utah, Salt Lake City 1, Utah
HEREN, Prof. MABEL M., M.S. (Northwestern) Knox Coll., Galesburg, Ill.
HERLIHY, F. W., B.S. (Chicago) Vice-Pres., Herlihy Mid-Continent Co., Chicago, Ill. *Box 81, Comstock, Mich.*
HERPEL, Asst. Prof. COLEMAN, A.M. (Harvard) Pennsylvania State Coll., State College, Pa. *Lt., U.S.N.R., Midshipmen's School, Fort Schuyler, New York 61, N.Y.*
HERR, Asso. Prof. GERTRUDE A., M.S. (Iowa) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
HERZOG, Asst. Prof. FRITZ, Ph.D. (Columbia) Michigan State Coll., East Lansing, Mich. *Dept. of Math.*
HESS, Prof. G. W., Ph.D. (Michigan) Howard Coll., Birmingham 6, Ala. *8009 Fourth Ave. S.*
HESSE, EMMA V., A.M. (Columbia) Teacher, Claremont Jr. High School, Oakland, Calif. *2412 Durant Ave., Berkeley 4, Calif.*
HESSELTINE, EVELYN, A.M. (Nebraska) Teacher, Black Hills Teachers Coll., Spearfish, S.D. *P.O. Box 309*
HESTENES, Asso. Prof. M. R., Ph.D. (Chicago) Univ. of Chicago, Chicago 37, Ill. *Dept. of Math.*
HEYDA, Asst. Prof. J. F., Ph.D. (Illinois) St. Ambrose Coll., Davenport, Iowa. *1403 W. 13th St.*
HICKERSON, Prof. T. F., A.M. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *108 Battle Lane*
HICKMAN, J. S., A.M. (Minnesota) Instr., Rochester Jr. Coll., Rochester, Minn. *Lt., U.S.N.R. 216 Avenue C, Redondo Beach, Calif.*
HICKSON, Asst. Prof. A. O., Ph.D. (Chicago) Duke Univ., Durham, N.C. *2712 Legion Ave.*
HIGGINS, Asso. Prof. T. J., Ph.D. (Purdue) Elec. Engg., Illinois Inst. of Tech., 3300 Federal St., Chicago, Ill.
HIGHTOWER, Prof. RUBY U., Ph.D. (Missouri) Shorter Coll., Rome, Ga.
HILDEBRANDT, Asst. Prof. E. H. C., Ph.D. (Michigan) Northwestern Univ., Evanston, Ill. *201 Golf Terrace*
HILDEBRANDT, Prof. T. H., Ph.D. (Chicago) Chm. of Dept., Univ. of Michigan, Ann Arbor, Mich. *1930 Cambridge Road*
HILDNER, Asst. Prof. R. C., Ph.D. (Ohio State) Coll. of Wooster, Wooster, Ohio. *709 N. Bever St.*
HILL, A. L., A.B. (Doane) Head of Dept., State Teachers Coll., Peru, Nebr. *Major, A.C., U.S. Army. 1287 Krameria St., Denver 7, Colo.*
HILL, Asso. Prof. J. D., Ph.D. (Brown) Michigan State Coll., East Lansing, Mich. *443 Grove St.*
HILL, Asso. Prof. L. S., Ph.D. (Yale) Hunter Coll., New York, N.Y. *22 Sagamore Road, Bronxville 8, N.Y.*
HILL, Prof. M. A., Jr., A.M. (North Carolina) Univ. of North Carolina, Chapel Hill, N.C. *333 Tenney Circle*
HILL, Asso. Prof. P. R., M.S. (Georgia) Univ. of Georgia, Athens, Ga. *190 Morton Ave.*

HILLE, Prof. EINAR, Ph.D. (Stockholm) Yale Univ., New Haven, Conn. *400 Prospect St., New Haven 11, Conn.*
HILLS, E. J., Ph.D. (Southern California) Instr., Los Angeles City Coll., Los Angeles 27, Calif. *1248 Holmby Ave., Los Angeles 24, Calif.*
HLAVATY, J. H., B.S. (C.C.N.Y.) Teacher, Bronx High School of Science, New York, N.Y. *250 Coligni Ave., New Rochelle, N.Y.*
HOARE, Prof. A. J., A.M. (Michigan) Emeritus, Univ. of Wichita, Wichita, Kans. *1717 N. Holyoke Ave.*
HOBBS, R. P., A.B. (Dartmouth) Acting Mgr., College Dept., Farrar and Rinehart, Inc., 232 Madison Ave., New York 16, N.Y.
HODGE, Asst. Prof. F. H., A.M. (Boston) Emeritus, Purdue Univ., West Lafayette, Ind. *820 N. Main St.*
HOEL, Asst. Prof. P. G., Ph.D. (Minnesota) Univ. of California at Los Angeles, Los Angeles 24, Calif.
HOHN, Asst. Prof. F. E., Ph.D. (Illinois) Guilford Coll., Guilford College, N.C.
HOLL, Prof. D. L., Ph.D. (Chicago) Head of Dept., Iowa State Coll., Ames, Iowa. *Dept. of Math.*
HOLLAND, Sister MARY CHARLOTTE, A.M. (Catholic Univ.) Registrar, St. Xavier Coll., 4900 Cottage Grove Ave., Chicago 15, Ill.
HOLLCROFT, Prof. T. R., Ph.D. (Cornell) Wells Coll., Aurora, N.Y.
HOLMES, Prof. C. T., Ph.D. (Harvard) Bowdoin Coll., Brunswick, Me. *60 Spring St.*
HOLT, E. W., M.S. (Chicago) Head of Dept., Lawrence Acad., Groton, Mass. *Lt. Comdr., U.S.N.R. 132 Hotchkiss St., Jamestown, N.Y.*
HOLTON, Asst. Prof. C. H., A.M. (Harvard) Georgia School of Tech., Atlanta, Ga. *167 Fourth St. N.W.*
HOLZINGER, J. R., B.S. (Franklin and Marshall) *Lt., U.S.N.R. Instr., Midshipmen's School, Fort Schuyler, N.Y. 120 Ruby St., Lancaster, Pa.*
HOOK, Asst. Prof. C. W., A.M. (North Carolina) Georgia School of Tech., Atlanta, Ga. *Lt. Comdr., U.S.N.R., U.S. Naval Acad., Annapolis, Md. Dept. of Math.*
HOOK, M. W., A.M. (North Carolina) Teacher, Elon Coll., Elon College, N.C. *Box 262*
HOOPES, M. F., A.B. (Oberlin) Teacher, Southern State Normal School, Springfield, S.D.
HOOPS, R. R. County Engr., Perry County, New Lexington, Ohio
HOOVER, Asso. Prof. B. P., Ph.D. (Illinois) Carnegie Inst. of Tech., Pittsburgh, Pa. *2040 Fairlawn St., Pittsburgh 21, Pa.*
HOPKINS, FANNIE, A.M. (Wisconsin) Teacher, High School, Waukesha, Wis. *138 S. East Ave.*
HOPKINS, Asso. Prof. L. A., Ph.D. (Chicago) Univ. of Michigan, Ann Arbor, Mich.
HORSFALL, Prof. I. O., Ph.D. (Cornell) Director, Extension Division, Univ. of Utah, Salt Lake City 1, Utah. *82 Virginia St., Salt Lake City 3, Utah*
HORVITZ, D. G., B.S. (Mass. State Coll.) *T/4, Engineers. 34 Jonathan St., New Bedford, Mass.*
HOSKINS, R. H., A.B. (Harvard) Teaching Fellow, Harvard Univ., Cambridge 38, Mass. *R.T. 3/C, U.S.N. 1245 Windsor Ave., Windsor, Conn.*
HOUGHTON, D. B., A.M. (Michigan) Aerodynamic Engr., Kellett Aircraft Corp., Upper Darby, Pa. *Engg. Div.*
HOUSEHOLDER, A. S., Ph.D. (Chicago) Fire Control Div., Naval Research Lab., Washington 25, D.C.
HOVE, E. MARIE, M.S. (Iowa) Instr., Univ. of New Mexico, Albuquerque, N.M. *Dept. of Math.*
HOVEY, Asst. Prof. B. K., Ph.D. (Göttingen) Elec. Eng., Univ. of Pittsburgh, Pittsburgh 13, Pa. *208 Thaw Hall*
HOWARD, Asso. Prof. AUGHTUM S. (Mrs. N. J.), Ph.D. (Kentucky) Kentucky Wesleyan Coll., Winchester, Ky.
HOWARD, Prof. C. M., E.Mines (Alabama Poly. Inst.) Acting Head of Dept., North Texas Agric. Coll., Arlington, Tex. *605 S. Center St.*
HOWARD, HARRIET, A.M. (Boston Univ.) Teacher, The Ethel Walker School, Simsbury, Conn.
HOWE, G. K., B.S. (Worcester Poly. Inst.) Bell Aircraft Corp., Marietta, Ga. *359 Fifth St. N.W., Atlanta, Ga.*
HOWE, T. D., Jr., B.S. (Harvard) T. D. Howe Construction Co., Houston, Tex. *R.F.D. 12, Box 324*
HOWELL, J. V., A.M. (North Carolina) Chm. of Dept., Mars Hill Coll., Mars Hill, N.C. *Lt., U.S.N.R.*
HOWELL, S. W., A.M. (South Dakota) Instr., Math. and Physics, Univ. of South Dakota, Vermillion, S.D.

HOWIE, Prof. J. M., A.M. (Columbia; Nebraska) Emeritus, Nebraska Wesleyan Univ., Lincoln, Nebr. *1350 Ogden St., Denver 3, Colo.*
HOWLAND, Prof. L. A., Ph.D. (Munich) Dean, Wesleyan Univ., Middletown, Conn.
HOY, Mrs. RUTH YAP, A.M. (Hawaii) 2800 Erie St. S.E., Apt. B-5, Washington 20, D.C.
HRATZ, Rev. J. A., B.S. (Iowa) Acting Head of Dept., St. Ambrose Coll., Davenport, Iowa
HUBBARD, J. F., Ed.M. (Teachers Coll. of Boston) Counselor, Rogers Intermediate School, Boston, Mass. *476 Beacon St., Boston 15, Mass.*
HUBBS, Prof. H. N., Ph.D. (Cornell) Treas. and Secretary, The Colleges of the Seneca, Geneva, N.Y.
HUBERT, Asso. Prof. W. G., Sc.D. (New York Univ.) Coll. of the City of New York, 139 St. and Convent Ave., New York 31, N.Y.
HUCK, RAYMOND, M.S. (Illinois) 115 Cayuga Heights Road, Ithaca, N.Y.
HUFF, Asso. Prof. G. B., Ph.D. (Illinois) Southern Methodist Univ., Dallas, Tex. *Lecturer, Univ. of Texas, Austin 12, Tex. 11 Waggener Hall, Univ. of Tex.*
HUFF, W. N., A.M. (Pennsylvania) Instr., Univ. of Rochester, Rochester 3, N.Y. *Box 24, Morey Hall*
HUFFER, Prof. R. C., Ph.D. (Chicago) Beloit Coll., Beloit, Wis. *729 Hobart Place*
HUGHES, Asso. Prof. H. K., Ph.D. (Michigan) Purdue Univ., Lafayette, Ind. *Dept. of Math.*
HUGHES, H. M., A.M. (Texas) Jr. Mathematician, Tennessee Valley Authority, Chattanooga, Tenn. *Lt., U.S.N.R. 800 N. Bridge St., Brady, Tex.*
HULL, RALPH, Ph.D. (Chicago) Chm. of Dept., Univ. of Nebraska, Lincoln, Nebr. *Dept. of Math., 300 M.A. Bldg.*
HUME, ALFRED, D.Sc. (Vanderbilt) Head of Dept. and Chancellor Emeritus, Univ. of Mississippi, University, Miss.
HUMMEL, Asst. Prof. P. M., Ph.D. (Ohio State) Univ. of Alabama, University, Ala. *Box 1251*
HUMPHREY, H. K., M.S. in E.E. (Union) Chm., Winnetka Trust and Savings Bank, Winnetka, Ill. *520 Ash St.*
HUMPHREYS, Asst. Prof. M. GWENETH, Ph.D. (Chicago) Sophie Newcomb Coll., New Orleans 18, La.
HUNT, Asst. Prof. G. H., C.E. (Cornell) Univ. of California at Los Angeles, 405 Hilgard Ave., Los Angeles 24, Calif.
HUNT, Prof. MILDRED, Ph.D. (Chicago) Illinois Wesleyan Univ., Bloomington, Ill. *406 E. Walnut St.*
HUNTER, Mrs. LOUISE S., Ed.M. (Harvard) Asst. Prof., Virginia State Coll., Ettrick, Va.
HUNTINGTON, Prof. E. V., Ph.D. (Strassburg) Emeritus, Mech., Harvard Univ., Cambridge, Mass. *48 Highland St.*
HURD, Dean C. C., Ph.D. (Illinois) Allegheny Coll., Meadville, Pa.
HURRY, Asso. Prof. J. A., A.M. (California) Physics, Colorado State Coll. of A. and M.A., Fort Collins, Colo. *650½ S. College Ave.*
HURST, Prof. J. W., Ph.D. (Illinois) Head of Dept., Montana State Coll., Bozeman, Mont. *522 S. Sixth St.*
HURWITZ, SOLOMON, Ph.D. (Columbia) Instr., Brooklyn Coll., Bedford Ave. and Avenue H, Brooklyn 10, N.Y.
HURWITZ, Prof. W. A., Ph.D. (Göttingen) Cornell Univ., Ithaca, N.Y. *White Hall*
HUTCHERSON, Prof. W. R., Ph.D. (Cornell) Head of Dept., Math. and Astr., Berea Coll., Berea, Ky.
HUTCHINSON, Prof. C. A., A.M. (Wittenberg) Head of Dept., Eng. Math., Univ. of Colorado, Boulder, Colo.
HUTCHINSON, L. C., Ph.D. (Mass. Inst. of Tech.) The Norden Laboratories Corp., 34 Varick St., New York 13, N. Y.
HUTCHINSON, Prof. R. O., Ph.D. (Chicago) Head of Dept., Tennessee Poly. Inst., Cookeville, Tenn. *Box 218, Tenn. Tech.*
HUTCHISON, L. P., Ph.D. (Kentucky) Instr., Pennsylvania State Coll., State College, Pa. *R.D. 1*
HYDE, Asso. Prof. EMMA, A.M. (Chicago) Kansas State Coll., Manhattan, Kans.
HYDEN, Prof. J. A., Ph.D. (Cornell) Vanderbilt Univ., Nashville 4, Tenn. *Box 111 Univ.*
HYERS, Asso. Prof. D. H., Ph.D. (Calif. Inst. of Tech.) Univ. of Southern California, Los Angeles 7, Calif. *5127 Parkglen Ave., Los Angeles 43, Calif.*
HYMAN, M. A., B.S. (Maryland) Mathematician, Naval Ordnance Lab., Washington, D.C. *4000 Kansas Ave. N.W., Washington 11, D.C.*

IKENBERRY, Prof. J. E., Ph.D. (Cornell) Head of Dept., Madison Coll., Harrisonburg, Va.
INGALLS, Asst. Prof. E. E., Ph.D. (Michigan) Albion Coll., Albion, Mich. *1111 Michigan Ave.*

INGRAHAM, Dean M. H., Ph.D. (Chicago) Coll. of Letters and Science, Univ. of Wisconsin, Madison, Wis.
INNIS, MARY E., A.M. (Smith) Head of Math. Dept., The Baldwin School, Bryn Mawr, Pa.
IRR, E. J. 3064 Lincoln Blvd., Cleveland Heights, Ohio
IVANOFF, V. F., A.M. (California) 777 Seventh Ave., San Francisco, Calif.

JABLONOWER, JOSEPH, Pd.M. (New York Univ.) Memb. Board of Examiners of the City of New York, New York, N.Y. *110 Livingston St., Brooklyn 2, N.Y.*
JACKSON, Prof. DUNHAM, Ph.D. (Göttingen) Univ. of Minnesota, Minneapolis, Minn. *119 Folwell Hall*
JACKSON, H. B., A.B. (Harvard) Head of Dept., Belmont Hill School, Belmont, Mass. *56 Smith Road, Milton, Mass.*
JACKSON, Prof. J. B., A.M. (Columbia) Univ. of South Carolina, Columbia, S.C. *227 S. Waccamaw Ave., Columbia 48, S.C.*
JACKSON, Prof. ROSA L., Ph.D. (Chicago) Head of Dept., Alabama Coll., Montevallo, Ala.
JACKSON, T. W., A.M. (Missouri) Head of Dept., Jamestown Coll., Jamestown, N.D. *729 Fifth St. N.E.*
JACOBSON, N. L., B.S. in Educ. (Oregon) Instr., Central Missouri State Teachers Coll., Warrensburg, Mo.
JAEGER, Prof. C. G., Ph.D. (Missouri) Pomona Coll., Claremont, Calif. *Army Univ. Center in Italy*
JAMES, Asso. Prof. GLENN, Ph.D. (Columbia) Univ. of California at Los Angeles, Los Angeles, Calif.
JAMES, Prof. R. D., Ph.D. (Chicago) Univ. of British Columbia, Vancouver, B.C., Can. *Dept. of Math.*
JAMISON, FREE, Ph.D. (Pittsburgh) Teacher, Public Schools; Lecturer, Univ. of Pittsburgh, Pittsburgh, Pa. *Lt., U.S.N.R., Navy Pre-flight School, St. Mary's College, Calif.*
JANES, Asst. Prof. W. C., A.M. (Nebraska) Kansas State Coll., Manhattan, Kans.
JASON, Prof. W. B., A.M. (Pennsylvania) Dean, Lincoln Univ., Jefferson City, Mo.
JAUTZ, Rev. M. L., M.S. (Marquette) Curate, St. Mary Church, Racine, Wis. *1933 N. 24th St., Milwaukee, Wis.*
JEFFERY, Prof. R. L., Ph.D. (Cornell) Queen's Univ., Kingston, Ont., Can.
JEFFERIES, J. B., M.S. (Chicago) Jr. Physicist, Metallurgical Lab., Univ. of Chicago, Chicago 37, Ill. *6332 S. Eberhart Ave.*
JENKINS, Asso. Prof. E. D., Ph.D. (Ohio State) Eastern Kentucky State Teachers Coll., Richmond, Ky. *Lt., U.S.N.R. 1405 Howard Ave., Utica, N.Y.*
JENNINGS, Asso. Prof. S. A., Ph.D. (Toronto) Univ. of British Columbia, Vancouver, B.C., Can.
JENSEN, C. M., Ph.D. (Minnesota) Lecturer, Univ. of Minnesota, Minneapolis, Minn. *3315-17th Ave. S., Minneapolis 7, Minn.*
JERBERT, Asso. Prof. A. R., Ph.D. (Washington) Univ. of Washington, Seattle, Wash.
JOFFE, S. A., M.S. (New York Univ.) Actuary, Retired, Mutual Life Ins. Co., New York, N.Y. *515 West 110 St., New York 25, N.Y.*
JOHANSON, Asst. Prof. R. N., Ph.D. (Chicago) Boston Univ., 688 Boylston St., Boston, Mass.
JOHN, Asso. Prof. FRITZ, Ph.D. (Göttingen) Univ. of Kentucky, Lexington, Ky. *Appl. Math. Group, New York Univ., 53 Washington Sq. S., New York, N.Y.*
JOHN, Prof. F. W., M.E. (Cornell) Washington Square Coll., New York Univ., New York, N.Y.
JOHNSON, Prof. A. FRANCES, Ph.D. (Minnesota) Math. and Physics, Rockford Coll., Rockford, Ill.
JOHNSON, A. L., Jr., A.B. (Nebraska Wesleyan Univ.) Secretary, The Crete Mills, Crete, Nebr.
JOHNSON, D. F., M.S. (Middlebury) Instr., Belmont Hill School, 350 Prospect St., Belmont 78, Mass.
JOHNSON, Asso. Prof. EVAN, Jr., Ph.D. (Chicago) Pennsylvania State Coll., State College, Pa. *345 S. Buckhout St.*
JOHNSON, E. H., Ph.D. (Michigan) *Analyst, War Food Admin., Washington, D.C. 3939 Pennsylvania Ave. S.E., Washington 20, D.C.*
JOHNSON, L. G., A.M. (Michigan) Mathematician, Chevrolet Gear and Axle, Detroit, Mich. *1677 Glendale Ave., Detroit 6, Mich.*
JOHNSON, O. H., M.S. (Iowa) *Elec. Engr., Signal Corps. Philadelphia Signal Corps Procurement District. 421 S. 15th St., Philadelphia 46, Pa.*
JOHNSON, Prof. R. A., Ph.D. (Harvard) Brooklyn Coll., Bedford Ave. and Avenue H, Brooklyn, N.Y.

JOHNSON, Asst. Prof. R. E., Ph.D. (Wisconsin) Mount Holyoke Coll., South Hadley, Mass.
JOHNSON, Asso. Prof. ROBERTA F., Ph.D. (Cornell) Wilson Coll., Chambersburg, Pa.
JOHNSON, Asso. Prof. R. P., Ph.D. (Pittsburgh) Carnegie Inst. of Tech., Pittsburgh, Pa. *Dept. of Math.*
JOHNSON, Prof. W. L., A.M. (Univ. of Tennessee) Head of Dept., Mississippi Southern Coll., Hattiesburg, Miss. *200 N. 25th Ave.*
JOHNSON, W. W. Instr., Huntington Poly. Inst., Cleveland, Ohio. *10722 Orville Ave., Cleveland 6, Ohio*
JOHNSTON, ERNEST, M.S. (Illinois) Asso. Mech. Engr., Naval Ordnance Lab., Washington, D.C. *3352 Brothers Place S.E., Washington 20, D.C.*
JOHNSTON, Prof. F. E., Ph.D. (Illinois) George Washington Univ., Washington 6, D.C.
JOHNSTON, Prof. L. S., A.M. (Missouri) Univ. of Detroit, Detroit 21, Mich. *227 Engineering Bldg.*
JOLIAT, Prof. J. S., M.S. (St. Louis Univ.) John Carroll Univ., Cleveland Heights, Ohio
JONAH, F. C., Ph.D. (Brown) Engr., Chance Vought Aircraft, Stratford, Conn. *381 Laughlin Road*
JONES, Prof. B. W., Ph.D. (Chicago) Cornell Univ., Ithaca, N.Y. *White Hall*
JONES, Prof. E. F. W., C.E. (Manhattan Coll.) Rollins Coll., Winter Park, Fla.
JONES, Prof. HARRIS, B.S. (Mass. Inst. of Tech.) Colonel, U.S. Army, U.S. Military Acad., West Point, N.Y.
JONES, Asst. Prof. MARGARET E., A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *164–13th Ave., Columbus 1, Ohio*
JONES, Prof. Olive M., A.M. (Columbia) Head of Dept., Queens Coll., Charlotte, N.C.
JONES, P. C., B.S. (Mass. Inst. of Tech.) Science Editor, Bell Telephone Labs., 463 West St., New York 14, N.Y.
JONES, PHILLIP S., A.M. (Michigan) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich. *315 W. Engineering Bldg.*
JONES, Asst. Prof. R. W., A.M. (Pennsylvania) Univ. of Delaware, Newark, Del. *121 Townsend Road*
JORDAN, Asso. Prof. H. E., Ph.D. (Chicago) Univ. of Kansas, Lawrence, Kans. *1600 Kentucky St.*
JORDAN, H. J. *Warrant Officer (j.g.), Army Air Force. 1415–45th St. W., Birmingham 8, Ala.*
JURDAK, Prof. M. H., M.A. (Amer. Univ. of Beirut) American Univ. of Beirut, Beirut, Syria
JUSTICE, Prof. H. K., Ph.D. (Cincinnati) Coll. of Eng. and Commerce, Univ. of Cincinnati, Cincinnati, Ohio. *927 Kreis Lane*
JUSTIN, E. M., M.S. (Case School) Instr., Engg., Univ. of California at Los Angeles, Los Angeles, Calif. *17138 Ventura Blvd., Encino, Calif.*

KAC, Asst. Prof. MARK, Ph.D. (John Casimir Univ., Lwow) Cornell Univ., Ithaca, N.Y. *Dept. of Math.*
KAELIN, G. R., A.M. (California) Instr., Los Angeles City Coll., 855 N. Vermont Ave., Los Angeles, Calif.
KALISCH, G. K., Ph.D. (Chicago) Instr., Cornell Univ., Ithaca, N.Y. *Dept. of Math.*
KALISH, AIDA, A.M. (Columbia) Fellow, Bryn Mawr Coll., Bryn Mawr, Pa. *Radnor Hall*
KALLFELZ, L. N., B.S. (Syracuse) *In Service*
KALTENBORN, Asso. Prof. H. S., Ph.D. (Michigan) Univ. of Idaho, Moscow, Idaho. *413 E. Seventh St.*
KAPLAN, E. L., B.S. (Carnegie Inst. of Tech.) *Asst. Physicist, Naval Ordnance Lab., Navy Yard, Washington 25, D.C.*
KAPLAN, SIDNEY, A.M. (Brooklyn Coll.) *Statistician, War Production Board, Washington, D.C. 2821–28th St. S.E., Washington 20, D.C.*
KAPLAN, Asst. Prof. WILFRED, Ph.D. (Harvard) Univ. of Michigan, Ann Arbor, Mich. *1308 Olivia Ave.*
KAPLANSKY, IRVING, Ph.D. (Harvard) Instr., Univ. of Chicago, Chicago, Ill. *Dept. of Math.*
KARAPETOFF, Prof. VLADIMIR, M.M.E. (Leningrad) Emeritus, Elec. Eng., Cornell Univ., Ithaca, N.Y. *39 Claremont Ave., Apt. 84, New York, N.Y.*
KARL, Sister MARY CORDIA, Ph.D. (Johns Hopkins) Head of Dept., Coll. of Notre Dame of Maryland, N. Charles St., Baltimore 10, Md.
KARLIN, MEYER, Ph.D. (Columbia) Instr., Brooklyn Coll. Evening Session, Brooklyn, N.Y. *1804 East 4th St., Brooklyn 23, N.Y.*
KARLIN, SAMUEL, B.S. (Ill. Inst. of Tech.) Grad. Asst., Illinois Inst. of Tech., Chicago, Ill. *3904 W. 18th St., Chicago 23, Ill.*
KARNES, Asso. Prof. H. T., Ph.D. (Peabody) Dean of Men, Louisiana State Univ., University Sta., Baton Rouge 3, La.
KARNOW, HERMAN, A.M. (Colorado) Teacher, Brooklyn Tech. High School, Brooklyn, N.Y. *210 Sixth Ave., New York 14, N.Y.*

KARPINSKI, Prof. L. C., Ph.D. (Strassburg) Univ. of Michigan, Ann Arbor, Mich. *1315 Cambridge Road*
KARR, Asso. Prof. LOIS, A.M. (Wisconsin) Lindenwood Coll., St. Charles, Mo.
KARST, O. J., M.S. (New York Univ.) Instr., Stevens Inst. of Tech., Hoboken, N.J. *136 Melrose Ave., Arlington, N.J.*
KASNER, Prof. EDWARD, Ph.D. (Columbia) Columbia Univ., New York, N.Y. *430 West 116 St., New York 27, N.Y.*
KATZ, MAX. Accountant, Katz, Zuckerman and Co., 50 Broad St., New York, N.Y.
KAUFMAN, NORBERT. *S 2/c, U.S. Navy.* 6930 Sheridan Road, Chicago 26, Ill.
KEARNEY, Asst. Prof. DORA E., A.M. (Minnesota) State Teachers Coll., Cedar Falls, Iowa. *1105 W. 23rd St.*
KECK, WINFIELD, A.M. (Pennsylvania) Instr., Physics, Muhlenberg Coll., Allentown, Pa.
KEELER, Prof. B. C., A.M. (Columbia) Webb Inst. of Naval Archit., New York, N.Y. *48 Sagamore Road, Bronxville, N.Y.*
KEFFER, RALPH, A.M. (Wisconsin) Actuary, Aetna Life Ins. Co., Hartford 15, Conn.
KELLER, E. R., A.M. (Tennessee) Teacher, Central High School, Fountain City, Tenn. *Route 3, Maryville, Tenn.*
KELLER, Asst. Prof. M. W., Ph.D. (Indiana) Purdue Univ., West Lafayette, Ind. *138 Sheetz St.*
KELLEY, J. L., Ph.D. (Virginia) Inst. for Advanced Study, Princeton, N.J. *R.D. 2*
KELLS, Asso. Prof. L. M., Ph.D. (Columbia) U.S. Naval Acad., Annapolis, Md. *23 Thompson St.*
KELLY, Asso. Prof. K. D., M.S. (Chicago) Fenn Coll., Cleveland 15, Ohio
KELLY, L. M., A.M. (Boston Univ.) *Lt., U.S.C.G.R.*, Instr., U.S. Coast Guard Acad., New London, Conn. *Dept. of Math.*
KELLY, MAY B., A.M. (Yale) Teacher, Bulkeley High School, Hartford, Conn. *93 Newport Ave., West Hartford 7, Conn.*
KEMPNER, Prof. A. J., Ph.D. (Göttingen) Head of Dept., Univ. of Colorado, Boulder, Colo.
KENDALL, Prof. CLARIBEL, Ph.D. (Chicago) Univ. of Colorado, Boulder, Colo. *1305 Euclid Ave.*
KENDALL, C. H., A.B. (Union Coll.), LL.B. (Buffalo) *Head Attorney, War Production Board, Washington, D.C. 3426–16th St. N.W.*
KENNA, Sister ESTHER MARIA, A.M. (St. Elizabeth) Prof., Coll. of St. Elizabeth, Convent Station, N.J.
KENNEDY, Asso. Prof. E. C., Ph.D. (Rice) Texas Coll. of Arts and Indus., Kingsville, Tex. *Major, Air Corps. Proof Division, Eglin Field, Fla.*
KENNEDY, E. S., Ph.D. (Lehigh) Instr., Univ. of Alabama, University, Ala. *238 Lincoln St., Easton, Pa.*
KENNEDY, EVELYN M., A.M. (Cincinnati) Instr., Univ. of Maryland, College Park, Md. *5911-16th St. N.W., Washington 11, D.C.*
KENNEDY, MADELINE (Mrs. C. C.), A.B. (Marquette) Instr., Marquette Univ., Milwaukee, Wis. *1443 N. 53rd St., Milwaukee 8, Wis.*
KENNEY, Asst. Prof. J. F., A.M. (Michigan) Univ. of Wisconsin at Milwaukee, 623 W. State St., Milwaukee 3, Wis.
KENNISON, Asst. Prof. L. S., Ph.D. (Calif. Inst. of Tech.) Brooklyn Coll., Bedford Ave. and Ave. H, Brooklyn, N.Y. *Lt. Comdr., U.S.N.R. 110-11 215 St., Queens Village 9, N.Y.*
KENT, J. R. F., M.A. (Queen's Univ.) Instr., Univ. of Arkansas, Fayetteville, Ark. *Lt. Comdr., R.C.N.V.R., Asst. Dir. of Naval Educ., Naval Service Hqs., Ottawa, Ont., Can.*
KEPPLER, KATHARINE B. (Mrs. Kurt), A.M. (Bryn Mawr) Head of Dept., Foxcroft School, Middleburg, Va.
KERR, F. J. Merrick St., Pleasantville, Pa.
KETCHUM, Asso. Prof. P. W., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *803 W. Illinois St.*
KEYFITZ, NATHAN, B.S. (McGill Univ.) Statistician, Dominion Bureau of Statistics, Ottawa, Ont., Can.
KIEFER, Prof. E. C., M.S. (Michigan) James Millikin Univ., Decatur 24, Ill.
KILLEN, Prof. C. G., M.S. (Louisiana State) Northwestern State Coll., Natchitoches, La. *429 Henry Ave.*
KIMBALL, Prof. S. H., Ph.D. (Harvard) Head of Dept., Univ. of Maine, Orono, Me. *135 Stevens Hall*
KIMBALL, T. C., A.M. (Princeton) Master, Lawrenceville School, Lawrenceville, N.J. *Box 40*
KINGSTON, Prof. H. R., Ph.D. (Chicago) Head of Dept., Math. and Astr., Univ. of Western Ontario, London, Ont., Can.
KINNEY, J. M., Ph.D. (Chicago) Emeritus, Wilson Jr. Coll., Chicago, Ill. *8058 Bennett Ave., Chicago 17, Ill.*
KINSMAN, BLAIR, B.S. (Chicago) Instr., Lake Forest Acad., Lake Forest, Ill.

KIPLINGER, C. C., M.S. (Ohio State) Chm., Chem. Dept., West Liberty State Coll., West Liberty, W.Va.
KIRBY, A. R., A.M. (Columbia) Lecturer, Math. and Commerce, School of Educ., Fordham Univ., New York 7, N.Y. *2921 Briggs Ave., New York 58, N.Y.*
KIRCHNER, Prof. W. H., B.S. (Worcester Poly. Inst.) Emeritus, Univ. of Minnesota, Minneapolis, Minn. *722 Tenth Ave. S.E., Minneapolis 14, Minn.*
KIRKHAM, W. J., Ph.D. (Indiana) *In Service*
KLAUBER, L. M., A.B. (Stanford) Vice-Pres. and Genl. Mgr., San Diego Gas and Elec. Co., San Diego, Calif. *233 W. Juniper St., San Diego 1, Calif.*
KLEENE, Asso. Prof. S. C., Ph.D. (Princeton) Amherst Coll., Amherst, Mass. *Lt., U.S.N.R. Naval Research Lab., Anacostia, Washington, D.C.*
KLEWANSKY, JULIUS, A.B. (Brooklyn) *Pvt. 877 Empire Blvd., Brooklyn, N.Y.*
KLINE, Prof. J. R., Ph.D. (Pennsylvania) Univ. of Pennsylvania, Philadelphia, Pa. *529 Riverview Road, Swarthmore, Pa.*
KLINE, MORRIS, Ph.D. (New York Univ.) Instr., New York Univ., New York, N.Y. *Engr., Signal Corps Engg. Labs. 1064 E. Fifth St., Brooklyn 30, N.Y.*
KLINGER, E. L., A.M. (Illinois) Instr., Purdue Univ., West Lafayette, Ind. *126 North St.*
KLIPHARDT, R. A., B.S. (Armour Inst. of Tech.) Instr., Civil Engg., Northwestern Univ., Evanston, Ill. *5116 N. Glenwood St., Chicago 40, Ill.*
KLIPPLE, Asso. Prof. E. C., Ph.D. (Texas) A. and M. Coll. of Texas, College Station, Tex.
KNEBELMAN, Prof. M. S., Ph.D. (Princeton) Head of Dept., State Coll. of Washington, Pullman, Wash. *2008 Indiana St.*
KNEDLER, Prof. P. A., A.M. (Pennsylvania) State Teachers Coll., Kutztown, Pa. *East Texas, Pa.*
KNIGHT, Asst. Prof. L. C., A.M. (Columbia) Emeritus, Coll. of Wooster, Wooster, Ohio. *1028 N. Bever St.*
KNIGHT, Asst. Prof. L. C., Jr., A.M. (Kent) Muskingum Coll., New Concord, Ohio. *156 Lakeside Drive*
KNOWLER, Asso. Prof. L. A., Ph.D. (Iowa) Univ. of Iowa, Iowa City, Iowa. *212 Physics Bldg.*
KNOX, Prof. J. J., M.S. (Chicago) Dakota Wesleyan Univ., Mitchell, S.D. *1219 W. University Blvd.*
KNOX, Asst. Prof. R. H., Jr., A.M. (Michigan) Virginia Milit. Inst., Lexington, Va. *313 Letcher Ave.*
KOCH, E. H., Jr., B.S. in E.E. (Pennsylvania) Retired. 253 Cumberland St., Brooklyn 5, N.Y.
KOCH, R. J. *A.S., U.S. Navy. NROTC, Willis Sweet Hall, Univ. of Idaho, Moscow, Idaho*
KOEHLER, Asst. Prof. FULTON, Ph.D. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *Dept. of Math. and Mech.*
KOEHLER, Asst. Prof. T. L., A.M. (Pennsylvania) Muhlenberg Coll., Allentown, Pa. *625 N. 24th St.*
KOKEN, J. C., A.B. (Missouri) Asst. Instr., Univ. of Missouri, Columbia, Mo. *1318 Ross St.*
KOLCHIN, E. R., Ph.D. (Columbia) *In Service*
KOLEY, Sister VINCENT DE PAUL, M.S. (Notre Dame) Head of Dept., Mary Manse Coll., Toledo, Ohio
KOPP, P. J., A.M. (Duke) Jr. Executive, Colgate-Palmolive-Peet Co., Jersey City, N.J. *Major, Chem. Warfare Service, Washington, D.C. 1305 N. Adams St., Arlington, Va.*
KORGEN, Asso. Prof. R. L., Ph.D. (Harvard) Bowdoin Coll., Brunswick, Me. *Prince's Point Road*
KOVARIK, Prof. A. F., Ph.D. (Minnesota), Sc.D. (Manchester) Physics, Yale Univ., New Haven, Conn. *Sloane Lab.*
KRABILL, D. M., Ph.D. (Ohio State) *Lt., U.S.N.R. U.S. Naval Acad., Annapolis, Md. Dept. of Math.*
KRAITCHIK, Asso. Prof. MAURICE, Dr.Math. (Liege; Brussels) New School for Social Research, New York, N.Y. *34-10 84th St., Jackson Heights, N.Y.*
KRALL, Asso. Prof. H. L., Ph.D. (Brown) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
KRAMER-LASSAR, EDNA E., Ph.D. (Columbia) Chm. of Dept., Thomas Jefferson High School, Brooklyn, N.Y. *Consultant, N.D.R.C. 32 Lenox Road, Brooklyn 26, N.Y.*
KRATHWOHL, Prof. W. C., Ph.D. (Chicago) Illinois Inst. of Tech., 3300 Federal St., Chicago 16, Ill.
KRAUS, G. R., M.S. (Carnegie Inst. of Tech.) Head of Dept., Math. and Physics, Gannon Coll. of Villa Maria Coll., Erie, Pa. *619 W. Ninth St.*
KRAVETZ, SAUL. Student, Harvard Univ., Cambridge, Mass. *3072 Brighton 13 St., Brooklyn 24, N.Y.*

KREIDER, Dean O. C., M.S. (Iowa State) Ellsworth Jr. Coll., Iowa Falls, Iowa. *Lt., U.S.N.R. 672 Mulberry St., Galesburg, Ill.*
KRONSBEIN, JOHN, Ph.D. (Leipzig) Research Engr., Hoosier-Cardinal Corp., Evansville, Ind.; Consultant, Illinois Tool Works, Chicago, Ill. *Claremont Apt. Hotel, Apt. 304, Locust Ave. and 2nd St., Evansville, Ind.*
KRUEGER, Asso. Prof. R. L., Ph.D. (Marquette) Head of Dept., Wittenberg Coll., Springfield, Ohio
KRYLOFF, Prof. NICOLAS, Dr. of Math. honoris causa (Kieff) Acad. of Sciences, Book Dept., Ul. Kropotkina 16, Moscow 34, USSR
KUBIS, Asst. Prof. J. F., Ph.D. (Fordham) Grad. School, Fordham Univ., New York, N.Y.
KUHN, Rev. B. J., A.M. (St. Bonaventure) Head of Dept. ,Siena Coll., Loudonville, N.Y.
KUHN, Prof. H. W., Ph.D. (Cornell) Emeritus, Ohio State Univ., Columbus, Ohio. *1179 Fairview Ave.*
KULLBACK, SOLOMON, Ph.D. (George Washington) Lecturer in Stat., George Washington Univ., Washington, D.C. *Lt. Col., Signal Corps. 1259 Van Buren St. N.W., Washington 12, D.C.*
KUNKEL, Prof. P. V., Ph.D. (Columbia) Head of Dept., Cedar Crest Coll., Allentown, Pa. *Box 104, Trexlertown, Pa.*
KURZIN, W. H., M.S. (Chicago) Instr., Herzl Jr. Coll., Chicago, Ill. *1631 W. Roosevelt Road, Chicago 8, Ill.*
KUTMAN, Mrs. HELEN L., A.M. (Columbia) Instr., Hunter Coll., New York, N.Y. *2825 Grand Concourse*

LABOCCETTA, ING. LETTERIO, C.E. (Naples) Patent Atty., Via S. Basilio 50, Rome (105), Italy
LADNER, Asst. Prof. A. C., A.M. (Brown) Math. and Engg., Denison Univ., Granville, Ohio. *P.O. Box 253*
LADUE, MARY E., Ph.D. (Columbia) Instr., Barnard Coll., Broadway and 119 St., New York 27, N.Y.
LADY, C. H., A.M. (Southern California) State Teachers Coll., Slippery Rock, Pa.
LAFON, Asst. Prof. J. E., A.M. (Oklahoma) Univ. of Oklahoma, Norman, Okla. *806 Monnett St.*
LAGERSTROM, P. A., Ph.D. (Princeton) Research Aerodynamicist, Bell Aircraft Corp., Niagara Falls, N.Y. *Engg. Dept.*
LALONDE, R. P. Lab. Asst., R.C.A. Victor Co., Montreal, P.Q., Can. *3910 Mentana St.*
LAMB, Prof. R. C., A.M. (Virginia) U.S. Naval Acad., Annapolis, Md. *Dept. of Math.*
LAMBERT, W. D., A.M. (Harvard) Chief, Sec. of Gravity and Astr., U.S. Coast and Geodetic Survey, Washington 25, D.C.
LAMPEN, Prof. A. E., A.M. (Michigan) Hope Coll., Holland, Mich. *86 E. 14th St.*
LAMPLAND, C. O., A.M. (Indiana) Lowell Observatory, Flagstaff, Ariz. *Bin 1640*
LANCASTER, Asst. Prof. O. E., Ph.D. (Harvard) Univ. of Maryland, College Park, Md. *Lt. Comdr., U.S.N.R. 4607-27th St., Mt. Rainier, Md.*
LANCKTON, A. L., A.M. (Duke) Socony Vacuum Oil Co., Istanbul, Turkey. *Box 660*
LANCZOS, Prof. CORNELIUS, Ph.D. (Szeged) Purdue Univ., Lafayette, Ind.
LANDERS, A. W., Ph.D. (Chicago) Instr., Brooklyn Coll., Brooklyn, N.Y. *Lt., U.S.N.R. 3760-39th St. N.W., Washington 16, D.C.*
LANDERS, MARY K. (Mrs. A. W.), Ph.D. (Chicago) Instr., Hunter Coll., New York, N.Y. *3760-39th St. N.W., Washington 16, D.C.*
LANDIN, JOSEPH, M.S. (New York Univ.) Instr., Univ. of Notre Dame, Notre Dame, Ind. *Dept. of Math.*
LANDRY, Prof. A. E., Ph.D. (Johns Hopkins) Catholic Univ. of America, Washington, D.C. *3624-13th St. N.E., Brookland, D.C.*
LANE, Prof. E. P., Ph.D. (Chicago) Chm. of Dept., Univ. of Chicago, Chicago 37, Ill *Dept. of Math.*
LANE, Prof. H. I., Ph.D. (Cornell) Hendrix Coll., Conway, Ark.
LANE, N. D., B.A. (Queen's Univ.) Head of Dept., St. Andrew's Coll., Aurora, Ont., Can.
LANE, Asso. Prof. RUTH O., Ph.D. (Iowa) Northwest Missouri State Teachers Coll., Maryville, Mo.
LANG, G. B., Ph.D. (Illinois) Instr., Univ. of Florida, Gainesville, Fla. *231 Ray St.*
LANGE, LUISE, Ph.D. (Göttingen) Wilson Jr. Coll., Chicago, Ill. *5851 Blackstone Ave.*
LANGER, Prof. R. E., Ph.D. (Harvard) Univ. of Wisconsin, Madison, Wis. *822 Miami Pass, Madison 5, Wis.*
LAPAZ, Prof. LINCOLN, Ph.D. (Chicago) Head of Dept., Univ. of New Mexico, Albuquerque, N.M. *Box 190 Univ.*

LAREW, Prof. GILLIE A., Ph.D. (Chicago) Head of Dept., Randolph-Macon Woman's Coll., Lynchburg, Va.
LAROE, RACHAEL A., A.M. (Tennessee) Instr., Mary Hardin-Baylor Coll., Belton, Tex. *Box 594 Baylor Station*
LARRIVEE, J. A., Ph.D. (Catholic Univ.) 2730 Wisconsin Ave. N.W., Washington 7, D.C.
LARSEN, Asso. Prof. H. D., Ph.D. (Wisconsin) Univ. of New Mexico, Albuquerque, N.M.
LARSON, Asso. Prof. OLGA, A.M. (Missouri) State Coll. for Women, Tallahassee, Fla.
LASHIER, Asst. Prof. H. M., A.M. (Southern California) Math. and Physics, Pacific Union Coll., Angwin, Calif.
LASLEY, Prof. J. W., Jr., Ph.D. (Chicago) Univ. of North Carolina, Chapel Hill, N.C. *523 E. Rosemary Lane*
LATIMER, Prof. C. G., Ph.D. (Chicago) Univ. of Kentucky, Lexington, Ky. *Army Univ. Center in England. 1112 Fontaine Road, Lexington, Ky.*
LATSHAW, ELMER. Tech. Engr., ACF-Brill Motors Co., Philadelphia, Pa. *1839 N. 60th St.*
LATSHAW, V. V., Ph.D. (Indiana) *Mathematician, N.A.C.A., Cleveland Airport. 4589 West 174th St., Cleveland 11, Ohio*
LAURENTINE MARIE, Sister, A.M. (Trinity Coll., Wash., D.C.) Head of Dept., Emmanuel Coll., 400 The Fenway, Boston 15, Mass.
LAUSMAN, ELLA E., A.M. (Michigan) Instr., Western Michigan Coll., Kalamazoo, Mich. *907 W. Lovell St., Kalamazoo 46, Mich.*
LAVOIE, E. S., A.B. (Brooklyn Coll.) Patrolman, New York, N.Y. *Lt. (j.g.) Asst. Shore Patrol Officer, Atlanta, Ga. 135 Sheridan Ave., Brooklyn, N.Y.*
LAY, L. C., A.M. (Southern California) Design Engr., Lockheed Aircraft Corp., Burbank, Calif. *2615 Sichel St., Los Angeles 31, Calif.*
LAYTON, W. I., M.S. (South Carolina) Head of Dept., Amarillo Coll., Amarillo, Tex.
LAZAR, NATHAN, Ph.D. (Columbia) Teacher, Midwood High School, Brooklyn, N.Y. *1055 Carroll St., Brooklyn 25, N.Y.*
LEAVENS, D. H., A.M. (Yale) Research Asso., Cowles Comm. for Research in Econ., Univ. of Chicago, Chicago 37, Ill.
LEE, Asst. Prof. H. L., Ph.D. (Duke) Univ. of Tennessee, Knoxville, Tenn. *Box 4052 Univ. Sta.*
LEE, MARY A., A.M. (Wisconsin) Grad. Asst., Cornell Univ., Ithaca, N.Y. *White Hall*
LEECH, J. S., M.S. (Oklahoma) Instr., Yale Univ., New Haven, Conn. *179 Dwight St., Apt. 408*
LEFSCHETZ, Prof. SOLOMON, Ph.D. (Clark) Chm. of Dept., Princeton Univ., Princeton, N.J. *Fine Hall*
LEHMAN, Mrs. MARGARET B., A.M. (U.C.L.A.) 2315-24th St., Santa Monica, Calif.
LEHMANN, C. H., A.M. (Columbia) Instr., Cooper Union, New York, N.Y. *144-17 29th Ave., Flushing, N.Y.*
LEHMER, Asso. Prof. D. H., Ph.D. (Brown) Univ. of California, Berkeley, Calif. *Mathematician, Ballistics Research Lab., Aberdeen Proving Ground, Md.*
LEHNER, JOSEPH, Ph.D. (Pennsylvania) Mathematician, Kellex Corp., 233 Broadway, New York, N.Y. *Room 1482*
LEHR, Asso. Prof. MARGUERITE, Ph.D. (Bryn Mawr) Bryn Mawr Coll., Bryn Mawr, Pa. *Cartref*
LEIFER, H. R., A.B. (Pittsburgh) 5500 Hays St., Pittsburgh, Pa.
LEIGHTON, WALTER, Ph.D. (Harvard) Lecturer, Rice Inst., Houston, Tex. *Director, Applied Math. Group, Northwestern Univ., Evanston, Ill. 1704 Judson Ave., Evanston, Ill.*
LELEIKO, MAX, B.S. (New York Univ.) Instr., Rutgers Univ., New Brunswick, N.J.
LEMME, Asso. Prof. M. M., A.M. (Indiana) Univ. of Toledo, Toledo, Ohio. *Lt., U.S.N.R.*
LENNAHAN, C. M., M.S. (Mass. Inst. of Tech.) Meteorologist, U.S. Weather Bureau, Washington 25, D.C.
LEONARD, Prof. H. B., Ph.D. (Colorado) Univ. of Arizona, Tucson, Ariz. *Box 4024 Univ. Sta.*
LEONE, F. C., M.S. (Georgetown Univ.) Instr., Purdue Univ., West Lafayette, Ind. *Ensign, U.S.N.R.*
LESTER, CAROLINE A., Ph.D. (Wisconsin) Instr., New York State Coll. for Teachers, Albany, N.Y. *Lt., U.S. Coast Guard Women's Reserve.*
LESTOURGEON, Asso. Prof. F. ELIZABETH, Ph.D. (Chicago) Univ. of Kentucky, Lexington, Ky.
LEVEQUE, W. J., A.M. (Cornell) Grad. Asst., Cornell Univ., Ithaca, N.Y. *422 Chestnut St.*
LEVINE, EMANUEL, Ed.M. (Rutgers) Teacher, Gen. Sci., High School, Freeport, N.Y.
LEVINE, Asso. Prof. JACK, Ph.D. (Princeton) North Carolina State Coll., Raleigh, N.C. *Box 5548. In Service*

LEVINE, L. D., A.B. (Brooklyn) *Pvt., U.S. Army. 376 Hinsdale St., Brooklyn 7, N.Y.*
LEVY, Asst. Prof. HARRY, Ph.D. (Princeton) Univ. of Illinois, Urbana, Ill. *358 Math. Bldg.*
LEVY, HERMAN, A.B. (New York Univ.) *Physicist, Naval Air. Exp. Station. 1139 N. 41st St., Philadelphia 4, Pa.*
LEWIS, Prof. A. J., Ph.D. (Colorado) Univ. of Denver, Denver 10, Colo. *2136 S. Josephine St.*
LEWIS, Asst. Prof. ANNE L., Ph.D. (Chicago) Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *Dept. of Math.*
LEWIS, Asso. Prof. C. F., M.S. (Kansas State) Kansas State Coll., Manhattan, Kans.
LEWIS, C. J., M.S. (Georgetown Univ.) Theological Student, Weston Coll., Weston, Mass.
LEWIS, Asso. Prof. D. C., Jr., Ph.D. (Harvard) Univ. of New Hampshire, Durham, N.H. *Research Mathematician, N.D.R.C. 401 West 118 St., New York 27, N.Y.*
LEWIS, Prof. F. A., Ph.D. (Ohio State) Head of Dept., Univ. of Alabama, University, Ala. *Box 1444*
LEWIS, Prof. FLORENCE P., Ph.D. (Johns Hopkins) Goucher Coll., Baltimore, Md. *2435 N. Charles St., Baltimore 18, Md.*
LEWIS, J. H., A.M. (Washington & Jefferson) *Research Mathematician, N.D.R.C., Columbia Univ., New York, N.Y. 400 West 118 St., New York 27, N.Y.*
LIEBER, LILLIAN R. (Mrs. H. G.), Ph.D. (Clark) Dir., Galois Inst. of Math., Long Island Univ., Brooklyn, N.Y. *258 Clinton Ave., Brooklyn 5, N.Y.*
LIEBOWITZ, BENJAMIN, A.B. (Brooklyn) *Capt., A.C. Radar Officer.*
LINARES, ENRIQUE, Jr., B.S. in E.E. (Univ. of Santa Clara) Gen. Agt., Cerveceria Nacional, David, R.P. *P.O. Box 540, Panama City, R.P.*
LINDQUIST, Prof. THEODORE, Ph.D. (Chicago) Head of Dept., State Normal Coll., Ypsilanti, Mich. *103 Elm St.*
LINEHAN, Prof. P. H., Ph.D. (Columbia) Coll. of the City of New York, New York, N.Y. *924 West End Ave., New York 25, N.Y.*
LINFIELD, Asso. Prof. B. Z., Ph.D. (Harvard) Univ. of Virginia, University, Va. *1324 Hilltop Road, Charlottesville, Va.*
LINSCHEID, Asst. Prof. H. W., A.M. (Oklahoma) Eastern New Mexico Coll., Portales, N.M.
LIOLIOS, LEO. Student, Washington Univ., St. Louis, Mo. *3001 Olive St.*
LIPSICH, H. D., A.M. (Cincinnati) Jr. Fellow, Princeton Univ., Princeton, N.J. *Grad. College*
LITTAUER, Prof. S. B., Ph.D. (Mass. Inst. of Tech.) Chm. of Dept., Newark Coll. of Engineering, Newark 2, N.J.
LITTERICK, W. S., Ed.D. (Rutgers) Chm. of Dept.; Dir. of Studies, The Peddie School, Hightstown, N.J. *340 S. Main St.*
LITZINGER, Prof. MARIE, Ph.D. (Chicago) Mount Holyoke Coll., South Hadley, Mass.
LIVINGSTON, Prof. G. R., A.M. (California) Head of Dept., San Diego State Coll., San Diego, Calif. *4641-55th St., San Diego 5, Calif.*
LLOYD, OLWEN (Mrs. George), M.A. (Cambridge, England) Headmistress, Mount Vernon Seminary, 4340 Fordham Road, Washington 16, D.C. *3301 University Ave.*
LOCH, JOSEPH. *Cpl., U.S. Army. 2027 N. Sheffield Ave., Chicago 14, Ill.*
LOCKE, Prof. J. F., Ph.D. (Illinois) Head of Dept., Memphis State Coll., Memphis, Tenn. *Lt., U.S.N.R., U.S. Naval Acad., Annapolis, Md. Dept. of Math.*
LOCKWOOD, E. C., A.M. (Brown) Head of Dept., Mount Hermon School, Mount Hermon, Mass.
LOCKWOOD, NEIL, A.B. (Baker Univ.) Operator, Red River Broadcast Co., Duluth, Minn. *703 S. 65th Ave. W.*
LOFLIN, Prof. Z. L., M.S. (Louisiana) Acting Head of Dept., Southwestern Louisiana Inst., Lafayette, La. *Box 353 S.L.I.*
LOGSDON, Mrs. MAYME I., Ph.D. (Chicago) Asso. Prof., Univ. of Chicago, Chicago, Ill.
LONDON, LIONEL, B.S.C.E. (Illinois) Analyst, A. O. Smith Corp., Milwaukee, Wis. *1315 E. Elmdale Court, Milwaukee 11, Wis.*
LONG, Asso. Prof. FLORENCE, M.S. (Illinois) Earlham Coll., Earlham, Ind.
LONG, Asst. Prof. T. R., A.M. (Rochester) Univ. of Rochester, Rochester, N.Y.
LONGENECKER, J. V., M.S. (Iowa) Actuary, Farmers and Bankers Life Ins. Co., Wichita, Kans. *Box 580*
LONGFELLOW, NATALIE M., A.M. (Chicago) 47 East 64 St., New York 21, N.Y.
LONGLEY, Prof. W. R., Ph.D. (Chicago) Yale Univ., New Haven, Conn. *305 Lawrence St.*
LONSETH, Asst. Prof. A. T., Ph.D. (California) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
LORING, S. J., B.S. (Mass. Inst. of Tech.) Research Engr., Chance Vought Aircraft, Stratford, Conn. *29 Stiles St.*
DE LOSADA Y PUGA, Prof. CRISTÓBAL, D.Sc. (Lima) Catholic Univ. of Peru, Lima, Peru. *Apartado 2708*
LOTKIN, MARK, Ph.D. (Kiel, Germany) *Mathematician, Aberdeen Proving Ground, Ballistic Res. Lab., Computing Branch, Philadelphia, Pa. 3719 Locust St., Philadelphia, 4 Pa.*

LOVE, Prof. C. E., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *1915 Scottwood Ave.*
LOVETT, Pres. E. O., Ph.D. (Virginia; Leipzig) Retired, Rice Inst., Houston 1, Tex. *P.O. Box 1892*
LOVITT, Prof. W. V., Ph.D. (Chicago) Colorado Coll., Colorado Springs, Colo. *1927 Wood Ave.*
LOWENSTEIN, Prof. L. L., Ph.D. (Cornell) Chm. of Dept., Alfred Univ., Alfred, N.Y. *Box 292*
LUBBEN, Asso. Prof. R. G., Ph.D. (Texas) Univ. of Texas, Austin, Tex. *W.H. 16, Univ. Station*
LUBIN, Asso. Prof. C. I., Ph.D. (Harvard) Univ. of Cincinnati, Cincinnati, Ohio. *3612 Washington Ave.*
LUCKEY, Asso. Prof. R. R. R., Ph.D. (Cornell) Math. and Physics, Houghton Coll., Houghton, N.Y.
LUGINBUHL, R. T., B.S. (Ursinus) *S 1/c (FC). 112 Broadway, Malverne, N.Y.*
LUIPPOLD, R. C., A.M. (Buffalo) Instr., Univ. of Buffalo, Buffalo, N.Y. *111 Sussex Ave., Buffalo 15, N.Y.*
LUKACS, ELIZABETH C. (Mrs. Eugene), Teacher's Certif. (Vienna) Head, Sci. Dept., Hillsdale School, Cincinnati, Ohio. *337 McGregor Ave., Cincinnati 19, Ohio*
LUNDBERG, G. H., A.M. (Vanderbilt) Instr., Vanderbilt Univ., Nashville, Tenn. *2310 Highland Ave.*
LUNN, Prof. A. C., Ph.D. (Chicago) Emeritus, Univ. of Chicago, Chicago, Ill. *5211 Kenwood Ave.*
LUTEYN, Asst. Prof. PETER, M.S. (Iowa) Marquette Univ., Milwaukee, Wis. *8223 Portland Ave., Wauwatosa, Wis.*
LUTHER, Prof. C. F., Ph.D. (Stanford) Dean, Coll. of Liberal Arts, Willamette Univ., Salem, Ore.
LUTHER, Asso. Prof. H. A., Ph.D. (Iowa) A. and M. Coll. of Texas, College Station, Tex. *Research Engr., Westinghouse Research Labs., Ardmore Blvd., Pittsburgh, Pa.*
LYLE, Asso. Prof. G. A., M.S. (Lehigh) U.S. Naval Acad., Annapolis, Md. *227 Wardour Drive*

MACCOLL, L. A., Ph.D. (Columbia) Member of Tech. Staff, Bell Telephone Labs., 463 West St., New York 14, N.Y.
MACCREADIE, Asso. Prof. W. T., Ph.D. (Cornell) Bucknell Univ., Lewisburg, Pa. *104-13th St.*
MACCULLOUGH, Prof. R. H., M.S. (Lafayette) Defiance Coll., Defiance, Ohio
MACDONALD, A. W., A.M. (N.J. State T.C., Montclair) Supervisor, Foreman Trg. Dept., Curtiss-Wright Corp., Caldwell, N.J. *217 Greylock Parkway, Belleville 9, N.J.*
MACDONALD, Prof. S. L., A.M. (Columbia) Emeritus, Math. and Astr., Colorado State Coll. of A. and M.A., Fort Collins, Colo. *513 Rotary St., Suncrest, Morgantown, W.Va.*
MACDONALD, Prof. W. E., A.M. (Harvard) Lignan Univ., Canton, China
MACDOUGAL, Prof. H. B., M.S. (Iowa) Acting Head of Dept., South Dakota State Coll., Brookings, S.D.
MACDUFFEE, Prof. C. C., Ph.D. (Chicago) Univ. of Wisconsin, Madison, Wis. *1815 Summit Ave., Madison 5, Wis.*
MACEWEN, Asst. Prof. D. M., Ph.D. (New York Univ.) Coll. of the City of New York, New York, N.Y. *55 East 65 St.*
MACKIE, Prof. E. L., Ph.D. (Chicago) Dean of Men, Univ. of North Carolina, Chapel Hill, N.C. *702 Gimghoul Road*
MAC LANE, Asso. Prof. SAUNDERS, Ph.D. (Göttingen) Harvard Univ., Cambridge, Mass. *7 Avon St., Cambridge 38, Mass.*
MACNEILLE, Prof. H. M., Ph.D. (Harvard) Kenyon Coll., Gambier, Ohio. *1 Euclid Ave., Summit, N.J.*
MAC NEISH, Prof. H. F., Ph.D. (Chicago) Chm. of Dept., Brooklyn Coll., Brooklyn, N.Y. *48 Taunton Road, Scarsdale, N.Y.*
MACPHAIL, Prof. M. S., Ph.D. (Oxford) Acadia Univ., Wolfville, N.S., Can.
MADDAUS, INGO, Jr., Ph.D. (Michigan) *Staff Member, Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass. 53 High St., Natick, Mass.*
MADDEN, Sister TERESA MARIE, A.M. (Boston Coll.) Prof., Coll. of Our Lady of the Elms, Chicopee, Mass.
MADDOX, Prof. A. C., A.M. (Columbia) Northwestern State Coll., Natchitoches, La. *405 New Second St.*
MADDRILL, J. D., Ph.D. (California) *Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.*

MADISON, Asst. Prof. M. L., M.S. (Colorado) Colorado State Coll. of A. and M.A., Fort Collins, Colo. *Dept. of Math.*
MAHONEY, Asst. Prof. J. J., Ph.D. (California) Loyola Univ., Chicago, Ill. *1st Lt., A.U.S., Air Corps*
MALIN, Sister M. MECHTILDIS, M.S. (Marquette) Head of Dept., Viterbo Coll., LaCrosse, Wis.
MALLORY, Prof. A. E., Ph.D. (George Peabody Coll.) Colorado State Coll. of Educ., Greeley, Colo.
MALLORY, Prof. V. S., Ph.D. (Columbia) Head of Dept., State Teachers Coll., Montclair, N.J.
MALONEY, Sister MARY MICHAEL, A.M. (Catholic Univ.) Head of Dept., D'Youville Coll., 320 Porter Ave., Buffalo 1, N.Y.
MANCHESTER, Prof. R. E., A.M. (Michigan) Head of Dept., Dean of Men, Kent State Univ., Kent, Ohio. *208 N. Lincoln St.*
MANCILL, Asso. Prof. J. D., Ph.D. (Chicago) Univ. of Alabama, University, Ala.
MANDEL, JOHN, M.S. (Brussels) Research Chemist, The BG Corp., New York, N.Y. *45 Kew Gardens Road, Kew Gardens, N.Y.*
MANDELBROJT, SZOLEM, Dr.ès Sc. (Paris) Address unknown
MANNING, Prof. F. L., Ph.D. (Cornell) Ursinus Coll., Collegeville, Pa. *68 Sixth Ave.*
MANNING, Asso. Prof. H. P., Ph.D. (Johs Hopkins) Emeritus, Brown Univ., Providence, R.I. *148 Governor St., Providence 6, R.I.*
MANNING, M. L., M.S. (Pittsburgh) Asst. Chief Engr., Kuhlman Elec. Co., Bay City, Mich. *Engg. Dept.*
MANSFIELD, RALPH, M.S. (Chicago) Research Engr., Jos. Weidenhoff, Inc., Chicago 24, Ill. *8049 S. Maryland Ave., Chicago 19, Ill.*
MANSON, Prof. E. S., M.S. (Mass. Inst. of Tech.) Astr., Ohio State Univ., Columbus, Ohio. *Observatory*
MANY, ANNA E., A.M. (Tulane) Counselor to Women, Sophie Newcomb Coll., New Orleans 18, La.
MAPLE, C. G., A.M. (Cincinnati) *Ensign, U.S.N.R. 3440-38th St. N.W., Apt. E-413, Washington 16, D.C.*
MARBURGER, CLIFFORD, A.M. (Franklin and Marshall) Instr., Franklin and Marshall Acad., Lancaster, Pa. *Denver, Pa.*
MARCH, Prof. H. W., Ph.D. (Munich) Univ. of Wisconsin, Madison, Wis. *Chief Mathematician, U.S. Forest Products Lab. 1825 Summit Ave., Madison 5, Wis.*
MARCOU, R. J., B.S. (Colby) Math. Physicist, Raytheon Mfg. Co., Waltham, Mass.
MARDEN, Asso. Prof. MORRIS, Ph.D. (Harvard) Univ. of Wisconsin, Milwaukee, Wis. *403 E. Carlisle Ave., Milwaukee 11, Wis.*
MARIA, D. MAY HICKEY (Mrs. A. J.), Ph.D. (Rice) Brooklyn Coll., Brooklyn, N.Y. *Dept. of Math.*
MARM, Prof. ANNA, A.M. (Kansas) Bethany Coll., Lindsborg, Kans. *741 N. Second St.*
MARQUIS, Prof. R. H., Ph.D. (Chicago) Ohio Univ., Athens, Ohio. *Box 216*
MARRIOTT, Prof. R. W., Ph.D. (Pennsylvania) Swarthmore Coll., Swarthmore, Pa. *213 Lafayette Ave.*
MARTH, ELLA, Ph.D. (St. Louis Univ.) Teacher, Hadley Tech. High School, St. Louis, Mo. *5421a Clarence Ave., St. Louis 15, Mo.*
MARTIN, JEROME, Ph.D. (California) Research Dir., Commercial Solvents Corp., Terre Haute, Ind. *1334 S. Center St.*
MARTIN, Prof. M. H., Ph.D. (Johns Hopkins) Chm. of Dept., Univ. of Maryland, College Park, Md.
MARTIN, Asst. Prof. MARGARET P., Ph.D. (Minnesota) Biostatistics, School of Public Health, Univ. of Minnesota, Minneapolis, Minn. *Millard Hall*
MARTIN, W. A., A.M. (Alabama) *Lt., U.S.N.R.*
MARTIN, Prof. W. T., Ph.D. (Illinois) Chm. of Dept., Syracuse Univ., Syracuse 10, N.Y. *Dept. of Math.*
MARTYN, W. J., M.A. (New Zealand) Mathematical Master, Otago Boys' High School, Dunedin, New Zealand
MARY DANIEL, Sister, Ph.D. (Fordham) Prof., Marywood Coll., Scranton, Pa.
MARY ESTHER, Sister (Gladys-Mary E. Campbell), A.M. (California) Instr., Mundelein Coll., 6363 Sheridan Road, Chicago, Ill.
MARY FELICE, Sister, Ph.D. (Catholic Univ.) Prof., Mount Mary Coll., Milwaukee 13, Wis.
MARY GERTRUDE, Sister, M.S. (Marquette) Instr., St. Clare Coll., 3221 S. Lake Drive, Milwaukee 7, Wis.
MARY OF MERCY, Sister (Fitzpatrick), A.M. (Catholic Univ.) Instr., Incarnate Word Coll., San Antonio 2, Tex.

MARY PAULA, Sister, M.S. (Notre Dame) Prof., Marygrove Coll., 8425 W. McNichols Road, Detroit 21, Mich.
MASON, Asso. Prof. S. L., A.M. (Michigan) Univ. of North Dakota, Grand Forks, N.D. *2205 Fourth Ave. N.*
MASON, Asso. Prof. W. E., M.E. (Michigan) Engg., Univ. of California at Los Angeles, 405 Hilgard Ave., Los Angeles 24, Calif.
MASSEY, Asst. Prof. W. L., A.M. (Duke) Univ. of Chattanooga, Chattanooga, Tenn.
MATHEWSON, Prof. L. C., Ph.D. (Illinois) Dartmouth Coll., Hanover, N.H.
MATHIAS, Asst. Prof. H. R., A.M. (Indiana) Bowling Green State Univ., Bowling Green, Ohio. *123 E. Evers Ave.*
MAUCH, MARGARET, Ph.D. (Chicago) Univ. of Akron, Akron 4, Ohio
MAY, LIDA B., A.M. (Texas) Instr., Texas Tech. Coll., Lubbock, Tex. *2409-14th St.*
MAYOR, Prof. J. R., Ph.D. (Wisconsin) Southern Illinois Normal Univ., Carbondale, Ill.
MAYS, W. J., A.M. (Vanderbilt) Volunteer State Life Ins. Co., Chattanooga, Tenn. *507½ W. Fourth St., Chattanooga 3, Tenn.*
McBRIEN, Asst. Prof. V. O., Ph.D. (Catholic Univ.) Coll. of the Holy Cross, Worcester, Mass.
McCAMMAN, CAROL V., A.M. (California) Teacher, McKinley High School, Washington, D.C. *3500-14th St. N.W., Apt. 106, Washington 10, D.C.*
McCARTHY, ANNE S., A.M. (Boston Coll.) 32 Auburn St., Brookline, Mass.
McCARTHY, Asst. Prof. E. D., A.M. (Pennsylvania State) Eng. Coll., Univ. of Detroit, Detroit, Mich.
McCARTHY, J. J., M.S. (New York Univ.) *Capt., U.S. Army. 128 Marine Ave., Brooklyn 9, N.Y.*
McCARTY, A. L., A.M. (California) Retired, Instr., Jr. Coll., San Francisco, Calif. *Route 1, Box 481, Merced, Calif.*
McCLELLAN, ADA A., B.S. (Chicago) Teacher, High School, Long Beach, Calif. *313 N. New Hampshire Ave., Los Angeles, Calif.*
McCLELLAND, J. N., A.M. (Drake) Member, Res. Staff, Naval Ordnance Test Station, Inyokern, Calif. *Res. and Dev. Dept.*
McCLENON, Prof. R. B., Ph.D. (Yale) Grinnell Coll., Grinnell, Iowa. *Army Univ. Center in England*
McCOLGIN, GLADYS B. (Mrs. O. R.), Ed.M. (Harvard), A.M. (Radcliffe) 1556 Brookside Ave., Indianapolis, Ind.
McCONNELL, H. J. Engr., U.S. Engineer Office, Los Angeles, Calif. *3328 Madison Ave., San Diego 4, Calif.*
McCORMICK, Asso. Prof. C. T., Ph.D. (Indiana) Illinois State Normal Univ., Normal, Ill.
McCOY, DOROTHY, Ph.D. (Iowa) Head of Dept., Belhaven Coll., Jackson, Miss.
McCOY, Prof. N. H., Ph.D. (Iowa) Smith Coll., Northampton, Mass. *53 Ridgewood Terrace*
McCREA, Prof. W. H., Ph.D. (Cambridge) Univ. of London, Royal Holloway Coll., Englefield Green, Surrey, England
McCULLEY, W. S., M.S. (A. and M. Coll. of Texas) Instr., A. and M. Coll. of Texas, College Station, Tex. *Lt. Col., Chemical Warfare Service, A.A.F. Box 279 Faculty Exchange, College Station, Tex.*
McCUTCHEON, G. J., B.S. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn. *90 Malcolm Ave. S.E., Minneapolis 14, Minn.*
McDANIEL, Asso. Prof. W. C., Ph.D. (Wisconsin) Southern Illinois Normal Univ., Carbondale, Ill.
McDILL, Prof. R. M., A.M. (Indiana) Head of Dept., Hastings Coll., Hastings, Nebr. *129 E. Seventh St.*
McDONALD, SOPHIA LEVY (Mrs. J. H.), Ph.D. (California) Asso. Prof., Univ. of California, Berkeley, Calif. *2818 Webster St., Berkeley 5, Calif.*
McDONOUGH, D. L., Ph.D. (Pennsylvania) Head of Dept., South Philadelphia High School for Boys, Philadelphia, Pa. *1205 Belfield Ave., Drexel Hill, Pa.*
McDOUGLE, EDITH A., A.B. (Delaware) Instr., Women's Coll., Univ. of Delaware, Newark, Del.
McEWEN, Prof. G. F., Ph.D. (Stanford) Scripps Institution of Oceanography, La Jolla, Calif. *P.O. Box 109*
McEWEN, Prof. W. H., Ph.D. (Minnesota) Mount Allison Univ., Sackville, N.B., Can.
McEWEN, Asst. Prof. W. R., A.M. (Minnesota) Univ. of Minnesota, Minneapolis, Minn. *Dept. of Math. and Mech.*
McFADDEN, Asso. Prof. LEONARD, Ph.D. (Brown) Virginia Poly. Inst., Blacksburg, Va. *Box 65*
McFARLAN, Asso. Prof. L. H., Ph.D. (Missouri) Univ. of Washington, Seattle, Wash.

McFarland, Asso. Prof. Dora, Ph.D. (Chicago) Univ. of Oklahoma, Norman, Okla. *Faculty Exchange*
McFarland, Elsie J., Ph.D. (California) Instr., Head of Dept., Boise Jr. Coll., Boise, Idaho
McGavock, Prof. W. G., Ph.D. (Duke) Davidson Coll., Davidson, N.C. *On leave. Visiting Prof., Emory Univ., Emory University, Ga. Box 456*
McGaw, Prof. F. M., B.S. (Wesleyan Univ.) Emeritus, Math. and Eng. Drawing, Cornell Coll., Mt. Vernon, Iowa
McGrath, Rev. P. H., A.M. (Woodstock Coll.) Church of St. Ignatius Loyola, 980 Park Ave., New York 28, N.Y.
McInnis, Asst. Prof. S. W., A.M. (Florida) Univ. of Florida, Gainesville, Fla. *1417 W. McCormick St.*
McKelvey, Prof. J. V., Ph.D. (Cornell) Iowa State Coll., Ames, Iowa. *2117 Graeber St.*
McKelvey, Martha M. (Mrs. J. V.), M.S. (Iowa) 2117 Graeber St., Ames, Iowa
McKenna, Mary, A.M. (Columbia) Teacher, Washington Irving High School, New York, N.Y. *17 East 11 St., New York 3, N.Y.*
McKenney, Rev. J. L., A.M. (Manhattan) Prof., Providence Coll., Providence, R.I.
McLarren, Robert. Tech. Dir., Public Relations, Aeronautical Chamber of Commerce of America, 610 Shoreham Bldg., Washington 5, D.C.
McLaughlin, K. F., A.M. (Yale) *Lt. (j.g.), U.S.N.R., U.S.N.M.W.T.S., Solomons, Md.*
McLaughlin, W. A., Pet.E. (Colo. Mines) *In Service*
McMahon, F. A., A.M. (Brooklyn Coll.) *Lt., A.U.S. Fire Control Section, Ordnance School, Aberdeen Proving Ground, Md.*
McMaster, Asst. Prof. A. S., M.S. in C.E. (Colorado) Eng. Math., Univ. of Colorado, Boulder, Colo. *College of Eng.*
McMurtrie, K. A., B.S. (Brown) *Lt., U.S. Army Signal Corps. 246 Ohio Ave., Providence, R.I.*
McNair, J. S., M.S. (Chicago) Instr., Canal Zone Jr. Coll., Balboa Heights, Canal Zone. *Box 933, Balboa, Canal Zone*
McNeal, R. L., B.S. in E.E. (Purdue) Head of Tech. Data Section, General Motors Proving Ground, Milford, Mich. *P.O. Box 338*
McNeary, S. S., A.M. (Pennsylvania) Instr., Drexel Inst., Philadelphia 4, Pa. *127 Grandview Road, Ardmore, Pa.*
McNeil, Sister Marie Gertrude, M.S. (Notre Dame) Prof., Seton Hill Coll., Greensburg, Pa.
McNelly, Sister Presentation, A.M. (Catholic Univ.) Prof., Our Lady of the Lake Coll., San Antonio, Tex.
McShane, Prof. E. J., Ph.D. (Chicago) Univ. of Virginia, Charlottesville, Va. *209 Maury Ave.*
McSweeny, Prof. A. A., A.M. (Montana) Head of Dept., John Tarleton Agric. Coll., Stephenville, Tex. *Box 368, Tarleton Station, Tex.*
Mead, L. V., A.M. (Ohio State) Instr., Univ. of Florida, Gainesville, Fla. *345 W. McCormick St.*
Mead, Mrs. Sallie P., A.M. (Columbia) Member Tech. Staff, Bell Telephone Labs., 463 West St., New York, N.Y.
Meade, Asst. Prof. Mary E., A.M. (Virginia) Hollins Coll., Hollins College, Va. *Box 34*
Mears, Prof. Florence M., Ph.D. (Cornell) George Washington Univ., Washington 6, D.C.
Mebane, W. N., Jr., A.M. (Cornell) Davidson Coll., Davidson, N.C. *Research Scientist, S.A.M. Lab., New York, N.Y. 100 Morningside Drive, New York 27, N.Y.*
Meder, A. E., Jr., A.M. (Columbia) Dean of Admin., Rutgers Univ., New Brunswick, N.J.
Mehlenbacher, Asso. Prof. L. E., Ph.D. (Michigan) Univ. of Detroit, McNichols Road at Livernois, Detroit 21, Mich. *Dept. of Math.*
Mehr, Emanuel, A.M. (Brooklyn) 1418 President St., Brooklyn, N.Y.
Melville, Prof. C. E., A.B. (Northwestern) Clark Univ., Worcester, Mass. *950 Main St., Worcester 3, Mass.*
Mendelsohn, N. S., Ph.D. (Toronto) Lecturer, Queen's Univ., Kingston, Ont., Can. *Dept. of Math.*
Menger, Prof. Karl, Ph.D. (Vienna) Univ. of Notre Dame, Notre Dame, Ind. *1421 E. La Salle St., South Bend 17, Ind.*
Menke, Asst. Prof. H. E., A.M. (Ohio State) Heidelberg Coll., Tiffin, Ohio. *200 Jefferson St.*
Mercedes, Sister M., A.B. (Coll. of St. Scholastica) Registrar, Coll. of St. Scholastica, Duluth 2, Minn.
Mercedes, Sister M., M.S. (Notre Dame) Mary Manse Coll., Toledo, Ohio. *2413 Collingwood Ave., Toledo 10, Ohio*
Mergendahl, Prof. T. E., M.S. (Tufts) Tufts Coll., Tufts College, Mass. *128 Professor's Row*

MERRILL, Prof. A. S., Ph.D. (Chicago) Montana State Univ., Missoula, Mont.
MERRILL, Prof. HELEN A., Ph.D. (Yale) Emeritus, Wellesley Coll., Wellesley, Mass. *6 Roanoke Road, Wellesley 81, Mass.*
MERRILL, L. L., Ph.D. (Rensselaer) Mathematician, Stromberg-Carlson Co., Rochester, N.Y. *35 Elmcroft Road, Rochester 9, N.Y.*
MERRIMAN, Asso. Prof. G. M., Ph.D. (Cincinnati) Univ. of Cincinnati, Cincinnati 21, Ohio
MERRISS, A. A. 4227 N.E. 23rd Ave., Portland 11, Ore.
MERTIE, J. B., Jr., Ph.D. (Johns Hopkins) Senior Geologist, U.S. Geological Survey, Washington, D.C.
MESSICK, C. A., Ph.D. (Chicago) Acting Prof., Missouri Valley Coll., Marshall, Mo. *1100 Tyler St., Topeka, Kans.*
MESSICK, Prof. J. F., Ph.D. (Johns Hopkins) Emory Univ., Emory University, Ga.
MEWBORN, Asst. Prof. A. B., Ph.D. (Calif. Inst. of Tech.) Univ. of Arizona, Tucson, Ariz. *Lt., U.S.N.R., Naval Air Station, Vero Beach, Fla. P.O. Box 1026*
MEYER, H. A., Ph.D. (Iowa) Acting Prof., Indiana Univ., Bloomington, Ind. *420 Eastside Drive*
MEYER, H. L., Jr., M.S. (Chicago) 86-16th St., Wheeling, W.Va.
MICHAEL, W. B., M.S. (Southern California) Lecturer, Univ. of Southern California, Los Angeles, Calif. *388 S. Oak Ave., Pasadena 8, Calif.*
MICHAL, Prof. A. D., Ph.D. (Rice) California Inst. of Tech., Pasadena 4, Calif.
MICHALUP, ERIC. Actuary, Compania Nac. Anonima de Seguros, La Previsora, Apartado 848, Caracas, Venezuela
MICHEL, R. J., Ph.D. (Missouri) Head of Dept., Southeast Missouri State Teachers Coll., Cape Girardeau, Mo. *225 N. Ellis St.*
MICHIE, Prof. J. N., A.M. (Michigan) Head of Dept., Texas Tech. Coll., Lubbock, Tex. *Box 91, Tech. Sta.*
MICKELSON, E. L., Ph.D. (Minnesota) *Lt. Col., Coast Artillery Corps, 414 Fifth Ave. S.E., Minneapolis 14, Minn.*
MICKLE, Asst. Prof. E. J., Ph.D. (Ohio State) Ohio State Univ., Columbus 10, Ohio. *Dept. of Math.*
MIDDLEMISS, Prof. R. R., M.S. (Colorado) Washington Univ., St. Louis 5, Mo.
MIKESH, J. S., A.B. (Minnesota) Head of Dept., Lawrenceville School, Lawrenceville, N.J. *Box 268*
MIKSA, F. L. Switchman, Illinois Bell Telephone Co., Aurora, Ill. *613 Spring St.*
MILES, Asso. Prof. E. J., Ph.D. (Chicago) Yale Univ., New Haven, Conn. *87 Marvel Road*
MILES, E. P., Jr., A.M. (Duke) Instr., North Carolina State Coll., Raleigh, N.C. *Lt. (SC), U.S.N.R. Supply Officer, NATTC, Ward Island, Corpus Christi, Tex.*
MILES, Asst. Prof. H. J., Ph.D. (California) Univ. of Illinois, Urbana, Ill. *253 Math. Bldg.*
MILGRAM, Asso. Prof. A. N., Ph.D. (Pennsylvania) Univ. of Notre Dame, Notre Dame, Ind.
MILKMAN, JOSEPH, A.M. (Brooklyn) Tutor, Coll. of the City of New York, New York, N.Y. *299 Adelphi St., Brooklyn 5, N.Y.*
MILLER, A. L., Ph.D. (Harvard) Treas. and Genl. Mgr., C. N. Miller Co., Boston, Mass. *100 Massachusetts Ave.*
MILLER, Prof. C. E., Ph.D. (Toronto) Univ. of New Brunswick, Fredericton, N.B., Can.
MILLER, D. D., Ph.D. (Michigan) *Lt., U.S.N.R. 1028 Connecticut Ave. N.W., Apt. 215, Washington, D.C.*
MILLER, Prof. E. B., A.M. (Chicago) Illinois Coll., Jacksonville, Ill. *1252 W. College Ave.*
MILLER, E. D., A.B. (Stanford) Instr., Yuba Jr. Coll., Marysville, Calif.
MILLER, Prof. F. H., Ph.D. (Columbia) Head of Dept., Cooper Union, Cooper Square, New York 3, N.Y.
MILLER, Prof. G. A., Ph.D. (Cumberland) Emeritus, Univ. of Illinois, Urbana, Ill. *1203 W. Illinois St.*
MILLER, Asst. Prof. G. T., Ph.D. (Purdue) Purdue Univ., Lafayette, Ind. *601 Dodge St., West Lafayette, Ind.*
MILLER, K. W., B.S. in E.E. (Union Coll.) Chm., Elec. Engg., Armour Research Foundation, Illinois Inst. of Tech., Chicago 16, Ill. *10431 S. Oakley Ave., Chicago 43, Ill.*
MILLER, LEONARD, A.M. (Brooklyn) *T/3, Inf., U.S. Army. 1726 Union St., Brooklyn 13, N.Y.*
MILLER, M. H., B.S. (Anderson Coll., Ind.) Instr., Anderson Coll. and Theolog. Seminary, Anderson, Ind.
MILLER, Prof. NORMAN, Ph.D. (Harvard) Queen's Univ., Kingston, Ont., Can.
MILLER, Asst. Prof. W. I., Ph.D. (Pittsburgh) Bucknell Univ., Lewisburg, Pa. *220 S. Third St.*
MILLER, Rev. W. J., Ph.D. (Harvard) Prof., Woodstock Coll., Woodstock, Md.
MILLER, W. M., Ph.D. (Illinois) Head of Dept., Robert Coll., Istanbul, Turkey
MILLESON, HELEN K., M.S. (Iowa) Statistician, Bureau of the Census, Washington, D.C. *4716 B Hudson Ave. S.E., Suitland Manor, Washington 20, D.C*

MILLINGTON, Asst. Prof. H. G., C.E. (Rensselaer) Univ. of Vermont, Burlington, Vt. *Room 339 Waterman Bldg.*
MILLS, Prof. C. N., Ph.D. (Wisconsin) Illinois State Normal Univ., Normal, Ill. *304 Virginia Ave.*
MILLS, W. H., A.B. (Swarthmore) *RT/2, U.S.N.R. 460 Riverside Drive, New York 27, N.Y.*
MILNE, Prof. W. E., Ph.D. (Harvard) Head of Dept., Oregon State Coll., Corvallis, Ore.
MILO, GASPERINE, B.S. (New River State Coll.) *In Service*
MILOS, J. F., A.M. (Columbia) Teacher, Memorial High School, Pelham, N.Y. *Lt., U.S.N.R. U.S. Naval Acad., Annapolis, Md. Dept. of Math.*
MINO, Rev. P. M., A.B. (St. Francis Coll., Pa.) Teacher, St. Francis Prep. School, Loretto, Pa. *On leave. Grad. Student, Columbia Univ.*
MIRICK, G. R., A.M. (Michigan; Columbia) Asst. Dir., Div. of Teachers College Schools, Horace Mann-Lincoln School, 433 West 123 St., New York, N.Y.
MISER, H. J., M.S. (Ill. Inst. of Tech.) *Operations Analyst, 20th A.F. c/o E. W. Lehman, Route 2, Urbana, Ill.*
MISER, NELLIE P. (Mrs. W. L.), A.B. (Huron Coll.) 1702 Cedar Lane, Nashville, Tenn.
MISER, Prof. W. L., Ph.D. (Chicago) Vanderbilt Univ., Nashville, Tenn. *1702 Cedar Lane*
MITCHELL, Asst. Prof. A. K., Ph.D. (Johns Hopkins) Trinity Coll., Hartford, Conn. *56 LeMay St., West Hartford 7, Conn.*
MITCHELL, Prof. B. E., Ph.D. (Columbia) Millsaps Coll., Jackson, Miss.
MODE, Prof. E. B., A.M. (Harvard) Boston Univ., Boston, Mass. *9 Longmeadow Road, Wellesley 81, Mass.*
MOELLER, A. C., B.S. (Western Reserve) Instr., Marquette Univ., Milwaukee 8, Wis.
MOELLER, E. L., B.S. (St. Ambrose) *In Service*
MOISE, E. E., A.B. (Tulane) 2045 Walnut St., Boulder, Colo.
MOLINA, E. C. Retired. *Scientific Staff, N.D.R.C., New York, N.Y. 141 Dodd St., East Orange, N.J.*
MONTAGUE, Asso. Prof. HARRIET F., Ph.D. (Cornell) Univ. of Buffalo, Buffalo 14, N.Y.
MONTGOMERY, Prof. DEANE, Ph.D. (Iowa) Smith Coll., Northampton, Mass. *N.D.R.C., Fine Hall, Princeton, N.J.*
MOODY, Prof. W. A., A.M. (Bowdoin) Emeritus, Bowdoin Coll., Brunswick, Me. *60 Federal St.*
MOON, Asso. Prof. PARRY, M.S. (Mass. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge, Mass.
MOORE, Asst. Prof. B. C., A.M. (Princeton) A. and M. Coll. of Texas, College Station, Tex. *Lt. Comdr., U.S.N.R.*
MOORE, Prof. C. N., Ph.D. (Harvard) Dir., Grad. studies in math., Univ. of Cincinnati, Cincinnati, Ohio. *219 Woolper Ave.*
MOORE, D. C., A.M. (Emory) Instr., Emory Jr. Coll., Oxford, Ga.
MOORE, DEWEY, A.B. (Berea Coll.) *Pvt., U.S. Army*
MOORE, Dean E. F., A.M. (Texas) Head of Dept., Hannibal-LaGrange Coll., Hannibal, Mo. *College Heights*
MOORE, Prof. F. C., A.B. (Dartmouth) Massachusetts State Coll., Amherst, Mass.
MOORE, Asso. Prof. G. E., Ph.D. (Illinois) Univ. of Illinois, Urbana, Ill. *55 E. Chalmers St., Champaign, Ill.*
MOORE, LILLIAN, Ed.M. (New York Univ.) Teacher, Far Rockaway High School, New York, N.Y. *885 N. 28th St., Philadelphia 30, Pa.*
MOORE, Asso. Prof. L. T., Ph.D. (Johns Hopkins) Brooklyn Coll., Brooklyn, N.Y. *205 Hicks St.*
MOORE, Asst. Prof. M. G., Ph.D. (Illinois) Bradley Poly. Inst., Peoria, Ill. *503 W. Melbourne St.*
MOORE, Prof. R. L., Ph.D. (Chicago) Univ. of Texas, Austin, Tex.
MOORE, Asst. Prof. T. W., Ph.D. (Yale) U.S. Naval Acad., Annapolis, Md. *Dept. of Math.*
MOORE, Prof. W. A., A.M. (Chicago) Birmingham-Southern Coll., Birmingham, Ala.
MOORE, Asso. Prof. W. L., Ph.D. (Illinois) Univ. of Louisville, Louisville, Ky. *R.F.D. 1, Box 428, Coral Ridge, Ky.*
MOORMAN, Asst. Prof. R. H., Ph.D. (Peabody) Tennessee Poly. Inst., Cookeville, Tenn.
MOOTS, Prof. E. E., Ph.D. (Iowa) Cornell Coll., Mt. Vernon, Iowa. *710 Eighth Ave. N.*
MORAN, C. W., Ph.D. (Illinois) Teacher, Lane Tech. School, Chicago, Ill. *2950 Jarlath Ave., Chicago 45, Ill.*
MORENO, Prof. JENARO, Grad. (Inst. Pedag.) Inst. Pedagogico, Santiago, Chile
MORENUS, Prof. EUGENIE M., Ph.D. (Columbia) Sweet Briar Coll., Sweet Briar, Va.
MORGAN, AGNES L., A.M. (Columbia) Teacher, Brackenridge High School, San Antonio, Tex. *232 Florida St., San Antonio 3, Tex.*
MORGAN, EDITH L. (Mrs. Joseph), A.B. (Texas Christian) Teaching Fellow, Texas Christian Univ., Fort Worth 9, Tex. *2106 Forest Park Blvd., Fort Worth 4, Tex.*

MORGAN, F. M., Ph.D. (Cornell) Headmaster, Clark School, Hanover, N.H. *Part-time teaching at Dartmouth Coll.*
MORGAN, W. D. 1764 St. Anthony Ave., St. Paul 4, Minn.
MORIN, Sister M. PRUDENTIA, A.B. (Catholic Univ.) Coll. of St. Scholastica, Duluth 2, Minn. *Dept. of Math.*
MORLEY, Prof. R. K., Ph.D. (Clark) Worcester Poly. Inst., Worcester, Mass. *43 Laconia Road, Worcester 5, Mass.*
MORREL, Asst. Prof. J. S., Ph.D. (Illinois) Vanderbilt Univ., Nashville, Tenn. *Lt., U.S.N.R.*
MORRILL, Asst. Prof. W. K., Ph.D. (Johns Hopkins) Johns Hopkins Univ., Baltimore 18, Md.
MORRIS, Prof. C. C., A.M. (Harvard) Emeritus, Ohio State Univ., Columbus, Ohio
MORRIS, Prof. F. R., Ph.D. (California) Head of Dept., Math. and Eng., Fresno State Coll., Fresno 4, Calif.
MORRIS, Asso. Prof. MAX, Ph.D. (Chicago) Case School of Appl. Sci., Cleveland, Ohio
MORRIS, Prof. RICHARD, Ph.D. (Cornell) Emeritus, Rutgers Univ., New Brunswick, N.J. *12 Johnson St., Highland Park, N.J.*
MORRISON, Sister CHARLES MARY, Ph.D. (Catholic Univ.) Dean, Nazareth Coll., Louisville, Ky. *851 S. Fourth St., Louisville 3, Ky.*
MORROW, Asso. Prof. D. C., Ph.D. (Chicago) Wayne Univ., Detroit 1, Mich.
MORSE, Prof. D. S., Ph.D. (Cornell) Union Coll., Schenectady 8, N.Y.
MORSE, Prof. MARSTON, Ph.D. (Harvard) Inst. for Advanced Study, Princeton, N.J. *Tech. Consultant, Office of Chief of Ordnance.*
MORSE, Dean W. P., A.M. (Maine) Ricker Jr. Coll., Houlton, Me. *Lt., U.S.N.R.*
MOSES, IRMA R., A.B. (Cornell) Grad. Asst., Cornell Univ., Ithaca, N.Y. *White Hall*
MOSESSON, Z. I., Ph.D. (Harvard) Prudential Ins. Co. of America, Newark, N.J. *Warrant Officer (j.g.), U.S. Army. P.O. Box 192, Ft. Monroe, Va.*
MOSKOVITZ, Asso. Prof. DAVID, Ph.D. (Brown) Carnegie Inst. of Tech., Pittsburgh 13, Pa.
MOSSMAN, Asst. Prof. THIRZA A., A.M. (Chicago) Kansas State Coll., Manhattan, Kans.
MOSTELLER, F. C., M.S. (Carnegie Inst. of Tech.), A.M. (Princeton) Address unknown
MOSTON, Dean L. T., Ph.D. (Harvard) Prof., Waynesburg Coll., Waynesburg, Pa. *498 Second Ave.*
MOULTON, Prof. E. J., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *1114 Colfax St.*
MOULTON, F. R., Ph.D. (Chicago) Permanent Sec'y, A.A.A.S., Washington, D.C. *Smithsonian Institution Bldg.*
MOURSUND, Prof. A. F., Ph.D. (Brown) Head of Dept., Univ. of Oregon, Eugene, Ore.
MOUZON, Prof. E. D., Jr., Ph.D. (Illinois) Head of Dept., Southern Methodist Univ., Dallas, Tex.
MOYE, Asst. Prof. W. B., M.S. (Mercer Univ.) Georgia Teachers Coll., Collegesboro, Ga.
MUEHLMANN, Rev. PAUL, A.M. (St. Louis Univ.) Prof., West Baden Coll., West Baden, Ind.
MUGGLI, ETHEL C., Ph.D. (Univ. of Washington) St. Joseph, Minn.
MULLAN, C. E., A.M. (Duquesne) Senior Engr., Duquesne Light Co., Planning and Development Dept., 435 Sixth Ave., Pittsburgh 19, Pa.
MULLEMEISTER, Asst. Prof. HERMANCE, Ph.D. (Utrecht, Holland) Univ. of Washington, Seattle 5, Wash. *1900 E. 68th St.*
MULLINGS, Prof. M. E., Ph.D. (Cincinnati) Head of Dept., Abilene Christian Coll., Abilene, Tex. *Field Engr., Radio Div., Western Elec. Co.*
MULLINS, Prof. G. W., Ph.D. (Columbia) Barnard Coll., Columbia Univ., New York, N.Y.
MUMMERY, C. R., B.S. (Toronto) Head, Mechanical and Cleaner Research Sec., The Hoover Co., North Canton, Ohio. *Engineering Dept.*
MUNDHJELD, SIGURD, A.B. (Concordia Coll.) Treas., Concordia Coll., Moorhead, Minn.
MUNRO, W. D., A.M. (Minnesota) Research Engr., Minneapolis-Honeywell Aero., Minneapolis, Minn. *56 Clarence Ave. S.E., Minneapolis 14, Minn.*
MUNROE, FLORENCE L., A.B. (Wellesley) Head of Dept., Retired, High School, Northampton, Mass. *51 Henshaw Ave.*
MUNSHOWER, Prof. C. W., Ph.D. (New York Univ.) Colgate Univ., Hamilton, N.Y.
MURDOCH, Asso. Prof. D. C., Ph.D. (Toronto) Univ. of British Columbia, Vancouver, B. C., Can. *Dept. of Math.*
MURNAGHAN, Prof. F. D., Ph.D. (Johns Hopkins) Johns Hopkins Univ., Baltimore 18, Md.
MURRAY, Prof. C. A., A.M. (Texas) West Texas State Coll., Canyon, Tex. *Box 81*
MURRAY, Asst. Prof. S. B., M.S. (Chicago) Mississippi State Coll., State College, Miss. *Lt. Col.*
MURRAY, V. F., B.S. (St. Andrew's, Scotland) 524 Hudson St., Hoboken, N.J.
MURRAY, Asso. Prof. W. R., M.S. (Cornell) Head of Dept., Franklin and Marshall Coll., Lancaster, Pa.
MUSSELMAN, Prof. J. R., Ph.D. (Johns Hopkins) Chm. of Dept., Western Reserve Univ., Cleveland 6, Ohio
MYERS, W. H., Ph.D. (Stanford) Head of Dept., San Jose State Coll., San Jose, Calif.

NAGLE, J. L. Div. Engr., Public Bldgs. Admin., Chicago, Ill. *1254 W. Columbia Ave., Apt. 1, Chicago 26, Ill.*
NÁPOLES, G. A., Ph.D. (National Univ., Mexico) Dir., Instito de Matematicas, Tacuba 5, Mexico. *Mar Negro 204, Mexico, D.F.*
NASH, Asst. Prof. E. E., M.S. (Rensselaer) Rensselaer Poly. Inst., Troy, N.Y. *Averill Park, N.Y.*
NASH, F. P., A.M. (Columbia) Teacher, Groton School, Groton, Mass.
NASSAU, J. J., Ph.D. (Syracuse) Dir. of Observatory, Case School of Appl. Sci., Cleveland 6, Ohio
NASTUCOFF, P. M., Diploma (Univ. of Moscow) Instr., Univ. of Notre Dame, Notre Dame, Ind. *202 S. Scott St., South Bend 2, Ind.*
NAUL, G. M., B.Ch.E. (Pratt Inst.) Chemist, Plastics Div., General Elec. Co., Pittsfield, Mass. *51 Holmes Road*
NEELLEY, Prof. J. H., Ph.D. (Yale) Carnegie Inst. of Tech., Pittsburgh, Pa. *Army Univ. Center in England. 300 Broadmoor Ave., Pittsburgh 16, Pa.*
NEFF, Prof. I. F., M.S. (Chicago) Emeritus, Drake Univ., Des Moines, Iowa. *2801 Brattleboro Ave.*
NEHRBAS, C. J., A.B. (C.C.N.Y.) Moses, Nehrbas and Tyler, 20 Pine St., New York 5, N.Y.
NEISIUS, W. V., B.S.Ch.E. (Georgia Tech.) Engr., Firestone Tire and Rubber Co., Atlanta, Ga. *597 St. Charles Ave. N.E.*
NELSON, Prof. A. L., Ph.D. (Chicago) Chm. of Dept., Wayne Univ., Detroit 1, Mich. *Army Univ. Center*
NELSON, Prof. C. A., Ph.D. (Chicago) New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
NELSON, Asso. Prof. H. E., Ph.M. (Wisconsin) Head of Dept., Gustavus Adolphus Coll., St. Peter, Minn.
NELSON, Q. E. *In Service*
NELSON, Prof. SARA L., Ph.D. (Cornell) Georgia State Coll. for Women, Milledgeville, Ga. *512 N. Columbia St.*
NELSON, Asso. Prof. W. K., M.S., E.E. (Colorado) Engg. Math., Univ. of Colorado, Boulder, Colo. *925 Grandview Ave.*
NEMÉNYI, Asst. Prof. P. F., D.Sc. (Berlin) State Coll. of Washington, Pullman, Wash. *Dept. of Math.*
NESS, MARIE M., A.M. (Minnesota) Research Asso., Univ. of Minnesota, Minneapolis, Minn. *2530 Dupont Ave. S.*
NEUBAUER, Asst. Prof. GRETA, A.M. (Wyoming) Univ. of Wyoming, Laramie, Wyo. *Engineering Bldg.*
NEUSTADTER, S. F., A.M. (California) Asso., Univ. of California, Berkeley 4, Calif. *Dept. of Math.*
NEVINS, Asst. Prof. W. V., III, A.M. (Columbia) Alfred Univ., Alfred, N.Y. *Box 33*
NEWHOUSE, ALBERT, Ph.D. (Chicago) Instr., Rice Inst., Houston, Tex. *1562 Danville St., Houston 6, Tex.*
NEWSOM, Prof. C. V., Ph.D. (Michigan) Oberlin Coll., Oberlin, Ohio. *174 Forest St.*
NEWSOM, Prof. MARY W., Ph.D. (Göttingen) Emeritus, Eureka Coll., Eureka, Ill. *Lake Dalecarlia, Lowell, Ind.*
NEWTON, Asst. Prof. ABBA V., Ph.D. (Chicago) Vassar Coll., Poughkeepsie, N.Y. *12 Collegeview Ave.*
NEWTON, Prof. G. A., A.M. (Trinity Univ.) Trinity Univ., San Antonio, Tex.
NICHOLLS, R. S., B.Sc. (London) Asst. to Chief Chemist, Miles Laboratories, Inc., Elkhart, Ind. *13 St. Joseph Manor*
NICHOLS, Prof. I. C., Ph.D. (Michigan) Louisiana State Univ., University Sta., Baton Rouge, La. *Nicholson Hall*
NICKOL, Prof. J. P., Ph.D. (Fribourg) Physics, Rensselaer Poly. Inst., Troy, N.Y. *Box 837, Maxwell Road, Newtonville, N.Y.*
NICOLET, JUSTIN. Structural Engr., Dept. of Subways and Highways, Chicago, Ill. *4148 N. Paulina St., Chicago 13, Ill.*
NIERSBACH, P. M., A.M. (Southern California) Chm. of Dept., High School, Bell, Calif. *6925 Passaic St., Huntington Park, Calif.*
NIVEN, Asst. Prof. IVAN, Ph.D. (Chicago) Purdue Univ., Lafayette, Ind.
NOBLE, Prof. C. A., Ph.D. (Göttingen) Emeritus, Univ. of California, Berkeley 4, Calif. *2224 Piedmont Ave.*
NOEL MARIE, Sister, A.M. (Catholic Univ.) Head of Dept., Coll. of Saint Rose, Albany 3, N.Y.
NOLAN, Asst. Prof. IRENE, M.S. (Louisiana) Southwestern Louisiana Inst., Lafayette, La. *508 E. Vermillion St.*
NORDGAARD, Prof. M. A., Ph.D. (Columbia) Head of Dept., Upsala Coll., East Orange, N.J.

NORDHAUS, E. A., Ph.D. (Chicago) Engr., Boeing Aircraft Corp., Seattle, Wash. *1716-28th Ave., Seattle 22, Wash.*
NORDSTROM, C. H., A.M. (Lehigh) Instr., Dartmouth Coll., Hanover, N.H. *9 College St.*
NORIEGA, P. L. Student, St. Vincent Coll., Latrobe, Pa.
NORMAN, P. B., A.M. (California) Instr., Columbia Univ., New York 27, N.Y. *2175 Cedar Ave., New York 53, N.Y.*
NORRIS, Prof. R. E., A.M. (Illinois) Head of Dept., State Teachers Coll., Milwaukee 11, Wis.
NORRIS, W. H., Jr., A.M. (Pennsylvania) 817 Manteo St., Apt. 6, Norfolk 7, Va.
NORTHROP, Asso. Prof. E. P., Ph.D. (Yale) Univ. of Chicago, Chicago, Ill. *5464 Cornell Ave., Chicago 15, Ill.*
NOTLEY, LLEWELLYN, A.M. (Texas) Supt. of Schools, Teague, Tex. *P.O. Box 830*
NOVAK, Asso. Prof. J. D., M.S. (Chicago) Univ. of South Carolina, Columbia, S.C. *Dept. of Math.*
NOWLAN, Prof. F. S., Ph.D. (Chicago) Univ. of British Columbia, Vancouver, B.C., Can.
NOWLAN, MABEL I., M.S. (Michigan) Clerk, War Dept., Washington, D.C. *5810 N.W. First Place, Miami 38, Fla.*
NYSWANDER, Asso. Prof. J. A., Ph.D. (Chicago) Univ. of Michigan, Ann Arbor, Mich.

OAKLEY, Prof. C. O., Ph.D. (Illinois) Chm. of Dept., Haverford Coll., Haverford, Pa.
OBERG, E. N., Ph.D. (Minnesota) Instr., Univ. of Iowa, Iowa City, Iowa. *Army Air Forces Proving Ground, Eglin Field, Fla. Physics Bldg., Univ. of Iowa, Iowa City, Iowa*
OBRIST, Sister JEANETTE, Ph.D. (Catholic Univ.) Instr., Mount St. Scholastica Coll., Atchison, Kans.
O'CALLAHAN, J. T., A.M. (Boston Coll.) *In Service*
O'CONNOR, H. J., M.A. (Toronto) Admin. Asst., Union Carbide and Carbon Res. Labs., Inc., Niagara Falls, N.Y. *1016 Cleveland Ave.*
O'DONNELL, Rev. G. A., Ph.D. (St. Louis Univ.) Dean of Grad. School, Boston Coll., Chestnut Hill 67, Mass.
OEHLER, Prof. CHRISTIAN, A.M. (Columbia) Accounting, Fordham Univ.; Oehler and Sanford, Cert. Pub. Accountants, 92 Liberty St., New York 6, N.Y.
OERGEL, C. T., B.S. in M.E. (Pennsylvania State) Test Engr., General Electric Co., Lynn, Mass. *135 Marble St., Stoneham, Mass.*
OGAWA, GEORGE, A.M. (State Coll. of Washington) *Pvt., Signal Corps*
OGDEN, Prof. E. B., Ph.D. (Boston Univ.) Head of Dept., Union Coll., Lincoln, Nebr. *4626 Bancroft St.*
OGLESBY, Prof. E. J., A.M. (Virginia) Eng. Math., Univ. of Virginia, University, Va. *P.O. Box 1032*
O'HARA, Rev. C. W. Prof., Math. and Physics, Heythrop Coll., Chipping Norton, Oxon, England
OLDENBURGER, Prof. RUFUS, Ph.D. (Chicago) Illinois Inst. of Tech., 3300 Federal St., Chicago 16, Ill. *Mathematician, Woodward Governor Co., Rockford, Ill.*
OLDS, Asst. Prof. C. D., Ph.D. (Stanford) San Jose State Coll., San Jose, Calif. *P.O. Box 462, Los Altos, Calif.*
OLDS, Asso. Prof. E. G., Ph.D. (Pittsburgh) Carnegie Inst. of Tech., Pittsburgh, Pa. *222 Gladstone Road, Pittsburgh 17, Pa.*
O'LEARY, Prof. A. J., M.S. (Catholic Univ.) St. Anselm's Coll., Manchester, N.H. *Major, U.S. Chaplain Corps.*
OLLIVIER, Asst. Prof. ARTHUR, Ph.D. (Iowa) Mississippi State Coll., State College, Miss. *Dept. of Math.*
OLLMANN, Prof. L. F., Ph.D. (Michigan) Head of Dept., Hofstra Coll., Hempstead, N.Y.
OLMSTED, Asst. Prof. J. M. H., Ph.D. (Princeton) Univ. of Minnesota, Minneapolis 14, Minn. *100 Malcolm Ave. S.E.*
OLMSTED, Asso. Prof. MARGARET, A.M. (Illinois) Latin and Math., Augustana Coll., Rock Island, Ill. *330-19th St.*
OLPIN, J. L., M.S. (Colorado) Instr., Chm. Sci. Com., Gila Jr. Coll., Thatcher, Ariz.
OLSON, Asst. Prof. EMMA J., Ph.D. (Chicago) Kent State Univ., Kent, Ohio. *537 S. Lincoln St.*
OLSON, F. C. W., B.S. (Chicago) Research Asso., Northwestern Tech. Inst., Evanston, Ill. *O.S.R.D. 1225 Lake Ave., Wilmette, Ill.*
OLSON, H. L., Ph.D. (Chicago) Head of Dept., Western Union Coll., Le Mars, Iowa. *827 Second Ave. S.E.*
OPATOWSKI, ISAAC, D.Eng. (R. Inst. of Engg., Turin), D.Math. (R. Univ., Turin) Armour Research Foundation, Illinois Inst. of Tech., 35 West 33 St., Chicago, Ill. *1047 W. North Shore Ave., Chicago 26, Ill.*
OPPENHEIM, Prof. ALEXANDER, B.A. (Oxon), Ph.D. (Chicago) Raffles Coll., Singapore, Straits Settlements

O'Quinn, Asso. Prof. R. L., Ph.D. (Peabody) Louisiana State Univ., University Sta., Baton Rouge, La.
Orange, William, A.M. (California) Chm. of Dept., Los Angeles City Coll., 855 N. Vermont Ave., Los Angeles 27, Calif.
Ore, Prof. Oystein, Ph.D. (Oslo) Yale Univ., New Haven, Conn.
O'Shaughnessy, Prof. Louis, Ph.D. (Pennsylvania) Applied Mech., Dir. of Grad. Studies, Virginia Poly. Inst., Blacksburg, Va. *Box 93*
Otis, F. F., B.Ed. (St. T.C., River Falls, Wis.) Instr., Engg., Marquette Univ., Milwaukee, Wis. *3712 S. 46th St., Milwaukee 14, Wis.*
O'Toole, A. L., Ph.D. (Michigan) *Lt., U.S.N.R.*
Ott, E. R., Ph.D. (Illinois) Executive Engr., National Union Radio Corp., Newark 2, N.J. *199 Sterling Drive, Orange, N.J.*
Otteson, Elli, A.M. (Wisconsin) Chm. of Dept., High School, Eau Claire, Wis. *705 Whipple St.*
Oursler, C. C., M.S. (Chicago) Teacher, High School, Gary, Ind. *Lt. (j.g.), U.S.N.R. Cynthiana, Ind.*
Overman, Prof. J. R., Ph.D. (Michigan) Dean of Coll. of Lib. Arts, Bowling Green State Univ., Bowling Green, Ohio
Owens, Prof. F. W., Ph.D. (Chicago) Head of Dept., Pennsylvania State Coll., State College, Pa. *462 E. Foster Ave.*
Owens, Asst. Prof. Helen B. (Mrs. F. W.), Ph.D. (Cornell) Pennsylvania State Coll., State College, Pa. *462 E. Foster Ave.*
Oxtoby, Asso. Prof. J. C., A.M. (California) Bryn Mawr Coll., Bryn Mawr, Pa.

Pall, Asso. Prof. Gordon, Ph.D. (Chicago) McGill Univ., Montreal, P.Q., Can. *Engineering Bldg.*
Palladino, James, B.S. (Long Island Univ.) Instr., Physics, Long Island Univ., Brooklyn, N.Y. *Pvt., A.A.F. 498 Graham Ave., Brooklyn 22, N.Y.*
Palmer, H. A., B.S. (Coll. of Idaho) Grad. Student, Univ. of Oklahoma, Norman, Okla. *Box 554, Sand Springs, Okla.*
Park, Prof. R. S., Ph.D. (Kentucky) Eastern Kentucky State Teachers Coll., Richmond, Ky. *213 Burnam Court*
Parker, Bob, A.M. (Texas Tech.) Teacher, High School, Lubbock, Tex. *2606-28th St.*
Parker, Asst. Prof. J. E., A.M. (Fisk) Knoxville Coll., Knoxville 16, Tenn.
Parker, Asst. Prof. S. T., M.A. (Univ. of British Columbia) Univ. of Louisville, Louisville 8, Ky.
Parker, Prof. W. V., Ph.D. (Brown) Head of Dept., Louisiana State Univ., University Sta., Baton Rouge, La.
Parkinson, Asso. Prof. G. A., Ph.D. (Wisconsin) Chm. of Dept., Univ. of Wisconsin in Milwaukee, 623 W. State St., Milwaukee, Wis. *In Service*
Parrish, H. C., M.S. (North Texas St. T.C.) *In Service*
Patten, W. E., C.E. (Cornell) Hydraulic Engr., U.S. Geological Survey, Charlottesville, Va. *312 E. High St.*
Patterson, D. O., Ph.D. (Minnesota) Head of Dept., State Teachers Coll., Valley City, N.D. *211 Fifth Ave.*
Patterson, G. W., A.M. (Columbia) Physicist, Natl. Bureau of Standards, Washington, D.C. *N.D.R.C. 504 N. Glebe Road, Arlington, Va.*
Patterson, Asst. Prof. K. B., A.M. (Princeton) Duke Univ., Durham, N.C. *1024 Monmouth Ave.*
Pawley, M. G., Ph.D. (Cornell) Radio Engr., Physicist, Naval Research Lab., Washington, D.C.
Paxton, Asso. Prof. E. K., A.M. (Columbia) Washington and Lee Univ., Lexington, Va. *P.O. Box 754*
Paydon, J. F., Ph.D. (Northwestern) *Lt. (j.g.), U.S.N.R. c/o J. S. Paydon, Plainfield, Ill.*
Payne, Asso. Prof. C. K., Ph.D. (New York Univ.) Washington Square Coll., New York Univ., 100 Washington Sq. E., New York, N.Y. *24 Valley Road, Butler, N.J.*
Pease, D. K. *T/Sgt., A.A.F. 261 Ridgewood Road, West Hartford 7, Conn.*
Peebles, N. A. Engr., Office of President, Atlantic Coast Line RR Co., Wilmington, N.C. *20 N. Eighth St.*
Pegram, Prof. Annie M., A.M. (Duke) Greensboro Coll., Greensboro, N.C.
Pehrson, Prof. E. W., A.M. (California) Univ. of Utah, Salt Lake City 1, Utah—
Peiser, A. M., Ph.D. (Cornell) *Head, Statistics Res. Group, N.A.C.A., Langley Field, Va. 9 Elizabeth Road, Hampton, Va.*
Pelletier, Prof. Arthur. Emerité, Higher Alg., École Poly., Montreal, P.Q., Can. *8456 Drolet St.*

PENCE, Asst. Prof. SALLIE E., Ph.D. (Illinois) Univ. of Kentucky, Lexington, Ky. *635 Maxwelton Court*

PENNELL, W. O., B.S. (Mass. Inst. of Tech.) Chief Engr., Retired, Southwestern Bell Tel. Co., St. Louis, Mo. *69 Court St., Exeter, N.H.*

PENNEY, WALTER. Jr. Cryptanalyst, Navy Dept., Washington, D.C. *1819 F Street N.W.*

PENNINGTON, J. V., Ph.D. (Rice) Consulting Engr., Houston, Tex. *1937 Portsmouth St., Houston 6, Tex.*

PEPPER, ECHO D., Ph.D. (Chicago) Asso., Univ. of Illinois, Urbana, Ill. *356 Math. Dept.*

PEPPER, Asso. Prof. P. M., Ph.D. (Cincinnati) Univ. of Notre Dame, Notre Dame, Ind. *Dept. of Math.*

PEPPER, R. I., A.M. (Columbia) Instr., Univ. of Tennessee, Knoxville 16, Tenn. *Box 4081 U.T. Station*

PERISHO, C. R., A.M. (Haverford Coll.) Instr., Jr. Coll., McCook, Nebr. *607 First St. W.*

PERKINS, Prof. F. W., Ph.D. (Harvard) Dartmouth Coll., Hanover, N.H. *8 Prospect St.*

PERKINS, Prof. H. A., A.M. (Wisconsin) Hampton Inst., Hampton, Va. *Lt., U.S.N.R., ASW Dept., Pier 2, USNTC, Miami, Fla.*

PERKINS, Prof. L. R., A.M. (Tufts) Emeritus, Middlebury Coll., Middlebury, Vt. *12 Hillcrest Road*

PERLIN, I. E., Ph.D. (Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill. *Lt., U.S.N. 3285 E. 121st St., Cleveland, Ohio*

PERRY, C. W., B.S. (McGill) Clerk, Sun Life Assur. Co. of Canada, Montreal, P.Q., Can. *In Service*

PERRY, D. B., A.M. (Stanford) *Lt. (j.g.), U.S.N.R. 14321 Valerio St., Van Nuys, Calif.*

PESHKIN, MURRAY. *T/4, U.S. Army. Special Engineer Detachment, P.O. Box 180, Santa Fe, N.M.*

PETERHANS, Prof. W. A. Visual Training, Illinois Inst. of Tech., Chicago, Ill. *1346 E. 49th St., Chicago 15, Ill.*

PETERS, A. S., Ph.D. (New York Univ.) Instr., New York Univ., University Heights, New York, N.Y. *2 Lafayette St., New Rochelle, N.Y.*

PETERS, I. D., M.S. (West Virginia) Instr., West Virginia Univ., Morgantown, W.Va. *P.O. Box 554*

PETERS, J. W., Ph.D. (Johns Hopkins) Instr., Univ. of Illinois, Urbana, Ill. *253 Math. Bldg.*

PETERS, MAX, M.S. (C.C.N.Y.) Teacher, New Utrecht High School, Brooklyn, N.Y. *845 West End Ave., New York, N.Y.*

PETERS, Prof. RUTH M., Ph.D. (Radcliffe) Math. and Physics, Lake Erie Coll., Painesville, Ohio

PETERSEN, J. S., Jr., M.S. (St. Louis Univ.) Office of Chief Signal Officer, Washington, D.C. *1819 G Street N.W.*

PETERSON, Prof. O. J., Ph.D. (Michigan) Head of Dept., State Teachers Coll., Emporia, Kans. *1417 West St.*

PETERSON, Asso. Prof. T. S., Ph.D. (Ohio State) Univ. of Oregon, Eugene, Ore. *1364 E. 18th Ave.*

PETRIE, G. W., III, M.S. (Carnegie Inst. of Tech.) *Lt. Comdr., U.S.N.R. 1439 Lehigh Parkway, Allentown, Pa.*

PETTIS, B. J., Ph.D. (Virginia) *2nd Lt. Box 673, Spartanburg, S.C.*

PETTIS, C. R., Ph.D. (Michigan) Head of Dept., Mississippi State Coll., State College, Miss. Colonel, Engineers, U.S. Army, Retired. *1651 Wyandotte Road, Columbus 8, Ohio*

PETTIT, Prof. H. P., Ph.D. (Illinois) Head of Dept., Marquette Univ., Milwaukee, Wis. *Waterford, Wis.*

PFLAUM, C. W., A.M. (Pennsylvania) Master, Darrow School, New Lebanon, N.Y. *Box 22, Dale, N.Y.*

PHALEN, Asso. Prof. H. R., Ph.D. (Chicago) Coll. of William and Mary, Williamsburg, Va.

PHELPS, C. R., Ph.D. (Harvard) *Lt., U.S.N.R., U.S. Naval Acad., Annapolis, Md. Dept. of Math.*

PHILLIPS, Rev. E. C., Ph.D. (Johns Hopkins) Treas., New York Province of the Society of Jesus, 501 E. Fordham Road, New York 58, N.Y.

PHILLIPS, E. J., M.S. in Educ. (Oklahoma) U.S. Veterans Admin. Guidance Center, San Jose State Coll., San Jose, Calif. *460 Garden St., Palo Alto, Calif.*

PHIPPS, Prof. C. G., Ph.D. (Minnesota) Univ. of Florida, Gainesville, Fla. *Armament Office, U.S.S.T.A.F. Box 2132 Univ. Sta., Univ. of Florida, Gainesville, Fla.*

PIERCE, Asst. Prof. JESSE, Ph.D. (Michigan) Case School of Appl. Sci., Cleveland, Ohio. *3364 East Blvd., Cleveland 4, Ohio*

PIERSON, A. D., A.M. (Missouri) Chm. of Dept., Jr. Coll., Kansas City, Mo. *7217 Summit Ave., Kansas City 5, Mo.*

PINKERTON, Prof. R. M., B.S. (Bradley Poly. Inst.) Acting Head of Dept., A. and M. Coll. of Texas, College Station, Tex. *Aero. Dept.*

PIRANIAN, GEORGE, Ph.D. (Rice) Instr., Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*

PIRENIAN, Asso. Prof. Z. M., M.S. (Florida) Univ. of Florida, Gainesville, Fla. *Box 2457, University Station*

PITCHER, Asst. Prof. A. E., Ph.D. (Harvard) Lehigh Univ., Bethlehem, Pa. *Capt., A.U.S., Ordnance. 422 W. Broad St., Bethelehem, Pa.*

PITTS, Asso. Prof. R. J., A.M. (Michigan) Fort Valley State Coll., Fort Valley, Ga. *908 Fuller St., Ann Arbor, Mich*

PIXLEY, Asso. Prof. H. H., Ph.D. (Chicago) Wayne Univ., Detroit 1, Mich.

PLANT, Prof. L. C., M.S. (Chicago) Emeritus, Michigan State Coll., East Lansing, Mich. *231 Oakhill Ave.*

PLOENGES, Asso. Prof. E. W., A.M. (Michigan) James Millikin Univ., Decatur, Ill.

PODMELE, THERESA L., A.M. (Buffalo) Teacher, East High School, Buffalo, N.Y. *356 Lisbon Ave.*

POLAN, Asso. Prof. L. R., M.S. (West Virginia) Alfred Univ., Alfred, N.Y.

POLLARD, Asso. Prof. H. S., Ph.D. (Wisconsin) Miami Univ., Oxford, Ohio. *350 Patterson Ave.*

POLLARD, Prof. W. G., Ph.D. (Rice) Physics, Univ. of Tennessee, Knoxville 16, Tenn.

POLLEY, Prof. J. C., Ph.D. (Cornell) Wabash Coll., Crawfordsville, Ind.

POLYA, Asso. Prof. GEORGE, Ph.D. (Budapest) Stanford Univ., Stanford University, Calif. *660 Dartmouth St., Palo Alto, Calif.*

POOLE, Asst. Prof. A. R., Ph.D. (Calif. Inst. of Tech.) Montana State Coll., Bozeman, Mont. *Dept. of Math.*

POPPEN, H. A., A.M. (Northwestern) Statistician, Stevenson, Jordan and Harrison, Chicago, Ill. *4240 Linden St., Western Springs, Ill.*

PORGES, ARTHUR, M.S. (Ill. Inst. of Tech.) 2743 W. San Marino St., Los Angeles 6, Calif.

PORITSKY, HILLEL, Ph.D. (Cornell) Consulting Mathematician, General Elec. Co., 1 River Road, Schenectady, N.Y.

PORTER, Asso. Prof. D. H., A.M. (Indiana) Math. and Physics, Marion Coll., Marion, Ind.

PORTER, RUTH E., A.M. (Oregon), M.S. (Oregon State Coll.) Instr., Math. and Physics, Grays Harbor Jr. Coll., Aberdeen, Wash. *Instr., Willamette Univ., Salem, Ore. 249 S. Cottage St., Apt. 10, Salem, Ore.*

POST, Asso. Prof. E. L., Ph.D. (Columbia) Coll. of the City of New York, New York 31, N.Y. *610 West 173 St., New York 32, N.Y.*

POTOR, VALENTINA, A.M. (Indiana) Instr., Berea Coll., Berea, Ky. *Box 426 Berea Coll.*

POULIOT, Dean ADRIEN, M.A. (Laval) Dean of Faculty of Sciences, Laval Univ., Blvd de l'Entente, Quebec, P.Q., Can.

POUND, Prof. V. E., Ph D. (Toronto) Univ. of Buffalo, Buffalo, N.Y. *190 Capen Blvd., Buffalo 14, N.Y.*

POUNDER, Prof. I. R., Ph.D. (Chicago) Univ. of Toronto, Toronto, Ont., Can.

POWELL, Prof. J. E., Ph.D. (Chicago) Michigan State Coll., East Lansing, Mich. *137 Bogue St.*

PRENOWITZ, Asst. Prof. WALTER, Ph.D. (Columbia) Brooklyn Coll., Bedford Ave. and Avenue H, Brooklyn 10, N.Y.

PRETZ, P. S., A.M. (St. John's Univ.) Head of Dept., St. Benedict's Coll., Atchison, Kans.

PRICE, Prof. G. B., Ph.D. (Harvard) Univ. of Kansas, Lawrence, Kans. *205 Frank Strong Hall*

PRICE, Prof. H. F., Ph.D. (Pennsylvania) Dean, Pacific Univ., Forest Grove, Ore. *303 Second St. S.*

PRICE, Asst. Prof. H. V., Ph.D. (Iowa) Univ. of Iowa; Head of Dept., University High School, Iowa City, Iowa

PRICE, Prof. IRENE, Ph.D. (Indiana) State Teachers Coll., Oshkosh, Wis. *Statistical Analyst, Army Air Forces, Wright Field. 158 E. Helena St., Dayton 5, Ohio*

PRICE, R. W., A.M. (Columbia) Principal, High School, Picture Rocks, Pa. *R.D. 1, Hughesville, Pa.*

PRIESTER, G. C., Ph.D. (Michigan) Head of Dept., Math. and Mech., Univ. of Minnesota, Minneapolis 14, Minn.

PUCKETT, W. T., Jr., Ph.D. (Virginia) Instr., Univ. of California at Los Angeles, Los Angeles, Calif. *Dept. of Math.*

PUGSLEY, Asso. Prof. D. W., M.S. (Michigan) Berea Coll., Berea, Ky. *Box 114, College P.O.*

PURCELL, Asso. Prof. E. J., Ph.D. (Cornell) Univ. of Arizona, Tucson, Ariz. *Route 5, Box 790*

PURDIE, Asso. Prof. K. S., B.S. (Va. Milit. Inst.) Virginia Military Inst., Lexington, Va. *Colonel*

PURDY, Asso. Prof. J. R., Ph.D. (Illinois) Southern Illinois Normal Univ., Carbondale, Ill. *Box 56*

PURVIANCE, R. A., A.B. (Wabash) Instr., Univ. of Tennessee, Knoxville, Tenn. *1st Lt., A.A.F.*
PUTNAM, A. L., Ph.D. (Harvard) 1005 E. 60th St., Chicago 37, Ill.
PUTNAM, Asso. Prof. R. G., Ph.D. (Chicago) New York Univ., New York, N.Y. *115 Riverview Ave., Tarrytown, N.Y.*
PYLE, Prof. H. R., Ph.D. (California) Whittier Coll., Whittier, Calif. *530 N. Bright Ave.*

QUAID, L. J., B.S. (Illinois) Instr., Univ. of Minnesota, Minneapolis, Minn. *236 Melbourne Ave. S.E.*
QUARLES, Asst. Prof. H. L., A.M. (Alabama) Univ. of Mississippi, University, Miss. *Lt. Col., Inspector General's Dept.*
QUERRY, Prof. J. W., Ph.D. (Iowa) Head of Dept., Sam Houston State Teachers Coll., Huntsville, Tex. *Capt., A.U.S., Air Corps. Hq. Preflight and O.C.S., Maxwell Field, Ala.*
QUILTY, PATRICK, C.E. (Cooper Union) Commissioner, Dept. of Water Supply, Gas and Elec., New York, N.Y. *884 Riverside Drive, New York 32, N.Y.*
QUINN, GRACE S. (Mrs. R. B.), Ph.D. (Ohio State) 3221 Wheeler Road S.E., Washington 20, D.C.

RADER, Asst. Prof. M.A., A.M. (Lehigh) Moravian Coll. and Theolog. Sem., Bethlehem, Pa.
RADÓ, Prof. TIBOR, Ph.D. (Szeged, Hungary) Ohio State Univ., Columbus, Ohio. *92 Walhalla Road, Columbus 2, Ohio*
RAHN, EDRIS P., A.M. (California) Teacher, Union High School, Hayward, Calif. *1456 Glen Drive, San Leandro, Calif.*
RAINE, P. W. A., A.M. (Virginia) Teacher, High School, Newport News, Va. *106 Galax St., Hampton, Va.*
RAINICH, Prof. G. Y., Master in Pure Math. (Kazan) Univ. of Michigan, Ann Arbor, Mich. *602 Oswego St.*
RAINVILLE, Asst. Prof. E. D., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *1459 Rosewood St., R.F.D. 7*
RAKER, P. H., A.M. (Michigan) Instr., General Motors Inst., Flint, Mich. *818 W. Dartmouth St., Flint 4, Mich.*
RAMBO, Prof. SUSAN M., Ph.D. (Michigan) Smith Coll., Northampton, Mass. *71 Ridgewood Terrace*
RAMLER, Prof. O. J., Ph.D. (Catholic Univ.) Catholic Univ. of America, Washington, D.C. *12 Girard St. N.E., Washington 2, D.C.*
RAMSDELL, Prof. G. E., A.M. (Harvard) Bates Coll., Lewiston, Me. *40 Mountain Ave.*
RAMSEY, Asst. Prof. L. W., B.S. (A. and M. Coll. of Texas) Math. and Engg., Texas Christian Univ., Fort Worth 9, Tex. *Box 441, T.C.U.*
RAMSEY, MARGARET, A.M. (Oregon) Instr., Linfield Coll., McMinnville, Ore. *Box 449*
RANDELS, W. C., Ph.D. (Brown) Physicist, North American Aircraft Corp., Inglewood, Calif. *6009 Mansfield St., Los Angeles 43, Calif.*
RANDOLPH, Prof. J. F., Ph.D. (Cornell) Oberlin Coll., Oberlin, Ohio. *131 Forest St.*
RANKIN, F. MOZELLE, A.B. (Texas Christian) Grad. Asst., Ohio State Univ., Columbus, Ohio. *1948 N. High St.*
RANKIN, Prof. J. M., A.M. (California) Coll. of Idaho, Caldwell, Idaho. *1810 Ash St.*
RANSOM, Prof. W. R., A.M. (Tufts; Harvard) Tufts Coll., Medford, Mass. *29 Sawyer Ave., Medford 55, Mass.*
RAO, Prof. B. S. M. Head of Dept., Central Coll., Bangalore, India
RASMUSEN, RUTH B., Ph.D. (Chicago) Instr., Wilson Jr. Coll., Chicago, Ill. *6105 S. Woodlawn Ave.*
RASOF, BERNARD, M.S. (Calif. Inst. of Tech.) Research Engr., California Inst. of Tech., Pasadena 4, Calif.
RASOR, E. A., M.S. (Ohio State) *Lt. (j.g.), U.S.N.R.*
RASOR, Prof. S. E., M.S. (Chicago) Ohio State Univ., Columbus, Ohio. *1594 Neil Ave., Columbus 1, Ohio*
RATNER, L. T., A.B. (U.C.L.A.) Asst., Univ. of California at Los Angeles, Los Angeles, Calif. *152 S. Maple Drive, Beverly Hills, Calif.*
RAUCH, L. L., A.B. (Southern California) Instr., Princeton Univ., Princeton, N.J. *NDRC Project Leader. Fine Hall*
RAUDENBUSH, Asst. Prof. H. W., Ph.D. (Columbia) Queens Coll., Flushing, N.Y. *84-41 169 St., Jamaica, N.Y.*
RAWLINS, Prof. C. H., Jr., Ph.D. (Johns Hopkins) Postgraduate School, U.S. Naval Acad., Annapolis, Md. *13 Franklin St.*
RAYHER, EDWARD, A.M. (Columbia) Instr., Bergen Jr. Coll., Teaneck, N.J. *202-17 36th Ave., Bayside, New York, N.Y.*

RAYL, Asst. Prof. ADRIENNE S., Ph.D. (Chicago) Univ. of Alabama Center, 2131 Sixth Ave. N., Birmingham, Ala.
RAYNOR, Asso. Prof. G. E., Ph.D. (Princeton) Lehigh Univ., Bethlehem, Pa. *349 Eighth Ave.*
READ, Prof. C. B., Ph.D. (Colo. State Coll. of Educ.) Head of Dept., Univ. of Wichita, Wichita 6, Kans.
READE, MAXWELL, Ph.D. (Rice) Instr., Purdue Univ., Lafayette, Ind. *Consultant, O.S.R.D. 2409 N. Upton St., Arlington, Va.*
REAGAN, Prof. C. A., A.M. (Kansas) Friends Univ., Wichita, Kans.
REAGAN, Asst. Prof. L. M., A.M. (Kansas) Poly. Inst. of Brooklyn, 99 Livingston St., Brooklyn 2, N.Y.
REAVES, Prof. CAROLINE M., A.M. (Oklahoma) Coker Coll., Hartsville, S.C. *1100 N. Third St.*
REAVES, Prof. S. W., Ph.D. (Chicago) Univ. of Oklahoma, Norman, Okla.
RECHARD, Prof. O. H., Ph.D. (Wisconsin) Dean, Coll. of Liberal Arts, Univ. of Wyoming, Laramie, Wyo.
RECHT, Asso. Prof. A. W., Ph.D. (Chicago) Univ. of Denver, Denver 10, Colo. *2233 South St. Paul St.*
RECKZEH, J. K., A.M. (Kentucky) Instr., West Virginia Inst. of Tech., Montgomery, W.Va. *Lt., U.S.N.R. Bldg. 633, Room 202, N.A.S., Pensacola, Fla.*
REDDEN, Asso. Prof. J. E., M.S. (Iowa State) John Tarleton Agric. Coll., Stephenville, Tex. *Box 246, Tarleton Sta.*
REDDICK, Adj. Prof. H. W., Ph.D. (Columbia) New York Univ., University Heights, New York 53, N.Y. *Box 205 N.Y.U.*
REECE, Pres. R. H., A.M. (Colorado) New Mexico School of Mines, Socorro, N.M.
REED, Prof. F. W., Ph.D. (Virginia) Ohio Univ., Athens, Ohio. *61 Columbia Ave.*
REED, Prof. L. J., Ph.D. (Pennsylvania) Biostatistics, Dean, School of Hygiene and Pub. Health, Johns Hopkins Univ., Baltimore 5, Md.
REES, Prof. C. J., Ph.D. (Pennsylvania) Head of Dept., Univ. of Delaware, Newark, Del. *230 E. Main St.*
REES, Asso. Prof. MINA S., Ph.D. (Chicago) Hunter Coll., New York, N.Y. *Tech. Aide, Applied Math. Panel, N.D.R.C. 140 East 40 St., New York 16, N.Y.*
REES, Prof. P. K., Ph.D. (Rice) Southwestern Louisiana Inst., Lafayette, La.
REES, W. A., A.M. (Texas) Center Point, Tex.
REEVE, Prof. W. D., Ph.D. (Minnesota) Teachers Coll., Columbia Univ., 525 West 120 St., New York 27, N.Y.
REHBERG, C. F., A.M. (Columbia) Instr., Elec. Eng., New York Univ., New York, N.Y. *25-28 84th St., Jackson Heights, N.Y.*
REIBER, F. A., M.S. (Florida) YMCA Hotel, 826 S. Wabash Ave., Chicago, Ill.
REICHELDERFER, P. V., Ph.D. (Ohio State) *Ensign, U.S.N.R. R.F.D. 1, Box 50, Laurelville, Ohio*
REID, Prof. W. T., Ph.D. (Texas) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
REILLY, Sister MARY HENRIETTA, Ph.D. (Catholic Univ.) Head of Dept., Teachers Coll., 28 Calhoun St., Cincinnati 19, Ohio. *1409 Freeman Ave., Cincinnati 14, Ohio*
REINGOLD, Asst. Prof. HAIM, Ph.D. (Cincinnati) Illinois Inst. of Tech., Technology Center, Chicago 16, Ill. *Dept. of Math.*
REINSCH, Prof. B. P., Ph.D. (Illinois) Head of Dept., Florida Southern Coll., Lakeland, Fla. *Army Univ. Center in France. 191 Lake Morton Drive*
REISSNER, Asst. Prof. ERIC, Ph.D. (Mass. Inst. of Tech.) Massachusetts Inst. of Tech., Cambridge 39, Mass.
REKLIS, Mrs. VIRGINIA M., Ph.D. (Illinois) *Mathematician, Aberdeen Proving Ground, Md. 35 Defense Drive, Aberdeen, Md.*
REMICK, Prof. B. L., Ph.M. (Cornell Coll.) Emeritus, Kansas State Coll., Manhattan, Kans. *613 Houston St.*
RENWICK, ELIZABETH, A.M. (Indiana) Cornelius Printing Co., 912–918 Burlington Ave., Silver Spring, Md.
REVES, Asst. Prof. G. E., Ph.D. (Cincinnati) The Citadel, Charleston, S.C.
REX, E. C., M.S. (Washington) Instr., Compton Jr. Coll., Compton, Calif. *911 W. 35th Place, Los Angeles 7, Calif.*
REYNOLDS, Prof. C. N., Ph.D. (Harvard) West Virginia Univ., Morgantown, W.Va. *217 McLane Ave.*
REYNOLDS, Prof. J. B., Ph.D. (Moravian) Head of Dept., Math. and Astr., Lehigh Univ., Bethlehem, Pa. *721 W. Broad St.*
REYNOLDS, J. O., A.M. (North Carolina) Instr., Univ. of North Carolina, Chapel Hill, N.C. *Lt., U.S.N.R. Ground School, Naval Air Station, Ottumwa, Iowa*
REYNOLDS, LENA E., A.M. (California) Instr., Jr. Coll., Fullerton, Calif.

RHODES, A. P., A.M. (Stanford) Asso. Civil Engr., U.S. Navy. *1885 Sheridan Ave., San Diego 3, Calif.*
RHODES, Prof. C. E., Ph.D. (Cincinnati) Washington Coll., Chestertown, Md.
RHODES, Prof. M. C., Ph.D. (Peabody) Dean, Univ. of Tampa, Tampa 6, Fla.
RICE, Prof. HARRIS, A.M. (Harvard) Worcester Poly. Inst., Worcester, Mass.
RICE, Asst. Prof. H. L., M.S. (Iowa) Univ. of Omaha, Omaha, Nebr. *130 Fifth Ave., Council Bluffs, Iowa*
RICE, Asso. Prof. J. N., Ph.D. (Catholic Univ.) Catholic Univ. of America, Washington, D.C. *3326-13th St. N.E., Washington 17, D.C.*
RICE, R. B., A.B. (Wooster) Asso. Engr., Phillips Petroleum Co., Bartlesville, Okla. *120 N. Quapaw St.*
RICHARDSON, Prof. C. H., Ph.D. (Michigan) Chm. of Dept., Bucknell Univ., Lewisburg, Pa.
RICHARDSON, Asst. Prof. MOSES, Ph.D. (Columbia) Brooklyn Coll., Bedford Ave. and Ave. H, Brooklyn, N.Y.
RICHARDSON, Dean R. G. D., Ph.D. (Yale) Grad. School, Brown Univ., Providence, R.I.
RICHERT, Prof. D. H., A.M. (Colorado) Emeritus, Math. and Astr., Bethel Coll., North Newton, Kans.
RICHMOND, C. A., B.S. (Pomona) Tyngsboro, Mass.
RICHTMEYER, Prof. C. C., Ph.D. (Colo. State Coll. of Educ.) Head of Dept., Central Michigan Coll. of Educ., Mt. Pleasant, Mich.
RICKARD, Asst. Prof. HORTENSE, A.M. (Ohio State) Ohio State Univ., Columbus, Ohio. *79 W. Beaumont Road*
RICKEY, Asso. Prof. F. A., Ph.D. (Louisiana State) Louisiana State Univ., University Sta., Baton Rouge, La. *Dept. of Math.*
RIDER, Prof. P. R., Ph.D. (Yale) Washington Univ., St. Louis 5, Mo. *Army Univ. Center in England*
RIESS, Asst. Prof. J. K., Ph.D. (Brown) Tulane Univ., New Orleans 15, La. *17 Audubon Blvd.*
RINEHART, Asst. Prof. R. F., Ph.D (Ohio State) Case School of Appl. Sci., Cleveland, Ohio
RIORDAN, JOHN, B.S. (Yale) Member, Tech. Staff, Bell Telephone Labs., Inc., 463 West St., New York 14, N.Y.
RIPANDELLI, J. S., A.B. (Columbia) Statistical Clerk, Miles and Dawson, Consulting Act., New York, N.Y. *1st Lt., A.U.S. 284th Engr. Combat Bn. Box 425, Tryon, N.C.*
RITT, Prof. J. F., Ph.D. (Columbia) Columbia Univ., New York 27, N.Y.
ROBB, J. M., A.M. (Michigan) Retired, Univ. of Southern California, Los Angeles, Calif. *1198 W. 29th St., Los Angeles 7, Calif.*
ROBBINS, Asst. Prof. C. K., A.M. (Harvard) Purdue Univ., West Lafayette, Ind. *418 Vine St.*
ROBERTS, Prof. B. D., Ph.D. (Iowa) New Mexico Highlands Univ., Las Vegas, N.M.
ROBERTS, G. G., A.M. (Kentucky) Instr., Berea Coll., Berea, Ky. *6 Estill St.*
ROBERTS, Asso. Prof. J. H., Ph.D. (Texas) Duke Univ., Durham, N.C. *Lt., U.S.N.R.*
ROBERTS, W. C., A.M. (U.C.L.A.) Instr., Jr. Coll., Glendale, Calif. *403 E. Fairview Ave., Glendale 7, Calif.*
ROBERTSON, Asst. Prof. FRED, A.M. (Indiana) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
ROBINSON, Asso. Prof. G. DEB., Ph.D. (Cambridge) Univ. of Toronto, Toronto, Ont., Can. *Dept. of Math.*
ROBINSON, Prof. H. A., Ph.D. (Johns Hopkins) Agnes Scott Coll., Decatur, Ga. *Lt. Col., Field Artillery, U.S. Military Acad., West Point, N.Y.*
ROBINSON, Prof. ROBIN, Ph.D. (Harvard) Dartmouth Coll., Hanover, N.H. *16 Allen St.*
ROBINSON, Asst. Prof. R. M., Ph.D. (California) Univ. of California, Berkeley, Calif. *1518 Acton St., Berkeley 2, Calif.*
ROBINSON, SELBY, Ph.D. (Iowa) Coll. of the City of New York, Convent Ave. and 139th St., New York, N.Y. *41-38 Westmoreland St., Little Neck, N.Y.*
ROBINSON, Prof. W. J., Ph.D. (Ohio State) Centre Coll. of Kentucky, Danville, Ky.
ROCHFORD, Sister M. DEPAZZI, Ph.D. (Notre Dame) Dean, Registrar, Briar Cliff Coll., Sioux City 17, Iowa
RODABAUGH, L. D., Ph.D. (Ohio State) Statistician, Bureau of the Census, Washington 25, D.C.
RODGERS, Dean T. G., A.M. (Wisconsin) Emeritus, New Mexico Highlands Univ., Las Vegas, N.M. *1018 Fourth St.*
RODRÍGUEZ, Aux. Prof. MARGARITA, D.P.M.S. (Havana Univ.) Matanzas Inst., Matanzas, Cuba. *Basarrate 56, Havana, Cuba*
ROESSLER, Asso. Prof. E. B., Ph.D. (California) Asso. Statistician, Experiment Sta., Univ. of California, Coll. of Agric., Davis, Calif.

ROEVER, Prof. W. H., Ph.D. (Harvard) Head of Dept., Math. and Astr., Washington Univ., St. Louis, Mo.
ROGERS, Asst. Prof. H. P., A.M. (Illinois) Kent State Univ., Kent, Ohio. *Capt. Box 203, MAAF, Midland, Tex.*
ROLAND, Brother, B.S. (Laval) Prof., Secretary, Superior School of Commerce, 2 Cook St., Quebec, P.Q., Can.
ROLFE, KATHRYN B. (Mrs. R. W.), M.S. (Univ. of Washington) Asso., Univ. of California, Berkeley 4, Calif. *Dept. of Math.*
ROLL, ROSE, A.M. (Columbia) Chm. of Dept., Washington Irving High School, 40 Irving Place, New York, N.Y. *205 East 78 St.*
ROMAN, IRWIN, Ph.D. (Chicago) Senior Geophysicist, Dept. of Interior, Bureau of Mines, Baltimore, Md. *722 Hunting Place, Baltimore 29, Md.*
ROORDA, ETHEL, M.S. (Iowa) Instr., Buena Vista Coll., Storm Lake, Iowa
ROOT, Prof. R. E., Ph.D. (Chicago) Math. and Mech., Postgrad. School, U.S. Naval Acad., Annapolis, Md. *7 Franklin St.*
ROSE, G. F., B.S. (Case School) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *Route 4*
ROSEN, Asst. Prof. J. S., Ph.D. (Washington Univ.) Univ. of Kansas City, Kansas City 4, Mo.
ROSENBACH, Prof. J. B., M.S. (Illinois) Acting Head of Dept., Carnegie Inst. of Tech., Pittsburgh, Pa. *2550 Beechwood Blvd., Squirrel Hill, Pittsburgh 17, Pa.*
ROSENBAUM, JOSEPH, Ph.D. (Cornell) Bloomfield, Conn.
ROSENBAUM, Mrs. LOUISE J., Ph.D. (Colorado) Asst. Prof., Reed Coll., Portland 2, Ore.
ROSENBERG, G. S. Student, Illinois Inst. of Tech., Chicago, Ill. *1501 S. Millard Ave.*
ROSENBLOOM, P. C., Ph.D. (Stanford) Instr., Brown Univ., Providence, R.I. *Dept. of Math.*
ROSENTHAL, Asst. Prof. ARTHUR, Ph.D. (Munich) Univ. of New Mexico, Albuquerque, N.M. *314 S. Maple St.*
ROSENTHALL, EDWARD, Ph.D. (Calif. Inst. of Tech.) Lecturer, McGill Univ., Montreal, P.Q., Can. *Arts Bldg.*
ROSS, Asso. Prof. A. E., Ph.D. (Chicago) St. Louis Univ., St. Louis, Mo. *Dept. of Math.*
ROSSER, Prof. J. B., Ph.D. (Princeton) Cornell Univ., Ithaca, N.Y. *Research Supervisor, Allegany Ballistics Lab., Cumberland, Md. 617 Sheridan St. N.W., Washington 11, D.C.*
ROSSKOPF, M. F., Ph.D. (Brown) Head of Dept., John Burroughs School, Clayton, Mo. *706 Elm Tree Lane, Kirkwood, Mo.*
ROTH, R. E., B.S. (St. Bonaventure) *In Service*
ROTH, S. G., A.M. (North Carolina) Instr., Cooper Union, Cooper Square, New York 3, N.Y.
ROTHE, Asst. Prof. E. H., Ph.D. (Berlin) Univ. of Michigan, Ann Arbor, Mich. *1015 Vaughn St.*
ROTHROCK, Prof. D. A., Ph.D. (Leipzig) Emeritus, Indiana Univ., Bloomington, Ind. *1000 Atwater Ave.*
ROUSE, Asst. Prof. L. J., Ph.D. (Michigan) Univ. of Michigan, Ann Arbor, Mich. *1137 Michigan Ave.*
ROWLAND, Mrs. ANNIE N., M.S. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex. *2413-22nd St.*
ROWLAND, Prof. S. A., A.B. (Ouachita) Ohio Wesleyan Univ., Delaware, Ohio. *Lt. Col., U.S.A. Chief, Troop Movement Div., New York Port of Embarkation. 45 Oak Hill Ave., Delaware, Ohio*
RUDDICK, C. T., Ph.D. (Pennsylvania) Treasurer, Asso. Prof., Philosophy, Lake Erie Coll., Painesville, Ohio
RUDERMAN, H. D., A.M. (Columbia) Teacher, Sr. High School, New York, N.Y. *T/4. Ordnance, Aberdeen Proving Ground, Md. 1533 Townsend Ave., New York 52, N.Y.*
RUFUS, Prof. W. C., Ph.D. (Michigan) Astr., Univ. of Michigan, Ann Arbor, Mich. *215 Angell Hall*
RULON, Prof. P. J., Ph.D. (Minnesota) Educ., Harvard Univ., Cambridge 38, Mass. *40 Quincy St.*
RUMNEY, ETHEL A., M.S. (Chicago) 309 N. Main St., Sandwich, Ill.
RUNGE, Asst. Prof. LULU L., A.M. (Wisconsin) Univ. of Nebraska, Lincoln, Nebr.
RUNNING, Prof. T. R., Ph.D. (Wisconsin) Emeritus, Univ. of Michigan, Ann Arbor, Mich. *1019 Michigan Ave.*
RUPP, Prof. C. A., Ph.D. (Chicago) Pennsylvania State Coll., State College., Pa. *Capt., Signal Corps. 2807A S. Abingdon St., Fairlington, Arlington, Va.*
RUSH, JACOB, A.B. (Columbia) Teacher, Brooklyn Tech. High School, 29 Ft. Greene Place, Brooklyn 1, N.Y.

RUSK, Prof. EVELYN C. (Mrs. W. S.), Ph.D. (Cornell) Dean, Wells Coll., Aurora, N.Y.
RUSK, Prof. W. J., M.A. (Toronto) Emeritus, Grinnell Coll., Grinnell, Iowa. *1415 Park St.*
RUSS, ALEXANDER, A.M. (Duke) Physicist, Naval Aircraft Factory, Philadelphia, Pa. *2423 Curtin Terrace, Philadelphia 45, Pa.*
RUSSELL, Rev. B. M., A.B. (St. Viator's) 6219 Sheridan Road, Chicago 40, Ill.
RUSSELL, Asso. Prof. HELEN G., Ph.D. (Radcliffe) Wellesley Coll., Wellesley 81, Mass.
RUTT, N. E., Ph.D. (Pennsylvania) *Army Univ. Center in England. Sci. Sect., Acad. Div., AUC No. 1, APO 756, c/o Postmaster, New York, N.Y.*

SABEL, R. G. M., B.S. (Mass. Inst. of Tech.) Teacher, Bristol High School, Bristol, Conn. *7 Forestville Ave., Plainville, Conn.*
SACHS, J. M., Ph.D. (Chicago) Instr., Wilson Jr. Coll., 6800 Stewart Ave., Chicago, Ill. *Lt., U.S.N.R. 6915 Cornell Ave., Chicago, Ill.*
SADOWSKY, Asso. Prof. M. A., Ph.D. (Berlin) Illinois Inst. of Tech., 3300 Federal St., Chicago 16, Ill.
SAFFORD, Prof. F. H., Ph.D. (Harvard) Emeritus, Univ. of Pennsylvania, Philadelphia, Pa. *4527 Osage Ave., Philadelphia 43, Pa.*
SAIBEL, Asso. Prof. E. A., Ph.D. (Mass. Inst. of Tech.) Mech., Carnegie Inst. of Tech., Pittsburgh 13, Pa.
SALEM, Asst. Prof. RAPHAEL, Sc.D. (Paris) Massachusetts Inst. of Tech., Cambridge, Mass.
SALKIND, CHARLES, M.S. (C.C.N.Y.) Teacher, S. J. Tilden High School, Brooklyn 3, N.Y. *1304 New York Ave.*
SALTZER, CHARLES, A.M. (Nebraska), M.S. (Brown) Instr., Brown Univ., Providence, R.I. *351 Thayer St.*
SAMELSON, Asst. Prof. HANS, Dr.sc. math. (Fed. Inst. of Tech., Zurich) Syracuse Univ., Syracuse 10, N.Y. *110 Buckingham Ave.*
SANDERS, Prof. S. T., M.S. (Chicago) Emeritus, Louisiana State Univ., Baton Rouge, La. *1404 E. Linwood Drive, Mobile 8, Ala.*
SANFORD, Prof. VERA, Ph.D. (Columbia) State Teachers Coll., Oneonta, N.Y.
SANGER, Asst. Prof. R. G., Ph.D. (Chicago) Dean, Div. of Phys. Sci., Univ. of Chicago, Chicago 37, Ill. *209 Cobb Hall*
SARD, Asst. Prof. ARTHUR, Ph.D. (Harvard) Queens Coll., Flushing, N.Y. *Research Mathematician, N.D.R.C. 146-19 Beech Ave.*
SASULY, MAX, M.S. (Chicago) Social Security Bd., Washington, D.C. *1029 Vermont Ave. N.W., Room 106, Washington 5, D.C.*
SAUNDERS, R. B., A.M. (Minnesota) Instr., Inst. of Tech., Univ. of Minnesota, Minneapolis 14, Minn. *Dept. of Math. and Mech.*
SAUTÉ, Asso. Prof. GEORGE, A.M. (Brown) Rollins Coll., Winter Park, Fla.
SCAMMON, Prof. R. E., Ph.D. (Harvard) Univ. of Minnesota, Minneapolis, Minn. *172 Bedford St. S.E.*
SCARBOROUGH, Prof. J. B., Ph.D. (Johns Hopkins) U.S. Naval Acad., Annapolis, Md. *Ferry Farms*
SCHAAF, S. A., Ph.D. (California) *Research Mathematician, Applied Math. Group, New York Univ., 53 Washington Square S., New York 12, N.Y.*
SCHAEFFER, Sister M. ROSALIN, A.M. (Catholic Univ.) Head of Dept., Ursuline Coll., Louisville 6, Ky. *3115 Lexington Road*
SCHECTER, SAMUEL, A.B. (Brooklyn) Jr. Research Asst., Brown Univ., Providence, R.I. *121 Power St., Providence 6, R.I.*
SCHEFFÉ, Asst. Prof. HENRY, Ph.D. (Wisconsin) Syracuse Univ., Syracuse 10, N.Y. *Mathematician, N.D.R.C., Fine Hall, Princeton, N.J.*
SCHEIER, Rev. M. A., M.S. (Catholic Univ.) Dean, Div. Natural Sci. and Math., Head of Dept., St. Bonaventure Coll., St. Bonaventure, N.Y.
SCHELKUNOFF, S. A., Ph.D. (Columbia) Member, Tech. Staff, Bell Telephone Labs., 463 West St., New York 14, N.Y.
SCHELL, E. D., A.M. (Western Reserve) Statistician, U.S. Bureau of Labor Stat., Washington, D.C. *3440 North 12 Road, Arlington, Va.*
SCHERBERG, M. G., Ph.D. (Minnesota) Head, Projects Group in Aerodynamics, Chance Vought Aircraft, Stratford, Conn. *87 Locust St., Bridgeport, Conn.*
SCHEY, OLE, A.M. (Minnesota) *In Service*
SCHLAUCH, Prof. W. S., A.M. (Columbia) Emeritus, New York Univ., New York, N.Y. *14 Lohman Place, Dumont, N.J.*
SCHMIDT, O. H., Ph.D. (Brown) Instr., Brown Univ., Providence 12, R.I.
SCHNEPP, Prof. R. F., Ph.D. (Fribourg) Head of Dept., St. Mary's Univ. of San Antonio, San Antonio 7, Tex.

SCHOENBERG, Asst. Prof. I. J., Ph.D. (Jassy) Univ. of Pennsylvania, Philadelphia, Pa. *236 Seneca Ave., Havre de Grace, Md.*
SCHOR, HARRY, M.S. in Educ. (C.C.N.Y.) Chm. of Dept., Manhattan High School of Aviation Trades, New York, N.Y. *2348 Ocean Parkway, Brooklyn 23, N.Y.*
SCHORLING, Prof. RALEIGH, Ph.D. (Columbia) Univ. of Michigan, Ann Arbor, Mich. *403 Lenawee Drive*
SCHRAUT, Asso. Prof. K. C., Ph.D. (Cincinnati) Univ. of Dayton, Dayton 9, Ohio
SCHREIBER, Prof. E. W., A.M. (Chicago) Western Illinois State Teachers Coll., Macomb, Ill. *719 W. Adams St.*
SCHROEDER, Prof. H. F., M.S. (Louisiana) Louisiana Poly. Inst., Ruston, La. *Tech. Sta.*
SCHULTZ, O. T., M.S. (Chicago) Project Engr., Sperry Gyroscope Co., Inc., Great Neck, N.Y. *365 Stewart Ave., Garden City, N.Y.*
SCHWARTZ, Asst. Prof. ABRAHAM, Ph.D. (Mass. Inst. of Tech.) Pennsylvania State Coll., State College, Pa. *Dept. of Math.*
SCHWARTZ, A.A., M.S. in Educ. (C.C.N.Y.) Teacher, Textile Evening High School; Certifying Officer, Railroad Retirement Board, New York, N.Y. *55 Parade Place, Brooklyn 26, N.Y.*
SCHWARTZ, H. M., Ph.D. (Pennsylvania) Research Fellow, Bartol Research Foundation, Whittier Place, Swarthmore, Pa.
SCHWEITZER, A. R., Ph.D. (Chicago) 452 Oakdale Ave., Chicago, Ill.
SCHWID, Asso. Prof. NATHAN, Ph.D. (Wisconsin) Coll. of Mines and Met., El Paso, Tex. *2604 N. Mesa Ave.*
SCIBIORSKI, TRYPHENA HOWARD (Mrs. A. C.), A.M. (Michigan) 20027 Derby St., Detroit 3, Mich.
SCOBERT, W. G., M.S. (Oregon) *Pvt., U.S. Army*
SCOBLIC, Sister CLAUDETTE, A.M. (Minnesota) Head of Dept., Coll. of St. Benedict, St. Joseph, Minn.
SCOTT, E. J., A.M. (Vanderbilt) Teaching Asst., Cornell Univ., Ithaca, N.Y. *502 Dryden Road*
SCOTT, P. C., Ph.D. (Peabody) *In Service*
SCOTT, W. M., Ph.D. (Michigan) Address unknown.
SCOTT, W. T., Ph.D. (Rice) *1st Lt., Ordnance. 151 Oakmont St., San Antonio, Tex.*
SEALANDER, Asst. Prof. C. E., Ph.D. (Iowa) Univ. of South Dakota, Vermillion, S.D. *Ensign, U.S.N.R. Detroit Lakes, Minn.*
SEAMONS, R. S., A.B. (Utah) *Capt.*
SECADA, C. F., B.S. (Peruvian Naval Acad.) Grad. Student, Massachusetts Inst. of Tech.. Cambridge, Mass. Lt. Comdr., Peruvian Navy. *410 Memorial Drive*
SEDLAK, G. A. Instr., Cudahy Vocational School, Cudahy, Wis. *3737 E. Holmes Ave.*
SEEBECK, Asst. Prof. C. L., Jr., Ph.D. (North Carolina) Univ. of Alabama, University, Ala. *P.O. Box 814*
SEIDEL, Asso. Prof. WLADIMIR, Ph.D. (Munich) Univ. of Rochester, Rochester, N.Y.
SEIDLIN, Prof. JOSEPH, Ph.D. (Columbia) Dir., Grad. Div., Alfred Univ., Alfred, N.Y.
SELBY, Prof. SAMUEL, Ph.D. (Chicago) Head of Dept., Univ. of Akron, Akron, Ohio
SERGESCU, Prof. PETRE, Lic.ès Sc. (Paris), Dr.ès Sc. (Bucarest) Fac. des Sciences, Univ. of Cluj, Timisoara, Roumania
SEUBERT, G. A., B.S. (Pittsburgh) Box 107, Latrobe, Pa.
SEWELL, Asst. Prof. W. E., Ph.D. (Harvard) Georgia School of Tech., Atlanta, Ga. *Major, U.S. Army*
SEYBOLD, M. ANICE, A.M. (Illinois) Univ. of Illinois, Urbana, Ill. *1115 W. California St.*
SHAH, Prof. N. M., M.A. (Trin. Coll., Cambridge, Eng.) Prin. and Prof. of Math., M.T.B. Coll., Surat, India
SHANKS, M.E., Ph.D. (Iowa) Instr., Univ. of Missouri, Columbia, Mo. *Mathematician, Aircraft Engine Research Lab., Nat'l Advisory Comm. for Aeronautics. 4575 W. 174th St., Cleveland 11, Ohio*
SHAPIRO, AARON, A.M. (C.C.N.Y.) Teacher, Midwood High School, Brooklyn, N.Y. *120 Kenilworth Place, Brooklyn 10, N.Y.*
SHAUB, Prof. H. C., Ph.D. (Cornell) Head of Dept., Washington and Jefferson Coll., Washington, Pa.
SHEEHY, M. J., A.B. (U.C.L.A.) Asso. Physicist, Univ. of California Div. of War Research. U.S. Navy Radio and Sound Lab., San Diego 52, Calif.
SHEERAN, Sister M. LAURINE, A.M. (Catholic Univ.) Instr., Mount St. Joseph Jr. Coll., Maple Mount, Ky.
SHEFFER, Prof. I. M., Ph.D. (Harvard) Pennsylvania State Coll., State College, Pa. *212 Sparks Bldg.*
SHELDON, Prof. E. W., Ph.D. (Yale) Univ. of Alberta, Edmonton, Alberta, Can.

SHENTON, Prof. W. F., Ph.D. (Johns Hopkins) American Univ., Washington 16, D.C. *3605 Porter St. N.W.*
SHERER, Prof. C. R., A.M. (Nebraska) Personnel Dir., Texas Christian Univ., Fort Worth, Tex.
SHERIDAN, Prof. L. W., Ph.D. (Catholic Univ.) Chm. of Dept., Coll. of Mount St. Vincent, New York, N.Y. *Meteorologist, U.S. Weather Bureau, Washington, D.C. 3302 Ely Place S.E., Washington 19, D.C.*
SHERWOOD, Prof. G. E. F., Ph.D. (Chicago) Univ. of California at Los Angeles, Los Angeles, Calif. *400 N. Barrington Ave., Los Angeles 24, Calif.*
SHEWHART, W. A., Ph.D. (California) Research Engr., Bell Telephone Labs., Murray Hill, N.J. *158 Lake Drive, Mountain Lakes, N.J.*
SHIRES, C. J., M.S. (Michigan) Teacher, Public Schools, Detroit, Mich. *13579 Monica Ave.*
SHIRK, Prof. J. A. G., M.S. (Kansas) Head of Dept., Kansas State Teachers Coll., Pittsburg, Kans.
SHIVELY, Prof. L. S., Ph.D. (Chicago) Ball State Teachers Coll., Muncie, Ind. *2025 W. Jackson St.*
SHOBE, L. R., M.S. (Kansas State Coll.) Tech. Instr., General Motors Inst., Flint 2, Mich. *826 W. Dartmouth St., Flint 4, Mich.*
SHOOK, Asso. Prof. C. A., Ph.D. (Johns Hopkins) Lehigh Univ., Bethlehem, Pa. *1122 W. Broad St.*
SHORT, C. A., M.S. (Delaware) Head, Dept. of Math. and Sci., High School, Lewes, Del.
SHORT, Prof. W. T., A.M. (Oklahoma) Oklahoma Baptist Univ., Shawnee, Okla. *303 W. Georgia St.*
SHREVE, Asso. Prof. D. R., Ph.D. (Illinois) Univ. of Tulsa, Tulsa 3, Okla.
SHRINER, Prof. W. O., Ph.D. (Michigan) Indiana State Teachers Coll., Terre Haute, Ind. *2525 N. Ninth St.*
SHULER, Asso. Prof. C. EUCEBIA, Ph.D. (Peabody) Univ. of South Carolina, Columbia, S.C. *2508 Monroe St.*
SHUMWAY, Prof. R. R., A.B. (Minnesota) Asst. Dean, Univ. of Minnesota, Minneapolis, Minn.
SHUSTER, Prof. C. N., Ph.D. (Columbia) State Teachers Coll., Trenton, N.J. *2393 Pennington Road*
SICELOFF, Prof. L. P., Ph.D. (Columbia) Columbia Univ., New York 27, N.Y.
SIGLEY, Asso. Prof. D. T., Ph.D. (Illinois) Kansas State Coll., Manhattan, Kans. *Johns Hopkins Univ., Physics Lab., 8621 Georgia Ave., Silver Spring, Md.*
SILBER, JACK, B.S. (Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill. *4908 N. Springfield Ave.*
SILLER, HARRY, M.S. (New York Univ.) *Research Asso., N.D.R.C., George Washington Univ. 2817-28th St. S.E., Washington 20, D.C.*
SILVERMAN, Prof. L. L., Ph.D. (Missouri) Dartmouth Coll., Hanover, N.H.
SIMESTER, Asso. Prof. J. H., M.A. (Toronto) Speed Scientific School, Univ. of Louisville, Louisville, Ky.
SIMMONS, Asso. Prof. H. A., Ph.D. (Chicago) Northwestern Univ., Evanston, Ill. *Dept. of Math.*
SIMON, Prof. W. G., Ph.D. (Chicago) Vice-Pres., Dean of Faculties of Arts and Sci., Western Reserve Univ., Cleveland 6, Ohio
SIMONS, Prof. LAO G., Ph.D. (Columbia) Emeritus, Hunter Coll., New York, N.Y. *875 West End Ave., New York 25, N.Y.*
SIMPSON, R. C., Jr., A.M. (Syracuse) Grad. Asst., Univ. of Wisconsin, Madison, Wis. *430 N. Lake St., Madison 5, Wis.*
SIMPSON, Prof. T. M., Ph.D. (Wisconsin) Dean, Grad. School, Univ. of Florida, Gainesville, Fla.
SIMPSON, Prof. T. McN., Jr., Ph.D. (Chicago) Randolph-Macon Coll., Ashland, Va. *Box 345*
SINCLAIR, Prof. MARY EMILY, Ph.D. (Chicago) Emeritus, Oberlin Coll., Oberlin, Ohio. Visiting Prof., Berea Coll., Berea, Ky.
SINGER, Asst. Prof. JAMES, Ph.D. (Princeton) Brooklyn Coll., Brooklyn, N.Y. *3054 Bedford Ave., Brooklyn 10, N.Y.*
SINGER, MAURICE, B.S. (Tulane) *Pvt., U.S.A. 4600 Freret St., New Orleans, La.*
SINGER, Prof. S. A., D.Ed. (Capital) Capital Univ., Columbus, Ohio. *2322 E. Main St., Columbus 9, Ohio*
SISAM, Prof. C. H., Ph.D. (Cornell) Colorado Coll., Colorado Springs, Colo. *816 N. Weber St.*
SISK, Prof. AUGUSTUS, Ph.D. (Cornell) Math. and Physics, Maryville Coll., Maryville, Tenn. *117 Miller Ave.*

SKARSTEDT, MARCUS, Ph.D. (California) Librarian, San Francisco Jr. Coll., Phelan Ave., San Francisco, Calif.
SKELDING, A. Z., A.B. (C.C.N.Y.) Asst. Actuary, Natl. Council on Compensation Ins., New York, N.Y. *116 Stanton Ave., Baldwin, N.Y.*
SKELLY, Sister MARY LUCINA, Ph.D. (Georgetown) Prof., Georgetown Visitation Jr. Coll., 1500-35th St., Washington 7, D.C.
SLEIGHT, Prof. E. R., A.M. (Albion) Albion Coll., Albion, Mich. *410 Allen Place*
SLEPIN, BENJAMIN, LL.B. (South Jersey Law Sch.) Dir., Mathematical Lab. *422 S. 57th St., Philadelphia 43, Pa.*
SLOAN, Prof. A. R., A.M. (Vanderbilt) Carson-Newman Coll., Jefferson City, Tenn.
SLOBIN, Prof. H. L., Ph.D. (Clark) Dean, Grad. School, Univ. of New Hampshire, Durham, N.H.
SLOTNICK, M. M., Ph.D. (Harvard) Supervisor, Geophysical Interpretation, Humble Oil and Refining Co., Houston, Tex. *4421 Roseneath Drive, Houston 4, Tex.*
SMAIL, Prof. L. L., Ph.D. (Columbia) Lehigh Univ., Betlehem, Pa.
SMILEY, Asso. Prof. C. H., Ph.D. (California) Dir. of Ladd Observatory, Astr., Brown Univ., Providence 12, R.I.
SMILEY, M. F., Ph.D. (Chicago) *Lt., U.S.N.R. 23 State Circle, Annapolis, Md.*
SMITH, Asst. Prof. A. LEE, A.M. (Duke) Math. and Engg., Coll. of William and Mary, and Virginia Poly. Inst., Norfolk Div., Norfolk, Va. *Lt., U.S.N.R. 5209 Argall Ave., Norfolk 8, Va.*
SMITH, Asst. Prof. ADAM J., Ph.D. (Pennsylvania) Montana School of Mines, Butte, Mont.
SMITH, Prof. C. D., Ph.D. (Iowa) Head of Dept., Mississippi State Coll., State College, Miss. *Dept. of Math.*
SMITH, Asso. Prof. C. E., Ph.D. (California) Astr., San Diego State Coll., San Diego, Calif. *Lt. Comdr., U.S.N.R. L.C.N.T., Ground Training School, Nav. Air Sta., Corpus Christi, Tex.*
SMITH, C. V. L., Ph.D. (Harvard) *Lt. Comdr., U.S.N.R., Instr., Radio Engg., Postgrad. School, U.S. Naval Acad., Annapolis, Md.*
SMITH, C. W., A.M. (Wisconsin) Emeritus, State Teachers Coll., Superior, Wis. *401 E. Third St.*
SMITH, Prof. D. M., Ph.D. (Chicago) Head of Dept., Georgia School of Tech., Atlanta, Ga.
SMITH, Prof. EDWARD S., Ph.D. (Virginia) Univ. of Cincinnati, Cincinnati 21, Ohio
SMITH, Prof. EDWIN R., Ph.D. (Munich) Iowa State Coll., Ames, Iowa
SMITH, Prof. ELMER R., A.M. (Vanderbilt) Retired, State Coll. for Women, Tallahasssee, Fla. *648 W. Call St.*
SMITH, F. C., Ph.D. (Michigan) Actuarial Dept., Lincoln Natl. Life Ins. Co., Fort Wayne, Ind.
SMITH, Asst. Prof. F. E., Ph.D. (Catholic Univ.) Brooklyn Coll., Brooklyn, N.Y. *Box 49, Wantagh, N.Y.*
SMITH, GENEVA M., A.B. (Maine) Head of Dept., Plymouth Normal School, Plymouth, N.H.
SMITH, Prof. G. W., Ph.D. (Illinois) Univ. of Kansas, Lawrence, Kans. *1730 Illinois St.*
SMITH, Prof. H. L., Ph.D. (Chicago) Louisiana State Univ., University Sta., Baton Rouge, La. *208A Nicholson Hall*
SMITH, H. R., M.S. (South Carolina) Head of Dept., Baltimore Poly. Inst., Baltimore, Md. *Capt., Air Corps. Lowrys, S.C.*
SMITH, Asso. Prof. H. W., Ph.D. (Texas) Oklahoma A. and M. Coll., Stillwater, Okla. *132 Husband St.*
SMITH, Prof. I. W., A.M. (Illinois) Emeritus, North Dakota Agric. Coll.; Actuary, Ancient Order of United Workmen, Fargo, N.D. *203 Tenth St. N.*
SMITH, J. C., Ph.D. (Cornell) Teaching Asst., Cornell Univ., Ithaca, N.Y. *1651 W. Water St., Elmira, N.Y.*
SMITH, Rev. J. P., A.M. (Woodstock) Church of the Gesu, 18th and Thompson Sts., Philadelphia, Pa.
SMITH, LOIS B. (Mrs. R. B.), A.M. (Kansas) Box 415, La Mesa, Calif.
SMITH, Prof. L. W., Ph.D. (Washington and Lee) Washington and Lee Univ., Lexington, Va. *P.O. Box 744*
SMITH, R. E., A.M. (North Carolina) Instr., Allegheny Coll., Meadville, Pa.
SMITH, Prof. R. G., Ph.D. (Kansas) State Teachers Coll., Pittsburg, Kans.
SMITH, S. J., A.M. (Pittsburgh) Chm. of Dept., State Teachers Coll., Lock Haven, Pa.
SMITH, Asst. Prof. S. R., Ph.D. (Indiana) Carnegie Inst. of Tech., Pittsburgh, Pa. *In Service*
SMITH, W. F., A.M. (Kentucky) Box 418, Hurricane, W.Va.
SMITH, Prof. W. M., Ph.D. (Columbia) Head of Dept., Lafayette Coll., Easton, Pa.

SMITH, Asst. Prof. ZENS L., M.S. (Chicago) Asst. Dean, Univ. of Chicago, Chicago, Ill. *5621 Drexel Ave.*
SMYTH, RUTH B. (Mrs. B. J.), A.M. (Oberlin) Visiting Lecturer, Oberlin Coll., Oberlin, Ohio. *273 Morgan St.*
SMYTH, Asso. Prof. S. GRACE, A.M. (Columbia) Dean of Women, Knox Coll., Galesburg, Ill.
SNADER, Asst. Prof. D. W., Ed.D. (Columbia) Supervisor of Math., New York State Coll. for Teachers, Albany, N.Y. *Western Ave.*
SNEDECOR, Prof. G. W., A.M. (Michigan) Dir., Statistical Lab., Iowa State Coll., Ames, Iowa. *521 Forest Glen St.*
SNYDER, Asso. Prof. A. D., A.M. (Wisconsin) Union Coll., Schenectady 8, N.Y. *1592 Union St.*
SNYDER, Prof. VIRGIL, Ph.D. (Göttingen) Emeritus, Cornell Univ., Ithaca, N.Y. *214 University Ave.*
SOBCZYK, Asso. Prof. ANDREW, Ph.D. (Princeton) Oregon State Coll., Corvallis, Ore. *Dept. of Math.*
SOHON, Rev. F. W., Ph.D. (Georgetown) Dean of Grad. School, Dir., Seismolog. Observatory, Georgetown Univ., Washington, D.C.
SOKOLNIKOFF, Prof. I. S., Ph.D. (Wisconsin) Univ. of Wisconsin, Madison, Wis. *Tech. Aide, N.D.R.C.*
SOLLINS, A. D., A.B. (George Washington) Mathematician, U.S. Coast and Geodetic Survey, Washington, D.C. *1501-27th St. S.E., Washington 20, D.C.*
SORGENFREY, Asst. Prof. R. H., Ph.D. (Texas) Univ. of California at Los Angeles, 405 Hilgard Ave., Los Angeles 24, Calif.
SORRELLS, RUTH C. (Mrs. C. C.), A.M. (Columbia) Teacher, Highland Park High School, Dallas, Tex. *619 N. Montclair St.*
SOUSLEY, Prof. C. P., Ph.D. (Johns Hopkins) Rose Poly. Inst., Terre Haute, Ind. *92 Potomac Ave.*
SOUTH, Asso. Prof. D. E., Ph.D. (Michigan) Univ. of Kentucky, Lexington, Ky.
SOUTHARD, Asst. Prof. T. H., Ph.D. (Ohio State) Wayne Univ., Detroit 1, Mich.
SPARKS, Prof. F. W., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex. *Box 94, Tech. Station*
SPEAR, Prof. JOSEPH, A.M. (Boston Univ.) Chm. of Dept., Northeastern Univ., Boston, Mass.
SPEARMAN, ETHEL, A.M. (Stanford) Teacher, Union High School, Sanger, Calif. *P.O. Box 315*
SPECHT, J. R., A.M. (De Paul) Asst. Prin., Hyde Park High School, Chicago, Ill. *1252 W. 95th Place, Chicago 43, Ill.*
SPEEKER, Asso. Prof. G. G., A.M. (Indiana) Michigan State Coll., East Lansing, Mich.
SPEICHER, Asst. Prof. P. I., A.M. (Pennsylvania) Math. and Physics, Albright Coll., Reading, Pa.
SPENCELEY, Asso. Prof. G. W., A.M. (Harvard) Miami Univ., Oxford, Ohio. *400 E. Church St.*
SPENCER, Prof. H. E., Ph.D. (Cornell) Presbyterian Coll., Clinton, S.c. *21 Calhoun St.*
SPENCER, Prof. MARY C., M.S. (Cornell) Emeritus, Newcomb Coll., New Orleans, La. *1111 Lowerline St., New Orleans 18, La.*
SPENCER, R. S., M.S. (Michigan) Physicist, Dow Chemical Co., Midland, Mich. *216 William St.*
SPENCER, VIVIAN E., Ph.D. (Pennsylvania) Sr. Statistician, U.S. Bureau of the Census, Washington, D.C. *2905 Russell Road, Alexandria, Va.*
SPERRY, Asso. Prof. PAULINE, Ph.D. (Chicago) Univ. of California, Berkeley, Calif. *1194 Cragmont Ave., Berkeley 8, Calif.*
SPICER, Prof. C. A., Ph.D. (Johns Hopkins) Western Maryland Coll., Westminster, Md. *17 Ridge Road*
SPINKS, M. J. Wilmington, Ohio
SPITZ, HILLEL, M.S. (Brown) *Physicist, Naval Research Lab., Washington 20, D.C. 3711 Horner Place S.E.*
SPITZBART, ABRAHAM, Ph.D. (Harvard) Instr., Univ. of Wisconsin, Madison 6, Wis.
SPOONER, Prof. C. C., A.M. (Amherst) Head of Dept., Retired, Northern Michigan Coll. of Educ., Marquette, Mich. *117 E. Ridge St.*
SPRINGER, Prof. C. E., Ph.D. (Chicago) Univ. of Oklahoma, Norman, Okla.
SPRUITENBURG, ELSJE, Drs. Math. (Amsterdam) Teacher, High School, The Hague, Netherlands. *110 Columbus Straat*
STABLER, Asst. Prof. E. R., Ed.D. (Harvard) Hofstra Coll., Hempstead, N.Y.

STALEY, Prof. R. C., Ph.D. (Michigan) Head of Dept., Univ. of North Dakota, Grand Forks, N.D. *321 Cambridge St.*

STANLEY, E. L., M.S. (Tennessee) Clemson Coll., Clemson, S.C.

STANWICK, C. A., B.S. in E.E. (Washington) Elec. Engr., 131 Rynda Road, South Orange, N.J.

STARCHER, Prof. G. W., Ph.D. (Illinois) Ohio Univ., Athens, Ohio. *Route 3*

STARK, L. W., A.B. (Univ. of Kansas City) Acting Prof., Atlantic Christian Coll., Wilson, N.C. *Drawer C, Atlantic Christian Sta.*

STARK, Prof. MARION E., Ph.D. (Chicago) Wellesley Coll., Wellesley 81, Mass. *6 Waban St.*

STARKE, Prof. E. P., Ph.D. (Columbia) Rutgers Univ., New Brunswick, N.J.

STARR, Asst. Prof. D. W., Ph.D. (Illinois) Southern Methodist Univ., Dallas, Tex. *4524 Cowan Ave., Dallas 9, Tex.*

STARRETT, Asst. Prof. A. L., A.M. (Harvard) Georgia School of Tech., Atlanta, Ga.

STAUFFER, J. R. K., M.S. (Chicago) Rhode Island State Coll., Kingston, R.I. *Dept. of Math.*

STAYER, Asst. Prof. J. C., A.M. (Pittsburgh) Dean of Men, Juniata Coll., Huntington, Pa. *1618 Moore St.*

STEARN, J. L., M.S. (C.C.N.Y.) Mathematician, U.S. Coast and Geodetic Survey, Washington, D.C. *Capt., Ordnance. Proving Center, Aberdeen Proving Ground, Md.*

STEED, Prof. D. V., Ph.D. (California) Univ. of Southern California, Los Angeles 7, Calif. *1731 W. 80th St., Los Angeles 44, Calif.*

STEEN, Prof. F. H., Ph.D. (Harvard) Allegheny Coll., Meadville, Pa. *R.D. 1*

STEINBERG, H. L., A.B. (U.C.L.A.) Grad. Student, Univ. of California at Los Angeles, Los Angeles, Calif. *10796 Wellworth Ave., Los Angeles 24, Calif.*

STEINHAUS, H. W., Ph.D. (Göttingen) Research Asst., Equitable Life Assur. Soc., 393 Seventh Ave., New York, N.Y.

STELFORD, Asst. Prof. NORMA K., A.M. (Northwestern) Northern Illinois State Teachers Coll., De Kalb, Ill.

STELSON, Prof. H. E., Ph.D. (Iowa) Kent State Univ., Kent, Ohio. *540 S. Lincoln St.*

STEPHENS, C. F., M.S. (Michigan) Instr., Prairie View State Coll., Prairie View, Tex. *Sp(T) 1/C, U.S.N.R. 769-44th St., Newsome Park, Newport News, Va.*

STEPHENS, Prof. R. C., Ph.D. (Iowa) Knox Coll., Galesburg, Ill.

STEPHENS, Prof. R. P., Ph.D. (Johns Hopkins) Emeritus, Univ. of Georgia, Athens, Ga.

STETSON, Prof. J. M., Ph.D. (Princeton) Coll. of William and Mary, Williamsburg, Va. *232 Jamestown Road*

STEVENS, W. R., A.B. (George Washington) Meteorologist, U.S. Weather Bureau, New Orleans, La. *1124 Second St., New Orleans 13, La.*

STEVENSON, Prof. GUY, Ph.D. (Illinois) Head of Dept., Univ. of Louisville, Louisville, Ky.

STEWART, Asst. Prof. B. M., Ph.D. (Wisconsin) Michigan State Coll., East Lansing, Mich. *318 Albert St.*

STEWART, Asso. Prof. I. D., A.M. (California) Whitman Coll., Walla Walla, Wash.

STIGLER, Prof. G. J., Ph.D. (Chicago) Econ., Univ. of Minnesota, Minneapolis, Minn.

STILWELL, M. F., A.M. (Syracuse) Rensselaer Poly. Inst., Troy, N.Y. *59 Collins Ave.*

STOKES, ELLEN C., Ph.D. (Chicago) Dean of Women, New York State Coll. for Teachers, Albany 3, N.Y.

STOKES, Prof. RUTH W., Ph.D. (Duke) Winthrop Coll., Rock Hill, S.C. *Box 113, Coll.*

STONE, Prof. M. H., Ph.D. (Harvard) Harvard Univ., Cambridge, Mass. *Research Analyst, War Dept. 30 Hillside Ave., Cambridge 40, Mass.*

STONE, Asso. Prof. R. B., A.M. (Harvard) Registrar, Purdue Univ., Lafayette, Ind. *615 Russell St., West Lafayette, Ind.*

STORY, HELEN F., A.M. (Wellesley) Instr., St. Petersburg Jr. Coll., St. Petersburg, Fla. *Lt., U.S.N.R.*

STOUFFER, Prof. E. B., Ph.D. (Illinois) Dean, Univ. of Kansas, Lawrence, Kans. *1019 Maine St.*

STOWELL, Prof. C. J., Ph.D. (Illinois) Dean, McKendree Coll., Lebanon, Ill. *810 Belleville St.*

STRAND, H. W., A.M. (Michigan) Instr., Purdue Univ., Lafayette, Ind. *224 S. Grant St., West Lafayette, Ind.*

STRATTON, Prof. W. T., Ph.D. (Washington) Head of Dept., Kansas State Coll., Manhattan, Kans.

STRAW, J. A., A.M. (Michigan State) Evansville Coll., Evansville 4, Ind. *Dept. of Math.*

STREET, R. E., Ph.D. (Harvard) Engr., General Electric Co., Schenectady, N.Y. *1158 Millington Road, Schenectady 8, N.Y.*

STRIGHT, Asst. Prof. I. L., A.M. (Allegheny) Baldwin-Wallace Coll., Berea, Ohio. *209 Beech St.*

STROBEL, Asst. Prof. C. F., Ph.D. (Illinois) North Carolina State Coll., Raleigh, N.C. *3310 Pollock Place*

STROCK, E. E., A.M. (Yale) Asst. Mathematician, Prudential Ins. Co., Newark, N.J. *Lt. Comdr., U.S.N.R.*

STRONG, Prof. CORA, A.M. (Michigan) Woman's Coll., Univ. of North Carolina, Greensboro, N.C. *109 Adams St.*

STRUTTON, E. B., M.S. (Toledo) Instr., Univ. of California at Los Angeles, Los Angeles, Calif. *P.O. Box 4081, Los Angeles 24, Calif.*

STRUYK, ADRIAN, A.M. (Columbia) Teacher, High School, Clifton, N.J. *232 Jefferson St., Paterson 2, N.J.*

STUART, H. L., A.B. (Dickinson) *In Service. 402 S. Hanover St., Carlisle, Pa.*

STUBBE, J. S., Ph.D. (Cincinnati) Instr., Univ. of Illinois, Urbana, Ill. *360 Math. Bldg.*

STUCKEY, C. S., Ph.M. (Wisconsin) Supervisor, Metropolitan Life Ins. Co., 1 Madison Ave., New York 10, N.Y.

STULKEN, E. J., A.M. (Texas) Seismologist, Geophysical Service, Inc., 1311 Republic Bank Bldg., Dallas 1, Tex.

STURGES, Mrs. FALKA G., A.M. (California) Teacher, Public Schools, San Francisco, Calif. *790 Hopkins Ave., Redwood City, Calif.*

STURM, Prof. R. G., Ph.D. (Illinois) Engg. Mech., Purdue Univ., Lafayette, Ind. *School of Civil Engg.*

SUFFA, Prof. MARY C., A.M. (Brown) Math. and Astr., Elmira Coll., Elmira, N.Y.

SULLIVAN, Sister M. CLOTILDA, A.M. (Canisius) Instr., Mercyhurst Coll., Erie, Pa.

SULLIVAN, Sister M. HELEN, Ph.D. (Catholic Univ.) Prof., Chm. of Dept., Mount St. Scholastica Coll., Atchison, Kans.

SUMMERS, C. R., A.M. (State Coll. of Washington) *In Service*

SUMNER, Mrs. RUTH G., Ed.M. (Stanford) Teacher, Oakland High School, Oakland, Calif. *3000 Central Ave., Alameda, Calif.*

SUTHERLAND, ETHEL, Ph.D. (Columbia) Instr., East Carolina Teachers Coll., Greenville, N.C.

SVOBODA, A. F., M.S. (DePaul) Instr., DePaul Univ., 64 E. Lake St., Chicago, Ill.

SWAFFORD, Asst. Prof. E. G., A.M. (Syracuse) Fort Hays Kansas State Coll., Hays, Kans. *Physicist, Bureau of Ships, Navy Dept., Washington, D. C. 1912 R Street N.W., Washington 9, D.C.*

SWANK, J. W. Engr., Southern California Edison Co., Ltd., P.O. Box 351, Los Angeles 53, Calif.

SWANSON, A. G., Ph.D. (Michigan) 205 W. Taylor St., Flint 5, Mich.

SWANSON, Asst. Prof. E. L., A.M. (Colorado Coll.) South Dakota State School of Mines, Rapid City, S.D. *P.O. Box 1588*

SWANSON, Asst. Prof. L. W., A.M. (Minnesota) Coe Coll., Cedar Rapids, Iowa

SWEAZEY, Prof. G. B., A.M. (Wabash) Dean, Westminster Coll., Fulton, Mo.

SWIFT, Prof. ELIJAH, Ph.D. (Göttingen) Dean, Univ. of Vermont, Burlington, Vt. *415 S. Willard St.*

SWINGLE, Asso. Prof. P. M., Ph.D. (Michigan) New Mexico Coll. of A. and M. A., State College, N.M. *Box 277*

SYNGE, Prof. J. L., Sc.D. (Dublin) Ohio State Univ., Columbus, Ohio

SZÁSZ, Prof. OTTO, Ph.D. (Budapest) Research Lecturer, Univ. of Cincinnati, Cincinnati, Ohio

SZEGÖ, Prof. GABOR, Ph.D. (Vienna) Stanford Univ., Stanford University, Calif.

TALBOT, Prof. W. R., Ph.D. (Pittsburgh) Head of Dept., Lincoln Univ., Jefferson City, Mo.

TALIAFERRO, Prof. CARRIE B., A.M. (Columbia) State Teachers Coll., Farmville, Va. *Box 392*

TANZOLA, J. J., A.M. (Columbia) Instr., Cooper Union, New York, N.Y. *2041 Watson Ave., Bronx, N.Y.*

TAPPAN, Prof. A. HELEN, Ph.D. (Cornell) Western Coll., Oxford, Ohio

TATE, JENNIE L., A.M. (Wisconsin) Head of Dept., McMurry Coll., Abilene, Tex. *1301. Orange St.*

TAYLOR, Asso. Prof. A. E., Ph.D. (Calif. Inst. of Tech.) Univ. of California at Los Angeles, Los Angeles 24, Calif. *Dept. of Math.*

TAYLOR, B. P., A.M. (U.C.L.A.) Sr. Mathematician, Douglas Aircraft Co., Santa Monica, Calif. *11326 Gladwin St., Los Angeles 24, Calif.*

TAYLOR, C. F., B.S. (Northeastern Univ.) Photo. Engr., General Electric Co., Lynn, Mass. *69 Cross St., Randolph, Mass.*

TAYLOR, Prof. E. H., Ph.D. (Harvard) Emeritus, Eastern Illinois State Teachers Coll., Charleston, Ill. *885 Seventh St.*

TAYLOR, Asst. Prof. F. J., A.B. (St. Thomas) Coll. of St. Thomas, St. Paul 1, Minn.

TAYLOR, Prof. H. O., Ph.D. (Cornell) Chm. of Dept., Math. and Physics, Wheaton Coll., Wheaton, Ill.

TAYLOR, Prof. J. H., Ph.D. (Chicago) George Washington Univ., Washington 6, D.C.
TAYLOR, Prof. MILDRED E., Ph.D. (Illinois) Math. and Astr., Mary Baldwin Coll., Staunton, Va.
TAYLOR, Asst. Prof. W. C., Ph.D. (Wisconsin) Univ. of Cincinnati, Cincinnati 21, Ohio
TAYLOR, Prof. W. E., Ph.D. (Syracuse) Emeritus, Syracuse Univ., Syracuse, N.Y. *822 Irving Ave.*
TEMMER, H. E., M.S. (Illinois) c/o J. H. Chanler, 505 S. Gregory Place, Urbana, Ill.
TEMPLE, Prof. V. B., A.M. (Texas) Louisiana Coll., Pineville, La.
TEMPLETON, Prof. MARY M., A.M. (North Carolina) Lander Coll., Greenwood, S.C.
TEMPLIN, L. J., A.M. (Loyola Univ.) Instr., Loyola Univ., Chicago, Ill. *In Service*
THÉBAULT, VICTOR, Inspecteur honoraire d'Assurances. *"Le Paradis," Tennie, Sarthe, France*
THIELMAN, Asso. Prof. H. P., Ph.D. (Ohio State) Iowa State Coll., Ames, Iowa. *Dept. of Math.*
THOMAS, Prof. C. F., M.S. (Case) Case School of Appl. Sci., Cleveland, Ohio
THOMAS, EARL, M.S. (Louisiana State) Geophysical Service, Inc., Dallas 1, Tex. *P.O. Box 72, Maypearl, Tex.*
THOMAS, Prof. J. M., Ph.D. (Pennsylvania) Duke Univ., Durham, N.C. *2215 Cranford Road*
THOMAS, P. D., A.M. (Oklahoma) Jr. Geodetic Engr., U.S. Coast and Geodetic Survey, Washington, D.C. *Lt. (j.g.), U.S.N.R.*
THOMAS, R. W., Ph.D. (Pittsburgh) Dean of Students, Washington and Jefferson Coll., Washington, Pa. *College Campus*
THOMAS, Prof. T. Y., Ph.D. (Princeton) Indiana Univ., Bloomington, Ind. *Swain Hall*
THOMPSON, Prof. E. L., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex.
THOMPSON, Asso. Prof. J. E., A.M. (Columbia) Pratt Inst., Brooklyn, N.Y. *183 Steuben St.*
THOMPSON, LOUISE, A.M. (Columbia) Registrar, Instr., Shorter Coll., Rome, Ga.
THOMPSON, W. I., A.M. (California) Instr., Los Angeles City Coll., Los Angeles, Calif. *122 W. 50th St., Los Angeles 37, Calif.*
THOMSON, Asst. Prof. J. F., Ph.D. (Michigan) Tulane Univ., New Orleans 15, La.
THORNE, Asst. Prof. C. J., Ph.D. (Iowa State) Univ. of Utah, Salt Lake City, Utah. *Dept. of Math.*
THORNTON, H. B., A.M. (Cincinnati) Instr., Sumner Jr. Coll., Kansas City, Kans. *1225 Everett Ave.*
THORNTON, Mrs. MARIAN W., Ph.D. (Minnesota) 372 N. Cleveland Ave., St. Paul 4, Minn.
THORP, ELLA, A.B. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn. *656 Jefferson St. N.E.*
THRALL, Asst. Prof. R. M., Ph.D. (Illinois) Univ. of Michigan, Ann Arbor, Mich. *Staff Member, Radiation Lab., Room 24-413, Massachusetts Inst. of Tech., Cambridge 39, Mass.*
THROCKMORTON, V. C., A.M. (Southern California) Visiting Instr., Univ. of Southern California, Los Angeles, Calif. *821 N. Heliotrope Drive, Los Angeles 27, Calif.*
THUENER, Sister M. DOMITILLA, Ph.D. (Catholic Univ.) Trustee, Villa Madonna Coll., Covington, Ky. *R.R. 2, Box 50*
THULLEN, PETER, Ph.D. (Münster, Germany) Dir., Departmento Matematico Actuarial, Instituto Nacional de Previsión. *Apartado 636, Quito, Ecuador*
TIERNEY, J. A., A.M. (Columbia) Instr., Vanderbilt Univ., Nashville 4, Tenn. *Box 101 Calhoun Hall*
TILLEY, Asso. Prof. ARTHUR, Ph.D. (New York Univ.) New York Univ., New York, N.Y.
TINNER, J. C., M.S. (Chicago) Instr., Florida A. and M. Coll., Tallahassee, Fla.
TODD, JOHN, B.S. (Belfast) Lecturer, King's Coll., Strand, London W.C. 2, England. *Belvedere, 82 Bridge Road, East Molesey, Surrey, England*
TOLLE, Prof. L. F., A.M. (St. John's Univ.) Chm. of Dept., Teachers Coll., St. John's Univ., Brooklyn, N.Y. *240-23 141 Ave., Rosedale 10, N.Y.*
TOM, D. Y. C., B.S. (Univ. of Dayton) Grad. Student, Univ. of Illinois, Urbana, Ill. *1108 S. First St., Champaign, Ill.*
TOMPKINS, C. B., Ph.D. (Michigan) Instr., Princeton Univ., Princeton, N.J. *Lt., U.S.N.R. 9926 Sutherland Road, Silver Spring, Md.*
TONEY, Prof. H. S., A.M. (Ohio State) Wilberforce Univ., Wilberforce, Ohio. *Mathematician, Engg. Div., U.S.A.A.F., Wright Field, Dayton, Ohio. 316 E. Church St., Xenia, Ohio*
TOOPS, Prof. H. A., Ph.D. (Columbia) Psychology, Ohio State Univ., Columbus, Ohio. *1430 Cambridge Blvd.*
TOPEL, Prof. B. J., Ph.D. (Notre Dame) Carroll Coll., Helena, Mont.
TOPP, C. W., A.M. (Illinois) Instr., Fenn Coll., Cleveland, Ohio. *Lt. (j.g.), U.S.N.R. 5540 Wessling Lane, Bethesda 14, Md.*
TORALBALLA, L. V., Ph.D. (Michigan) Suite 102, The Grad. College, Princeton, N.J.
TORRANCE, Asso. Prof. C. C., Ph.D. (Cornell) Case School of Appl. Sci., Cleveland, Ohio. *3607 Randolph Road, Cleveland 21, Ohio*

TORREY, Prof. MARIAN M., Ph.D. (Cornell) Goucher Coll., Baltimore 18, Md.
TRACEY, Asso. Prof. J. I., Ph.D. (Johns Hopkins) Yale Univ., New Haven, Conn. *Army Univ. Center in England. 84 McKinley Ave., New Haven 15, Conn.*
TREIBER, H. I., A.B. (Brooklyn) Grad. Student, Princeton Univ., Princeton, N.J. *Grad. College*
TREMBLAY, Prof. ALTHÉOD, M.A. (Laval) Laval Univ., Quebec, P.Q., Can. *367 St. Cyrille St.*
TRIGG, C. W., A.M., M.S. (Southern California) Instr., Los Angeles City Coll., Los Angeles, Calif. *Lt., U.S.N.R., 10522 Butterfield Road, Los Angeles 34, Calif.*
TRIMBLE, Asst. Prof. H. C., Ph.D. (Wisconsin) Iowa State Teachers Coll., Cedar Falls, Iowa. *Operations Research Group, U.S. Fleet. 809 W. 18th St.*
TRIPP, Prof. M. O., Ph.D. (Columbia) Emeritus, Wittenberg Coll., Springfield, Ohio. *218 W. Cecil St.*
TRUMP, Asso. Prof. P. L., Ph.D. (Wisconsin) Teaching of Math., Univ. of Wisconsin, Madison, Wis. *2228 Eton Ridge, Madison 5, Wis.*
TUCKER, Asso. Prof. A. W., Ph.D. (Princeton) Princeton Univ., Princeton, N.J. *N.D.R.C., Statistical Res. Group, Princeton Univ., Fine Hall, Princeton, N.J.*
TUCKER, Prof. B. A., A.M. (Millsaps Coll.) Southeastern Louisiana Coll., Hammond, La.
TUCKER, Asst. Prof. C. B., M.S. (Brown) State Teachers Coll., Emporia, Kans. *Johns Hopkins Univ., Appl. Physics Lab., Silver Spring, Md. 8621 Georgia Ave., Silver Spring, Md.*
TUKEY, Asst. Prof. J. W., Ph.D. (Princeton) Princeton Univ., Princeton, N.J. Member, Tech. Staff, Bell Telephone Labs., Murray Hill, N.J. *Princeton Univ., Fine Hall, Princeton, N.J.*
TULLER, ANNITA, Ph.D. (Bryn Mawr) Instr., Hunter Coll., New York, N.Y. *139-62 Pershing Crescent, Jamaica 2, N.Y.*
TUNELL, GEORGE, Ph.D. (Harvard) Memb. of Staff, Geophysical Lab., Carnegie Institution of Washington, 2801 Upton St., Washington 8, D.C. *c/o H. A. Baum, 110 S. Dearborn St., Room 1617, Chicago 3, Ill.*
TURNER, Prof. A. B., Ph.D. (Pennsylvania) Emeritus, Coll. of the City of New York, New York, N.Y. *245 N. Mountain Ave., Montclair, N.J.*
TURNER, Prof. BIRD M., Ph.D. (Bryn Mawr) West Virginia Univ., Morgantown, W.Va. *61 Wilson Ave.*
TURNER, Mrs. LOIS K., M.S. (Virginia Poly. Inst.) Instr., University High School, Univ. of Minnesota, Minneapolis, Minn. *369 W. Broadway St., Winona, Minn.*
TURRITTIN, Asst. Prof. H. L., Ph.D. (Wisconsin) Math. and Mech., Univ. of Minnesota, Minneapolis, Minn. *4046 Beard Ave. S., Minneapolis 10, Minn.*

UHLER, Prof. H. S., Ph.D. (Johns Hopkins) Emeritus, Physics, Yale Univ., New Haven, Conn. *12 Hawthorne Ave., Hamden 14, Conn.*
ULLSVIK, Asso. Prof. B. R., Ph.D. (Wisconsin) Illinois State Normal Univ., Normal, Ill..
ULMER, Asst. Prof. GILBERT, Ph.D. (Kansas) Educ. and Math., Asst. Dean, Coll. of Liberal Arts and Sci., Univ. of Kansas, Lawrence, Kans.
ULRICH, F. E., Ph.D. (Harvard) Instr., Rice Inst., Houston 1, Tex.
UNDERWOOD, Prof. R. S., Ph.D. (Chicago) Texas Tech. Coll., Lubbock, Tex. *2220 Broadway*
UPTON, Prof. C. B., A.M. (Columbia) Emeritus, Teachers Coll., Columbia Univ., 525 West 120 St., New York 27, N.Y.
URNER, S. E., Ph.D. (Harvard) Instr., Los Angeles City Coll., Los Angeles 27, Calif. *807 N. Mariposa Ave.*
UTZ, W. R., A.M. (Missouri) Instr., Univ. of Virginia, Charlottesville, Va. *6 Hill Apts., 1121 Wertland St.*

VALENTINE, Asst. Prof. F. A., Ph.D. (Chicago) Univ. of California at Los Angeles, 405 Hilgard Ave., Los Angeles, Calif.
VALUCKAS, Mrs. CAROL C., A.M. (Cornell) Address unknown
VAN ARKEL, G. H., M.S. (Washington) Instr., Math. and Physics, Everett Jr. Coll., Everett, Wash.
VAN ARNAM, Asst. Prof. R. N., M.S. (Cornell) Math. and Astr., Lehigh Univ., Bethlehem, Pa. *705 First Ave.*
VAN BUSKIRK, Prof. H. C., Ph.B. (Cornell) Emeritus, California Inst. of Tech., Pasadena, Calif. *390 S. Holliston Ave., Pasadena 5, Calif.*
VANCE, E. P., Ph.D. (Michigan) Visiting Lecturer, Oberlin Coll., Oberlin, Ohio. *155 Elm St.*
VANDIVER, Prof. H. S. Univ. of Texas, Austin, Tex.
VAN ENGEN, HENRY, Ph.D. (Michigan) Head of Dept., Iowa State Teachers Coll., Cedar Falls, Iowa

VAN HORN, Prof. C. E., Ph.D. (Chicago) Fisk Univ., Nashville, Tenn. *916-17th Ave. N., Nashville 8, Tenn.*
VAN ORSTRAND, C. E., M.S. (Michigan) Geophysicist, Retired, U.S. Geol. Survey, Washington, D.C. *3906 Morrison St. N.W., Washington 15, D.C.*
VAN OS, JEANETTE A., A.B. (Harris T.C., Mo.) Computer, General American Life Ins. Co., St. Louis, Mo. *6625 Mitchell Ave., St. Louis 10, Mo.*
VAN SCHAACK, G. B., Ph.D. (Harvard) *Lt., U.S.N.R. West Coxsackie, N.Y.*
VAN VOORHIS, Asst. Prof. W. R., Ph.D. (Pennsylvania State) Fenn Coll., Cleveland, Ohio. *Capt., Air Corps*
VARINEAU, V. J., Ph.D. (Wisconsin) Instr., Univ. of Wyoming, Laramie, Wyo. *Lt., U.S. N.R., Postgrad. School, U.S. Naval Acad., Annapolis, Md.*
VARNHORN, Asst. Prof. MARY C., Ph.D. (Catholic Univ.) Trinity Coll., Washington, D.C. *1120 Poplar Grove Ave., Baltimore 16, Md.*
VARNUM, E. C., M.S. (Michigan) Math'l Physicist, Barber-Coleman Co., Rockford, Ill. *1214 Van Wie Ave.*
VASS, J. I., Ph.D. (Wisconsin) Instr., Univ. of Wisconsin in Milwaukee, Milwaukee, Wis. *5141 N. Santa Monica Blvd.*
VAUGHAN, H. E., Ph.D. (Michigan) Asso., Univ. of Illinois, Urbana, Ill. *907 S. Vine St.*
VAUSE, R. Z., Jr., B.S. (South Carolina) *Radar Technician, U.S.N.R. Naval Air Station, St. Simons Island, Ga.*
VEATCH, Asso. Prof. R. W., A.M. (Northwestern) Univ. of Tulsa, Tulsa, Okla. *332 N. Santa Fe St.*
VEBLEN, Prof. OSWALD, Ph.D. (Chicago) Inst. for Advanced Study, Princeton, N.J. *58 Battle Road*
VEDOVA, Prof. G. C., Ph.D. (Maryland) Head of Dept., Pennsylvania Military Coll., Chester, Pa.
VEHSE, Asso. Prof. C. H., Ph.D. (Brown) West Virginia Univ., Morgantown, W.Va. *Army Univ. Center. P.O. Box 98, Harmony, R.I.*
VEST, Asst. Prof. M. L., M.S. (West Virginia), A.M. (Michigan) West Virginia Univ., Morgantown, W.Va. *Dept. of Math.*
VON NEUMANN, Prof. JOHN, Ph.D. (Budapest) Inst. for Advanced Study, Princeton, N.J.

WADE, L. I., Jr., Ph.D. (Duke) Instr., Duke Univ., Durham, N.C. *Box 4731 Duke Station*
WADE, T. L., Jr., Ph.D. (Virginia) Head of Dept., Florida State Coll. for Women, Tallahassee, Fla. *1003 Washington St.*
WAGNER, R. D., Ph.M. (Wisconsin) Instr., Univ. of Wisconsin, Madison, Wis. *202 N. Brooks St., Madison 5, Wis.*
WAGNER, R. W., Ph.D. (Michigan) Instr., Oberlin Coll., Oberlin, Ohio. *Lt. (j.g.), U.S.N.R. U.S. Naval Acad., Annapolis, Md. 403 Dream's Landing*
WAHLERT, Asst. Prof. H. E., A.M. (Princeton) Washington Square Coll., New York Univ., New York, N.Y. *50 Strickland Pl., Manhasset, N.Y.*
WAHLIN, Prof. G. E., Ph.D. (Yale) Univ. of Missouri, Columbia, Mo. *1401 Anthony St.*
WAIDER, K. J., A.M. (California) Instr., Physics, Univ. of San Francisco, 2130 Fulton St., San Francisco, Calif.
WAITE, ETTA A., A.M. (Columbia) Teacher, Girls Commercial High School, Brooklyn, N.Y. *491 New Milford Ave., Oradell, N.J.*
WAKERLING, Asst. Prof. R. K., Ph.D. (California) Fresno State Coll., Fresno, Calif. *Mathematician, Radiation Lab., Univ. of California. 2834 Derby St., Berkeley 5, Calif.*
WALDER, Asso. Prof. O. E., A.M. (Nebraska) South Dakota State Coll., Brookings, S.D. *College Station*
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WALKER, Prof. HELEN M., Ph.D. (Columbia) Educ., Teachers Coll., Columbia Univ., 525 West 120 St., New York, N.Y.
WALKER, L. A., A.M. (Stanford) Teacher, Math. and Physics, High School, San Mateo, Calif. *S 1/C, U.S. Navy. 336 San Antonio St.*
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WALLACE, Asst. Prof. A. D., Ph.D. (Virginia) Univ. of Pennsylvania, Philadelphia 4, Pa. *Sunderland Court, 35th and Powelton Ave.*
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WALLACH, ISRAEL, M.S. (New York Univ.) Teacher, Thomas Jefferson High School, Brooklyn, N.Y. *816 St. John's Place, Brooklyn 16, N.Y.*

WALLICK, E. E., Ed.M. (Temple Univ.) Head of Dept., Senior High School, Lakewood, N.J. *426 Third St.*
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WARDWELL, Asso. Prof. J. F., Ph.D. (Johns Hopkins) Colgate Univ., Hamilton, N.Y. *62 Broad St.*
WARNOCK, Asso. Prof. W. G., Ph.D. (Illinois) Univ. of Alabama, University, Ala.
WARREN, K. L., Ph.D. (Michigan State) Chm., Physical Sci., Univ. of Grand Rapids, Grand Rapids, Mich. *Lt., U.S.N.R. 1315 N. Monroe St., Hutchinson, Kans.*
WARREN, Prof. L. A. H., Ph.D. (Chicago) Actuarial Sci., Univ. of Manitoba, Winnipeg, Man., Can. *64 Niagara St.*
WASHBURNE, A. C., Grad. (U.S. Milit. Acad.) Actuary Emeritus, Berkshire Life Ins. Co., Pittsfield, Mass.
WATKEYS, Prof. C. W., A.M. (Harvard) Chm. of Dept., Univ. of Rochester, Rochester, N.Y. *287 Dartmouth St.*
WATSON, G. C., A.M. (Virginia) Instr., North Carolina State Coll., Raleigh, N.C. *Dept. of Math.*
WATT, Asso. Prof. MARTHA W., A.M. (Columbia) Emeritus, Wheaton Coll., Norton, Mass. *35 Humboldt Ave., Providence 6, R.I.*
WATTS, C. B., A.B. (Indiana) Astronomer, U.S. Naval Observatory, Washington 25, D.C.
WAYNE, ALAN, M.S. in Educ. (C.C.N.Y.) Instr., High School, New York, N.Y. *141-21 78th Road, Flushing, N.Y.*
WEAR, Asso. Prof. L. E., Ph.D. (Johns Hopkins) California Inst. of Tech., Pasadena, Calif. *2247 Lambert Drive*
WEAVER, C. L., B.S. (Kent State Univ.) Asst. Actuary, New England Mut. Life Ins. Co., 501 Boylston St., Boston, Mass.
WEAVER, WARREN, Ph.D. (Wisconsin) Dir. for the Natural Sciences, Rockefeller Foundation, New York, N.Y. *Chief, Applied Math. Panel, N.D.R.C. 160 Brite Ave., Scarsdale, N.Y.*
WEBB, Asso. Prof. D. L., Ph.D. (Calif. Inst. of Tech.) Texas Tech. Coll., Lubbock, Tex.
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WILLIAMSON, Asso. Prof. JOHN, Ph.D. (Chicago) Queens Coll., Flushing, N.Y.

WILLIS, W. A., A.B. (Johns Hopkins) Mech. Engr., Bendix Aviation Corp., Towson 4, Md. *6907 Avondale Road, Baltimore 12, Md.*

WILSON, Prof. A. H., Ph.D. (Chicago) Haverford Coll., Haverford, Pa.

WILSON, Prof. E. B., Ph.D. (Yale) Vital Statistics, Harvard Univ., 55 Shattuck St., Boston 15, Mass.

WILSON, G. H., Ph.D. (Pennsylvania) Instr., Physics, Univ. of Pennsylvania, Philadelphia, Pa. *N. Rockland Road, Merion, Pa.*

WILSON, HAZEL S. (Mrs. L. T.), Ph.D. (Cornell) Teacher, High School, Annapolis, Md. *20 Thompson St.*

WILSON, R. H., Jr., Ph.D. (Pennsylvania) Jr. Astronomer, U.S. Naval Observatory, Washington, D.C. *Lt., U.S.N.R. Instr., U.S. Naval Acad., Dept. of Math., Annapolis, Md.*

WILSON, Prof. W. A., Ph.D. (Yale) Yale Univ., New Haven, Conn. *1960 Chapel St.*

WILSON, W. E., Ph.D. (Iowa) Consulting Engr. 1452 Scott Ave., Winnetka, Ill.

WILSON, Asso. Dean W. H., Ph.D. (Illinois) Coll. of Arts and Sci., Univ. of Florida, Gainesville, Fla. *Box 2227, University Sta.*

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WOLFE, Asso. Prof. H. E., Ph.D. (Indiana) Indiana Univ., Bloomington, Ind. *812 S. Fess Ave.*

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WOLFE, R. S., A.M. (Washington) Instr., Northwestern Univ., Evanston, Ill. *Dept. of Math.*
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WOODS, C. L., M.S. (Cincinnati) Instr., Emmanuel Missionary Coll., Berrien Springs, Mich. *Route 1, Box 42*
WOODS, Prof. F. S., Ph.D. (Göttingen) Emeritus, Massachusetts Inst. of Tech., Cambridge, Mass. *123 Sumner St., Newton Centre, Mass.*
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WRAY, W. D., Ph.D. (Cornell) U.S. Navy Dept., Washington, D.C. *R.F.D. 2, Box 375, Alexandria, Va.*
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WRESTLER, FERNA E., A.M. (Kansas) Instr., Math. and Physics, Jr. Coll., El Dorado, Kans. *401 N. Atchison St.*
WRIGHT, Mrs. ALICE KELSEY, A.M. (Illinois) Asst. Prof., Southern Illinois Normal Univ., Carbondale, Ill. *804 W. Main St.*
WRIGHT, FRANCES M., A.M. (Brown) Tech. Aide, Office of Scientific Personnel, Natl. Res. Council, Washington 25, D.C.
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WRIGHT, Pres. H. N., Ph.D. (California) Coll. of City of New York, New York, N.Y. *3900 Greystone Ave., New York 63, N.Y.*
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YANNEY, Prof. B. F., Ph.D. (Chicago) Emeritus, Coll. of Wooster, Wooster, Ohio. *354 E Bowman St.*
YANOSIK, Asso. Prof. G. A., C.E. (New York Univ.) New York Univ., New York, N.Y. *52 Greenvale Ave., Yonkers, N.Y.*
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YOUNG, Asso. Prof. J. W. A., Ph.D. (Clark) Emeritus, Univ. of Chicago, Chicago, Ill. *5422 Blackstone Ave., Chicago 15, Ill.*

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YOUNG, P. M., Ph.D. (Ohio State) *Lt., U.S.N.R. Box 86, Seaman, Ohio*
YOWELL, E. I., Ph.D. (Cincinnati) Acting Dir., Observatory, Univ. of Cincinnati, Cincinnati, Ohio. *3127 Griest Ave., Cincinnati 8, Ohio*

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SANTA MONICA. Adams, Lehman, Taylor.
STANFORD UNIVERSITY.
Stanford University. Bacon, Baez, Butter, Polya, Szegö.
STOCKTON. Corbin.
VALLEJO. Becker.
WHITTIER. Pyle.

CANAL ZONE

BALBOA. McNair.

COLORADO

BOULDER. Moise.
Univ. of Colorado. Britton, Hutchinson, Kempner, Kendall, McMaster, Nelson.
COLORADO SPRINGS.
Colorado Coll. Hansman, Lovitt, Sisam.
DENVER. Goodpasture, Gysland.
Univ. of Denver. Carmichael, Gorrell, Lewis, Recht.
FORT COLLINS.
Colorado State Coll. Clark, Guard, Gunder, Hayward, Hurry, Macdonald, Madison.
GOLDEN.
Colorado School of Mines. Brown, Duffner, Everett, Fitterer, Hebel.

GREELEY. Mallory.
LORETTO. Cook.

CONNECTICUT

BLOOMFIELD. Rosenbaum.
BRIDGEPORT. Bodnar.
BRISTOL. Sabel.
EAST HARTFORD. Hall.
GLENBROOK. Fraser.
HARTFORD. Bronstein, Donchian, Elston, Goins, Keffer, Kelly.
Trinity Coll. Dadourian, Mitchell, Wyckoff.
MIDDLETOWN.
Wesleyan Univ. Arnold, Camp, Howland.
NEW BRITAIN. Fuller, Weeber.
NEW HAVEN.
Yale Univ. Begle, Dunford, Engstrom, Hille, Kovarik, Leech, Longley, Miles, Ore, Tracey, Uhler, Whittemore, Wilson.
NEW LONDON. Bower, Cole, Dimick, Kelly.
SIMSBURY. Howard.
STAMFORD. Allen.
STORRS. Cheney.
STRATFORD. Jonah, Loring, Scherberg, Wood.
WATERTOWN. Gillette.
WEST HARTFORD. Pease.
WEST HAVEN. Anderton.

DELAWARE

LEWES. Short.
NEWARK. Jones, McDougle, Rees.

DISTRICT OF COLUMBIA

WASHINGTON. Adams, Aitchison, Anderson, Avers, Barten, Berry, Blanche, Botts, Branson, Clark, Cromwell, Crow, Daly, Darling, Duerksen, Federico, Flaherty, Fleming, Fox, Fry, Gabrielle, Gleason, Goldberg, Grad, Greville, Householder, Hoy, Hyman, Johnson, Johnston, E. L. Kaplan, S. Kaplan, Kendall, Lambert, Larrivee, Lennahan, Lloyd, McCamman, McLarren, Mertie, Milleson, Moulton, Nowlan, Patterson, Pawley, Penney, Petersen, Rodabaugh, Sasuly, Schell, Shenton, Siller, Skelly, Sohon, Sollins, Spencer, Spitz, Stearn, Thomas, Tunell, Van Orstrand, Varnhorn, Watts, Wilson, Winston, Woolard, Wray, Wright.
Catholic Univ. of America. Duffie, Finan, Landry, Ramler, Rice, Wrench.
George Washington Univ. Johnston, Kullback, Mears, Quinn, Taylor, Weida.

FLORIDA

GAINESVILLE.
Univ. of Florida. Blake, Davis, Dostal, Gager, Lang, McInnis, Mead, Phipps, Pirenian, Simpson, Wilson.
LAKELAND. Reinsch.
MIAMI. Boyle.
ST. PETERSBURG. Burt, Story.
TALLAHASSEE. Clarke, Tinner.
Florida State Coll. for Women. Larson, Smith, Wade.
TAMPA. Rhodes.
WINTER PARK. Jones, Sauté.

GEORGIA

ATHENS.
Univ. of Georgia. Barrow, Beckwith, Callaway, Fort, Hill, Stephens.
ATLANTA. Neisius.
Georgia School of Tech. Bailey, Bornmann, Field, Fulmer, Hefner, Holton, Hook, Sewell, Smith, Starrett.
COLLEGEBORO. Moye.
DAHLONEGA. Barnes.
DECATUR. Gaylord, Robinson.
EMORY UNIVERSITY. Messick.
FORT VALLEY. Pitts.
MACON. Bruce, Carlton.
MARIETTA. Howe.
MILLEDGEVILLE. Nelson.
OXFORD. Moore.
ROME. Hightower, Thompson.

IDAHO

BOISE. McFarland.
CALDWELL. Rankin.
MOSCOW. Kaltenborn.

ILLINOIS

ARLINGTON HEIGHTS. Godfrey.
AURORA. Furman, Miksa.
BLOOMINGTON. Hunt.
BLUFFS. Carter.
CARBONDALE.
Southern Illinois Normal Univ. Mayor, McDaniel, Purdy, Wright.
CARTHAGE. Boatman.
CHARLESTON. Hendrix, Taylor.
CHICAGO. Bay, Boardman, Campbell, Christman, Corliss, D'Arco, Davis, Esposito, Ettinger, Georges, Gore, Hadley, Herlihy, Holland, Kaufman, Kurzin, Loch, Mansfield, Mary Esther, Moran, Nagle, Nicolet, Poppen, Putnam, Reiber, Russell, Schweitzer, Specht, Svoboda, Werkman.
Illinois Inst. of Tech. Ballard, Bibb, Christian, DeCicco, Ford, Heins, Higgins, Karlin, Krathwohl, Miller, Oldenburger, Opatowski, Perlin, Peterhans, Reingold, Rosenberg, Sadowsky, Silber, Wall, Wiegand, Wilcox.
Loyola Univ. Gerst, Mahoney, Templin.
Univ. of Chicago. Albert, Barnard, Bartky, Bliss, Dickson, Everett, Graves, Hartung, Hestnes, Jeffries, Kaplansky, Lane, Leavens, Logsdon, Lunn, Northrop, Sanger, Smith, G. J. Young, J. W. A. Young.
Wilson Junior Coll. Feltges, Kinney, Lange, Rasmusen, Sachs.
DECATUR. Coulter, Kiefer, Ploenges.
DE KALB. Hellmich, Stelford, Storm.
DIXON. West.
EUREKA. Newson.
EVANSTON.
Northwestern Univ. Broyde, Buell, Curtiss, Durfee, Givens, Hellinger, Hildebrandt, Kliphardt, Moulton, Lonseth, Olson, Reid, Simmons, Wescott, Wolfe.

FREEPORT. Baumgartner.
GALESBURG.
Knox Coll. Heren, Smyth, Stephens.
JACKSONVILLE. Hallerberg, Miller.
JOLIET. Zeller.
LAKE FOREST. Kinsman, Curtis.
LEBANON. Stowell.
LINCOLN. Balof.
MACOMB.
Western Illinois State Teachers Coll. Ayre, Ginnings, Schreiber.
MAYWOOD. Alberti, Baumgart.
MONMOUTH. Beveridge.
NORMAL.
Illinois State Normal Univ. Atkin, Bey, Flagg, McCormick, Mills, Ullsvik.
PEORIA.
Bradley Poly. Inst. Comstock, Gault, Moore.
RIVER FOREST. Dobbin.
ROCKFORD. Johnson, Varnum.
ROCK ISLAND. Cederberg, Olmsted.
SANDWICH. Rumney.
SPRINGFIELD. Harman.
TAYLORVILLE. Dappert.
URBANA. Miser, Temmer.
Univ. of Illinois. Armstrong, Bailey, Bates, Carmichael, Chanler, Coble, Crathorne, Hartley, Hattan, Hazlett, Ketchum, Levy, Miles, Miller, Moore, Pepper, Peters, Seybold, Stubbe, Tom, Vaughan.
WHEATON. Boyce, Taylor.
WINNETKA. Gaffney, Humphrey, Wilson.

INDIANA

ANDERSON, Miller, Wiley.
BLOOMINGTON.
Indiana Univ. Artin, Bohnenblust, Hennel, Meyer, Rothrock, Thomas, Weyl, Williams, Wolfe.
COLLEGEVILLE. Zanolar.
CRAWFORDSVILLE.
Wabash Coll. Carscallen, Elder, Polley.
EARLHAM. Long.
EAST CHICAGO. Burns.
ELKHART. Nicholls.
EVANSVILLE, Kronsbein, Straw.
FORT WAYNE. Smith.
GARY. Copp, Oursler.
GREENCASTLE.
DePauw Univ. Arnold, Edington, Greenleaf.
HOLY CROSS. Edward.
INDIANAPOLIS. Beal, Dotterer, McColgin, Welchons, Zieroff.
LAFAYETTE.
Purdue Univ. Ayres, Black, Bolks, Burr, Carnahan, Crain, Golomb, Graves, Hadley, Hardman, Hazard, Hodge, Hughes, Keller, Klinger, Lanczos, Leone, Miller, Niven, Reade, Robbins, Stone, Strand, Sturm, Webster.
MARION. Porter.
MUNCIE. Edwards, Shively.
NOTRE DAME.
Univ. of Notre Dame. Caparó, Landin, Menger, Milgram, Nastucoff, Pepper.
REYNOLDS. Erwin.
TERRE HAUTE. Martin, Shriner, Sousley.
UPLAND. Draper.
WEST BADEN. Muehlmann.
WHITING. Groves.

IOWA

AMES. M. M. McKelvey.
Iowa State Coll. Anderson, Bancroft, Beach, Brandner, Daniells, Davis, Gouwens, Herr, Holl, J. V. McKelvey, Robertson, Smith, Snedecor, Thielman.
CEDAR FALLS.
Iowa State Teachers Coll. Kearney, Trimble, Van Engen.
CEDAR RAPIDS.
Coe Coll. Coffin, Gaddis, Swanson.
DAVENPORT. Heyda, Hratz.
DES MOINES. Neff.
DUBUQUE. Burns, Ernsdorff.
FAYETTE. Deming.
GRINNELL. McClenon, Rusk.
HOPKINTON. Earhart.
INDIANOLA. Emmons.
IOWA CITY. Price.
Univ. of Iowa. Carpenter, Chittenden, Conkwright, Craig, Knowler, Oberg, Ward, Woods, Wylie.
IOWA FALLS. Kreider.
LE MARS. Olson.
MOUNT VERNON.
Cornell Coll. Davis, McGaw, Moots.
SIOUX CITY. Rochford.
STORM LAKE. Roorda.
WAVERLY. Chellevold.

KANSAS

ATCHISON. Obrist, Pretz, Sullivan.
BALDWIN. Garrett.
EL DORADO. Wrestler.
EMPORIA. Peterson, Tucker.
HAYS. Grabbe, Swafford.
HESSTON. Driver.
KANSAS CITY. Dougherty, Thornton.
LAWRENCE.
Univ. of Kansas. Babcock, Black, Greer, Jordan, Price, Smith, Stouffer, Ulmer, Wheeler.
LINDSBORG. Marm.
MANHATTAN.
Kansas State Coll. Babcock, Buikstra, Hadley, Hyde, Janes, Lewis, Mossman, Remick, Sigley, Stratton, White.
NORTH NEWTON. Richert.
OTTAWA. Bemmels.
PITTSBURG.
Kansas State Teachers Coll. Curfman, Shirk, Smith.
SALINA. Arnoldy.
STERLING. Bell.
TOPEKA. Eberhart, Greene, Messick.
WICHITA. Longenecker, Reagan.
Univ. of Wichita. Hanna, Hoare, Read, Wedel.
XAVIER. Ann Elizabeth.

KENTUCKY

BEREA.
Berea Coll. Hutcherson, Potor, Pugsley, Roberts, Sinclair.
BOWLING GREEN. Yarbrough.
COVINGTON. Thuener.
DANVILLE. Robinson.
GEORGETOWN. Hatfield.
LEXINGTON. Wright.
Univ. of Kentucky. Boyd, Brown, Downing, John, Latimer, LeStourgeon, Pence, South.
LOUISVILLE. Bloom, Bullitt, Fields, Ford, Morrison, Schaeffer.
Univ. of Louisville. Moore, Parker, Simester, Stevenson.
MAPLE MOUNT. Sheeran.
MOREHEAD. Fair.
MURRAY. Carman.
RICHMOND. Jenkins, Park.
WINCHESTER. Howard.

LOUISIANA

BATON ROUGE.
Louisiana State Univ. Dorroh, Freas, Karnes, Nichols, O'Quinn, Parker, Rickey, Sanders, Smith, White, Yates.
HAMMOND. Tucker.
LAFAYETTE.
Southwestern Louisiana Inst. Buchanan, Loflin, Nolan, Rees.
LAKE CHARLES. Bradford.
MONROE. Currie.
NATCHITOCHES.
Northwestern State Coll. Blair, Killen, Maddox.
NEW ORLEANS. Frankenbush, Stevens.
Tulane Univ. Beard, Buchanan, Cramer, Duren, Humphreys, Many, Riess, Spencer, Thomson, Weiss.
PINEVILLE. Temple.
RUSTON.
Louisiana Poly. Inst. Garrison, Gentry, Schroeder.
SHREVEPORT. Hardin.

MAINE

BRUNSWICK.
Bowdoin Coll. Christie, Hammond, Holmes, Korgen, Moody.
HOULTON. Morse.
LEWISTON. Ramsdell, Wilkins.
ORONO. Bryan, Kimball.
WATERVILLE. Ashcraft.

MARYLAND

ABERDEEN PROVING GROUND. Blake, Dederick, Hailperin, Hart, Maddrill, Reklis.
ANNAPOLIS. Bingley, Buchanan, Wilson.
U. S. Naval Acad. Ayres, Ball, Bleick, Bramble, Church, Clements, Currier, Dillingham, Gere, Hammond, Kells, Krabill, Lamb, Lyle, Moore, Phelps, Rawlins, Root, Scarborough.
BALTIMORE. Karl, Roman, Smith.
Goucher Coll. Bacon, Lewis, Torrey.
Johns Hopkins Univ. Bourne, Cohen, Faulkner, Morrill, Murnaghan, Reed, Zariski.
CHESTERTOWN. Rhodes.
COLLEGE PARK.
Univ. of Maryland. Good, Hall, Kennedy, Lancaster, Martin.
EMMITSBURG. Ahl, Burke.
FREDERICK. Brown.
FROSTBURG. Hallett.
SILVER SPRING. Renwick.
TOWSON. Willis.
WESTMINSTER. Spicer.
WOODSTOCK. Miller.

MASSACHUSETTS

AMHERST. Boutelle, Moore.
Amherst Coll. Esty, Finkel, Graff, Kleene.
BELMONT. Johnson.
BOSTON. Brown, Combellack, Gould, Hemenway, Hubbard, Laurentine Marie, Miller, Spear, Weaver, Wilson.
Boston Univ. Bruce, Johanson, Mode.
BROOKLINE. McCarthy.
CAMBRIDGE. Boas.
Harvard Univ. Arnold, Beatley, Birkhoff, Buck, Coolidge, Emmons, Hoskins, Huntington, Kravetz, Mac Lane, Rulon, Stone, Walsh, Widder.
Massachusetts Inst. of Tech. Barker, Clifford, Douglass, Franklin, Harvey, Maddaus, Moon, Reissner, Salem, Secada, Woods, Zeldin.
CHESNUT HILL. O'Donnell.
CHICOPEE. Madden.
GROTON. Holt, Nash.
LYNN. Oergel, Taylor.
MEDFORD.
Tufts Coll. Ferguson, Mergendahl, Ransom.
MILFORD. Dennison.
MOUNT HERMON. Lockwood.
NEW BEDFORD. Horvitz.
NORTHAMPTON. Munroe.
Smith Coll. Busemann, McCoy, Montgomery, Rambo.
NORTON. Garabedian, Watt.
PITTSFIELD. Naul, Washburne.
SOUTHBORO. Harrison.
SOUTHBRIDGE. Boeder.
SOUTH HADLEY. Johnson, Litzinger.
SWAMPSCOTT. Evans.
TYNGSBORO. Richmond.
WALTHAM. Marcou.
WELLESLEY.
Wellesley Coll. Copeland, Merrill, Russell, Stark, Young.
WESTON. Burke, Lewis.
WILLIAMSTOWN.
Williams Coll. Agard, Beer, Wells.
WORCESTER. Burns, McBrien, Melville, Wheeler.
Worcester Poly. Inst. Brown, Gay, Morley, Rice.

MICHIGAN

ALBION. Ingalls, Sleight.
ANN ARBOR. Hamilton, Williams.
Univ. of Michigan. Arena, Bookston,

Bradshaw, Churchill, Coe, Copeland, Craig, Crispin, Dwyer, Eilenberg, Field, Fischer, Goldstine, Hay, Hildebrandt, Hopkins, Jones, Kaplan, Karpinski, Love, Nyswander, Piranian, Rainich, Rainville, Rothe, Rouse, Rufus, Running, Schorling, Thrall, Wilder.
BAY CITY. Manning.
BERRIEN SPRINGS. Woods.
DETROIT. Bagby, Denton, Johnson, Mary Paula, Scibiorski, Shires.
Univ. of Detroit. Hausmann, Johnston, McCarthy, Mehlenbacher.
Wayne Univ. Baldwin, Borgman, Folley, Morrow, Nelson, Pixley, Southard.
EAST LANSING.
Michigan State Coll. Barbour, Baten, Bissinger, Frame, Grove, Herzog, Hill, Plant, Powell, Speeker, Stewart, Welmers.
FLINT. DeMoss, Raker, Shobe, Swanson.
GRAND RAPIDS. Bellardo, Warren.
HART. Burdick.
HILLSDALE. Beeler.
HOLLAND. Lampen.
IRONWOOD. Field.
KALAMAZOO. Walton.
Western Michigan Coll. Ackley, Bartoo, Blair, Butler, Cain, Everett, Lausman.
KALKASKA. Dunlap.
MARQUETTE. Spooner.
MIDLAND. Spencer.
MILFORD. McNeal.
MOUNT PLEASANT. Foust, Richtmeyer.
MUSKEGON. Conger.
YPSILANTI. Erikson, Lindquist.

MINNESOTA

COLERAINE. Davis.
COLLEGEVILLE. Danzl, Winkelmann.
DULUTH. Caton, Cothran, Lockwood, Mercedes, Morin.
GILBERT. Schey.
HIBBING. Erickson.
MINNEAPOLIS. Munro.
Univ. of Minnesota. Amundson, Brink, Brooke, Bussey, Cameron, Campaigne, Carlson, Colson, Eggers, Fattu, Fischer, Gibbens, Hart, Hartig, Jackson, Jensen, Kirchner, Koehler, Martin, McCutcheon, McEwen, Ness, Olmsted, Priester, Quaid, Saunders, Scammon, Shumway, Stigler, Thorp, Turner, Turrittin.
MOORHEAD. Anderson, Mundhjeld.
NORTHFIELD. Carlson, Gingrich, Wegner.
ROCHESTER. Hickman.
ST. JOSEPH. Muggli, Scoblic.
ST. PAUL. Bracewell, Camp, Fisher, Gibbons, Morgan, Thornton.
Coll. of St. Thomas. Bush, Godderz, Taylor.
ST. PETER. Nelson.
VIRGINIA. Hancock.
WINONA. De LaSalle, Gregory.

MISSISSIPPI

CLEVELAND. Ward.
CLINTON. Barnes.
HATTIESBURG. Johnson.
JACKSON. Babbitt, McCoy, Mitchell.
MOORHEAD. Felder.
STATE COLLEGE.
Mississippi State Coll. Murray, Ollivier, Pettis, Smith.
UNIVERSITY.
Univ. of Mississippi. Bickerstaff, Hume, Quarles.

MISSOURI

CAPE GIRARDEAU. Michel.
CLAYTON. Rosskopf.
COLUMBIA. Callaway, Cosby.
Univ. of Missouri. Blumenthal, Ewing, Ferguson, Haynes, Koken, Wahlin, Wehausen.
FAYETTE. Helton.
FULTON. Sweazey.
HANNIBAL. Foreman, Moore.
JEFFERSON CITY. Jason, Talbot.
KANSAS CITY. Cutting, Doyle, Pierson, Rosen.
MARYVILLE. Lane.
PARKVILLE. Crull.
ROLLA. Erkiletian.
ST. CHARLES. Karr.
ST. LOUIS. Callaghan, Gatewood, Gove, Marth, Van Os.
St. Louis Univ. Bock, Case, Ross.
Washington Univ. Dunkel, Liolios, Middlemiss, Rider, Roever.
SPRINGFIELD. Beasley, Finkel, H'Doubler.
WARRENSBURG. Brown, Jacobson.
WEBSTER GROVES. Clarke.

MONTANA

BOZEMAN. Hurst, Poole.
BUTTE. Smith.
GARRISON. Canning.
HELENA. Topel.
MISSOULA. Carey, Merrill.

NEBRASKA

CHADRON. Berry.
CRETE. Johnson.
DUNBAR. Westbrook.
GILEAD. Erwin.
HASTINGS. Hadlock, McDill.
LINCOLN. Gass, Howie, Ogden.
Univ. of Nebraska. Basoco, Brenke, Camp, Candy, Congdon, Cox, Gaba, Harper, Hull, Runge.
McCOOK. Perisho.
OMAHA. Bettinger.
Univ. of Omaha. Coleman, Earl, Rice.
PERU. Hill.
WAYNE. Boyce.
YORK. Feemster.

NEVADA

RENO. Beesley, Wood.

NEW HAMPSHIRE

CONCORD. Conwell.

DURHAM. Lewis, Slobin.
EXETER. Adkins, Funkhouser, Pennell.
HANOVER. Morgan.
Dartmouth Coll. Brown, Doyle, Forsyth, Mathewson, Nordstrom, Perkins, Robinson, Silverman, Wilder.
KEENE. Goodrich.
MANCHESTER. O'Leary.
PLYMOUTH. Smith.
RYE BEACH. Williams.

NEW JERSEY

CALDWELL. MacDonald.
CLIFTON. Struyk.
CONVENT STATION. Kenna.
EAST ORANGE. Nordgaard.
HIGHTSTOWN. Litterick.
HOBOKEN. Karst, Murray.
JERSEY CITY. Kopp.
LAKEWOOD. Wallick.
LAWRENCEVILLE.
Lawrenceville School. Durell, Kimball, Mikesh.
MADISON. Battin.
MAPLEWOOD. Hazeltine.
NEWARK. Littauer, Mosesson, Ott, Strock.
NEW BRUNSWICK.
Rutgers Univ. Bolton, Bunyan, Galbraith, Grant, LeLeiko, Nelson, Meder, Morris, Starke, Walter.
ORANGE. Chacalos.
PATERSON. Daugherty.
PRINCETON. Toralballa.
Inst. for Advanced Study. Alexander, Kelley, Morse, Veblen, von Neumann.
Princeton Univ. Adams, Brock, Eisenhart, Gillespie, Lefschetz, Lipsich, Rauch, Tompkins, Treiber, Tucker, Tukey, Wedderburn, Wilks.
RIVER EDGE. Coleman.
SOUTH ORANGE. Stanwick.
SUMMIT. White.
TEANECK. Rayher.
TRENTON. Shuster.
UPPER MONTCLAIR. Campbell.
New Jersey State Teachers Coll. Clifford, Davis, Fehr, Mallory.
WEST ORANGE. Edison.

NEW MEXICO

ALBUQUERQUE. Bauer.
Univ. of New Mexico. Hove, La Paz, Larsen, Rosenthal.
LAS VEGAS. Roberts, Rodgers.
PORTALES. Linscheid.
ROSWELL. Harp.
SANTA FE. Hamming, Whitman.
SOCORRO. Reece.
STATE COLLEGE.
New Mexico Coll. of A. and M.A. Branson, Heinzman, Swingle, Wells.

NEW YORK

ALBANY. Noel Marie.
New York State Coll. for Teachers. Beaver, Birchenough, Lester, Snader, Stokes.
ALFRED.
Alfred Univ. Lowenstein, Nevins, Polan, Seidlin, Whitford.
AURORA.
Wells Coll. Clement, Hollcroft, Rusk.
BROOKLYN. Braverman, Charosh, Cowles, First, Gerst, Karnow, Klewansky, Koch, Kramer-Lassar, Lavoie, Lazar, Levine, Lieber, McCarthy, Mehr, Miller, Palladino, Peters, Rush, Salkind, Shapiro, Thompson, Tolle, Waite, Wallach.
Brooklyn Coll. Borofsky, Boyer, Douglas, Fleisher, Forman, Griffin, Harkin, Hurwitz, Johnson, Karlin, Kennison, Landers, Mac Neish, Maria, Moore, Prenowitz, Richardson, Singer, Smith, Wolfe, Woodbridge.
Poly. Inst. of Brooklyn. Berry, Forray, Foster, Reagan, Whitford.
BUFFALO. Bartram, Browne, Erickson, Giese, Maloney, Podmele.
Univ. of Buffalo. Brendel, Civin, Gehman, Luippold, Montague, Pound.
CLINTON. Brown, Ferry.
ELMIRA. Suffa.
FARMINGDALE. Goodman.
FLUSHING. Bakst.
Queens Coll. Archibald, Brown, Cairns, Cope, Dean, Eaton, Feld, Raudenbush, Sard, Williamson.
FREEPORT. Levine.
GARDEN CITY. Bowden.
GENEVA. Durfee, Hubbs.
GREAT NECK. Schultz.
HAMILTON.
Colgate Univ. Aude, Munshower, Wardwell.
HAYT CORNERS. Ford.
HEMPSTEAD. Ollmann, Stabler.
HOUGHTON. Davison, Luckey.
ITHACA. Huck.
Cornell Univ. Agnew, Beinert, Carver, Curtiss, Feller, Firestone, Flexner, Gunderson, Hurwitz, Jones, Kac, Kalisch, Karapetoff, Lee, LeVeque, Moses, Rosser, Scott, Smith, Snyder, Walker.
JAMAICA. Bergstresser.
KINGS POINT. Guelpa.
LOUDONVILLE. Kuhn.
MALVERNE. Lugihbuhl.
NEW LEBANON. Pflaum.
NEW YORK. Alfieri, Berger, Berkeley, Bernard, Alfred, Boehm, Burgess, Coleman, Conlan, Crane, Darraugh, D'Atri, A. M. Ginsburg, J. Ginsburg, Gray, Grossman, Hastings, Heath, Hlavaty, Hobbs, Holzinger, Hutchinson, Jablonower, Joffe, Katz, Keeler, Kirby, Kratchik, Kubis, Lehner, Longfellow, Mandel, Mandelbrojt, McGrath, McKenna, Mirick, Molina, Moore, Nehrbas, Oehler, Phillips, Quilty, Ripandelli, Roll, Ruderman, Rutt, Schor, Schwartz, Sheridan, Skelding, Steinhaus, Stuckey, Wayne, Weaver.
Bell Telephone Labs. Fry, Gray, Harvey, Jones, MacColl, Mead, Riordan, Schelkunoff, Shewhart.
Coll. of the City of New York. Allen, Fager-

strom, Gill, Griffin, Grove, Hubert, Linehan, MacEwen, Milkman, Post, Robinson, Turner, Whitford, Wirth, Wright.
Columbia Univ. Aurora, Comer, Fite, Gentzler, Kasner, Ladue, Lewis, Mullins, Norman, Reeve, Ritt, Siceloff, Upton, Walker.
Cooper Union. Lehmann, Miller, Roth, Tanzola.
Hunter Coll. Anderson, Aroian, Bradley, Mrs. J. H. Bushey, J. H. Bushey, Cooper, Darkow, Eisele, Hill, Kutman, Landers, Rees, Simons, Tuller, Weisner, Whelan.
New York Univ. Adler, Bernstein, Cooley, Courant, Graham, Kline, Payne, Peters, Putnam, Reddick, Rehberg, Schaaf, Schlauch, Tilley, Wahlert, Yanosik.
NIAGARA FALLS. Lagerstrom, O'Connor.
NIAGARA UNIVERSITY. Banks.
ONEONTA. Callahan, Sanford.
PELHAM. Milos.
POTSDAM. Buxton, Waltz.
POUGHKEEPSIE.
Vassar Coll. Baker, Durand, Newton, Wells.
ROCHESTER. Chesna, Eastham, Foard, Harding, Merrill.
Univ. of Rochester. Atkins, Bernstein, Betz, Gale, Huff, Long, Seidel, Watkeys.
ST. BONAVENTURE. Hanhauser, Scheier.
SCHENECTADY. Poritsky, Street
Union Coll. Burkett, Fox, Morse, Snyder.
SPRINGVILLE. Harrington.
SYRACUSE.
Syracuse Univ. Carroll, Decker, Gelbart, Halmos, Harwood, Martin, Samelson, Scheffé, Taylor.
TROY.
Rensselaer Poly. Inst. Allen, Biggerstaff, Brown, Campbell, Nash, Nickol, Stilwell.
WEST POINT.
U. S. Military Acad. Farnell, Hazlewood, Jones.
WYOMING. Hartnell.

NORTH CAROLINA

CHAPEL HILL. Carroll.
Univ. of North Carolina. Brauer, Browne, Cameron, Garner, Henderson, Hickerson, Hill, Lasley, Mackie, Reynolds.
CHARLOTTE, Jones, Woodson.
DAVIDSON. McGavock, Mebane.
DURHAM.
Duke Univ. Dressel, Elliott, Gergen, Hickson, Patterson, Roberts, Thomas, Wade.
ELON COLLEGE. Hook.
GREENSBORO. Pegram.
Woman's Coll. of the Univ. of North Carolina. Barton, Lewis, Strong.
GREENVILLE. Graham, Sutherland.
GUILFORD COLLEGE. Hohn.
HIGH POINT. Adams.
MARS HILL. Howell.
RALEIGH. Downing, Harris.
North Carolina State Coll. Bullock, Cell, Levine, Miles, Strobel, Watson.
SALISBURY. Dearborn.
WILMINGTON. Peebles.
WILSON. Stark.

NORTH DAKOTA

FARGO. Smith.
GRAND FORKS. Mason, Staley.
JAMESTOWN, Jackson.
VALLEY CITY. Patterson.

OHIO

ADA. Whitted.
AKRON. Mauch, Selby.
ALLIANCE. Freese.
ATHENS.
Ohio Univ. Denbow, Marquis, Reed, Starcher.
BEREA. Annear, Stright.
BLUFFTON. Hartzler.
BOWLING GREEN. Mathias, Overman.
CHILLICOTHE. Clinton.
CINCINNATI. Antony, Lukacs, Reilly.
Univ. of Cincinnati. Barnett, Brand, Justice, Lubin, Merriman, Moore, Smith, Szász, Taylor, Yowell.
CLEVELAND. Burwell, Garvin, Green, Joliat, Johnson, Latshaw, Shanks.
Case School of Appl. Sci. Brown, Burington, Focke, Morris, Nassau, Pierce, Rinehart, Thomas, Torrance.
Fenn Coll. Brown, Kelly, Topp, Van Voorhis.
Western Reserve Univ. Boyce, Musselman, Simon.
CLEVELAND HEIGHTS. Irr.
COLUMBUS. Baker, Singer, Wildermuth.
Ohio State Univ. Albert, Bamforth, Bareis, Beatty, Blumberg, Caris, Fawcett, Helsel, Jones, Kuhn, Manson, Mickle, Morris, Radó, Rankin, Rasor, Rickard, Synge, Toops, Wylie.
DAYTON. Schraut.
DEFIANCE. MacCullough.
DELAWARE. Crane, Rowland.
GAMBIER. Bumer, MacNeille.
GRANVILLE. Ladner, Wiley.
HAMILTON. Baird.
HIRAM. Clarke.
KENT.
Kent State Univ. Brooks, Harshbarger, Manchester, Olson, Rogers, Stelson.
LAKESIDE. Wolfe.
LAURELVILLE. Reichelderfer.
MARIETTA. Bennett.
MARTINSVILLE. Eagle.
MOUNT ST. JOSEPH. Corona.
NAPOLEON. Yeager.
NEW CONCORD. Knight.
NEW LEXINGTON. Hoops.
NORTH BALTIMORE. Blackall.
NORTH CANTON. Mummery.
OBERLIN. Marie Yeaton.
Oberlin Coll. Cairns, Carr, Newsom,

Randolph, Smyth, Wagner, Vance, Yeaton.
OXFORD. Tappan.
Miami Univ. Anderson, Pollard, Spenceley, Wolfe.
PAINESVILLE. Peters, Ruddick.
SEAMAN. Young.
SPRINGFIELD. Krueger, Tripp.
TIFFIN. Menke.
TOLEDO. Koley, Mercedes.
Univ. of Toledo. Brandeberry, Dancer, Lemme, Welker.
WESTERVILLE. Glover.
WOOSTER.
Coll. of Wooster. Fobes, Hildner, Knight, Williamson, Yanney.
WILBERFORCE. Toney.
WILMINGTON. Spinks.
YELLOW SPRINGS. Astrachan.

OKLAHOMA

ADA. Heimann, Winn.
ALVA. Hall.
BARTLESVILLE. Rice.
NORMAN.
Univ. of Oklahoma. Brixey, Court, Hassler, LaFon, McFarland, Palmer, Reaves, Springer.
SHAWNEE. Doerfler, Short.
STILLWATER.
Oklahoma A. and M. Coll. Allen, Barnett, Flanders, Smith, Zant.
TULSA. Argue, Doll, Duncan, Ellis, Shreve, Veatch.

OREGON

CORVALLIS.
Oregon State Coll. Beaty, Hammer, Milne, Sobczyk, Williams.
EUGENE.
Univ. of Oregon. DeCou, Moursund, Peterson, Wood.
FOREST GROVE. Price.
MCMINNVILLE. Ramsey.
PORTLAND. Griffin, Merriss, Rosenbaum.
SALEM. Luther.

PENNSYLVANIA

ALLENTOWN. Billig, Kunkel.
Muhlenberg Coll. Deck, Keck, Koehler.
ANNVILLE. Black.
ASHLEY. Davis.
BEAVER FALLS. Cleland,.
BETHLEHEM. Ashbaugh, Rader.
Iehigh Univ. Celauro, Cutler, Pitcher, Raynor, Reynolds, Shook, Smail, Van Arnam.
BRISTOL. Downing.
BRYN ATHYN. Allen.
BRYN MAWR. Atkinson, Innis.
Bryn Mawr Coll. Kalish, Lehr, Oxtoby, Wheeler.
CARLISLE. Ayres, Stuart.
CHAMBERSBURG. Johnson.
CHESTER. Helms, Vedova.
COLLEGEVILLE.
Ursinus Coll. Clawson, Dennis, Manning.
EASTON.
Lafayette Coll. Benner, Cawley, Hatch, Smith.
EAST PITTSBURGH. Epstein.
ELIZABETHTOWN. Heilman.
ERIE. Benedicta, Kraus, Sullivan.
GETTYSBURG. Clutz.
GREENSBURG. McNeil.
GROVE CITY. Carpenter.
HAVERFORD.
Haverford Coll. Allendoerfer, Oakley, Wilson.
HERSHEY. Haag.
HUNTINGDON. Stayer.
KUTZTOWN. Knedler.
LANCASTER. Marburger, Murray.
LATROBE. Noriega, Seubert.
LEWISBURG.
Bucknell Univ. Gold, MacCreadie, Miller, Richardson.
LOCK HAVEN. Smith.
LORETTO. Mino.
MEADVILLE.
Allegheny Coll. Hurd, Smith, Steen.
MILLERSVILLE. Boyer.
NEW WILMINGTON. Black.
OXFORD. Haviland.
PHILADELPHIA. Brown, Campbell, Cavalli, Constable, Davis, DeCleene, Eggert, Fine, Fudge, Hearn, Johnson, Latshaw, Levy, Lotkin, McDonough, McNeary, Russ, Slepin, Smith, Walker.
Univ. of Pennsylvania. Caris, Evans, Gottschalk, Kline, Safford, Schoenberg, Wallace, Wilson.
PICTURE ROCKS. Price.
PITTSBURGH. Arnold, Briant, Buker, Calkins, Frankel, Harmon, Jamison, Leifer, Mullan, Wells.
Carnegie Inst. of Tech. Dines, Hoover, Johnson, Moskovitz, Neelley, Olds, Rosenbach, Saibel, Smith, Whitman.
Univ. of Pittsburgh. Blumberg, Bryson, Foraker, Hovey.
PLEASANTVILLE. Kerr.
READING. Speicher.
SHENANDOAH. Bauser.
SCRANTON. Bertrand, Daniel.
SLIPPERY ROCK. Lady.
STATE COLLEGE.
Pennsylvania State Coll. Cohen, Cunningham, Curry, Dunlap, Frink, Gordon, Gravatt, Graves, Hagen, Harrington, Herpel, Hutchison, Johnson, Krall, F. W. Owens, H. B. Owens, Rupp, Schwartz, Sheffer.
SWARTHMORE. Schwartz.
Swarthmore Coll. Brinkmann, Dresden, Fried, Marriott, Walton.
UPPER DARBY. Houghton.
WASHINGTON.
Washington and Jefferson Coll. Bert, Dorwart, Shaub, Thomas.
WAYNESBURG. Moston.
YORK. Baker.

RHODE ISLAND

KINGSTON.
Rhode Island State Coll. Bender, Brown, Stauffer.
NEWPORT. Chase.
PROVIDENCE. McKenney, McMurtrie.
Brown Univ. Adams, Archibald, Bennett, Carlen, Gaskell, Gilman, Manning, Richardson, Rosenbloom, Saltzer, Schecter, Schmidt, Smiley, Western.

SOUTH CAROLINA

CHARLESTON.
The Citadel. Dye, Hair, Reves.
CLEMSON. Stanley.
CLINTON. Spencer.
COLUMBIA.
Univ. of South Carolina. Dinkines, Jackson, Novak, Shuler, Williams.
GREENVILLE. Blackwell.
GREENWOOD. Templeton.
HARTSVILLE. Reaves.
NEWBERRY. Gaver.
ROCK HILL. Stokes.
SPARTANBURG. Pettis.

SOUTH DAKOTA

BROOKINGS.
South Dakota State Coll. MacDougal, Walder, Wente.
HURON. Cramer.
MITCHELL. Knox.
RAPID CITY. Swanson.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.
VERMILLION.
Univ. of South Dakota. Bedwell, Ekman, Howell, Sealander.

TENNESSEE

CHATTANOOGA. Hughes, Massey, Mays.
COOKEVILLE. Hutchinson, Moorman.
FOUNTAIN CITY. Keller.
HARROGATE. Bowling.
JEFFERSON CITY. Sloan.
JOHNSON CITY. Carson.
KNOXVILLE. Parker.
Univ. of Tennessee. Cooley, Eaves, Ficken, Gillis, Lee, Pepper, Pollard, Purviance.
MARYVILLE. Sisk.
MEMPHIS. Coker, Locke.
NASHVILLE. Boswell, Mrs. W. L. Miser, Van Horn, Wren.
Vanderbilt Univ. Blair, Hyden, Lundberg, W. L. Miser, Morrell, Tierney.

TEXAS

ABILENE. Burnam, Mullings, Tate.
AMARILLO. Layton, Whetstone.
ARLINGTON. Howard.
AUSTIN.
Univ. of Texas. Anderson, Batchelder, Coleman, Craig, Decherd, Ettlinger, Greenwood, Lubben, Moore, Vandiver.
BELTON. LaRoe.
BROWNSVILLE. de la Garza.
CANYON. Murray.
CENTER POINT. Rees.
COLLEGE STATION.
A. and M. Coll. of Texas. Basye, Blumberg, Edmonson, Klipple, Luther, McCulley, Moore, Pinkerton, Wapple.
DALLAS. Sorrells, Stulken, Thomas.
Southern Methodist Univ. Huff, Mouzon, Starr.
DENTON. White, Willey.
North Texas State Teachers Coll. Beeman, Brown, Cooke, Hanson.
EL PASO. Schwid.
FORT WORTH.
Texas Christian Univ. Morgan, Ramsey, Sherer.
HOUSTON. Blau, Howe, Pennington, Slotnick.
Rice Inst. Bray, Dean, Leighton, Lovett, Newhouse, Ulrich.
HUNTSVILLE. Querry.
KINGSVILLE. Kennedy.
LUBBOCK. Cross, Parker.
Texas Tech. Coll. Heineman, May, Michie, Rowland, Sparks, Thompson, Underwood, Webb, Woodward, Whyburn.
PRAIRIE VIEW. Stephens.
SAN ANGELO. Bright.
SAN ANTONIO. Mary of Mercy, McNelly, Morgan, Newton, Schnepp.
STEPHENVILLE. McSweeny, Redden.
TEAGUE. Notley.
WACO. Hedberg.
WICHITA FALLS. Adams.

UTAH

LOGAN. Bird.
ROYAL. Calvert.
SALT LAKE CITY.
Univ. of Utah. Biesele, Bridger, Hayes, Henriques, Horsfall, Pehrson, Thorne.

VERMONT

BURLINGTON.
Univ. of Vermont. Bullard, Butterfield, Millington, Swift.
MIDDLEBURY.
Middlebury Coll. Ballou, Bowker, Hazeltine, Perkins.
NORTHFIELD. Dix.
RANDOLPH. Alliot.
SAXONS RIVER. Belding.

VIRGINIA

ARLINGTON. Getchell.
ASHLAND. Blincoe, Simpson.
BLACKSBURG.
Virginia Poly. Inst. Hatcher, McFadden, O'Shaughnessy.
BUENA VISTA. Durham.
CHARLOTTESVILLE. Patten.
Univ. of Virginia. Aylor, Hedlund, Linfield, McShane, Oglesby, Utz, Whyburn.
ETTRICK. Hunter.
FARMVILLE. Taliaferro.
FREDRICKSBURG. Frick.
HAMPTON. Perkins.
HARRISONBURG. Ikenberry.
HOLLINS. Falvey, Meade.
LANGLEY FIELD. Peiser.

LEXINGTON. Paxton, Smith.
Virginia Military Inst. Byrne, Knox, Purdie.
LYNCHBURG.
Randolph-Macon Woman's Coll. Baker, Larew, Wiggin.
MIDDLEBURG. Keppler.
NEWPORT NEWS. Raine.
NORFOLK. Norris.
RICHMOND. Drew.
Univ. of Richmond. Babcock, Gaines, Grable, Harris, Wheeler.
SALEM. Carpenter.
STAUNTON. Taylor.
SWEET BRIAR. Morenus.
WILLIAMSBURG.
Coll. of William and Mary. Calkins, Gregory, Phalen, Smith, Stetson.

WASHINGTON

ABERDEEN. Porter.
CHENEY. Bell.
EVERETT. Van Arkel.
LACEY. Cebula.
PULLMAN.
State Coll. of Washington. Butler, Hacker, Knebelman, Neményi.
SEATTLE. Beegle, Frederickson, Nordhaus.
Univ. of Washington. Ballantine, Beaumont, Cramlet, Haller, Jerbert, McFarlan, Mullemeister, Winger, Zuckerman.
SPOKANE. Carlson.
TACOMA. de Regt, Eves.
WALLA WALLA. Stewart.
YAKIMA. Whitney.

WEST VIRGINIA

HURRICANE. Smith.
MONTGOMERY. Reckzeh.
MORGANTOWN.
West Virginia Univ. Davis, Eiesland, Peters, Reynolds, Turner, Vehse, Vest.
RONCEVERTE. Bauserman.
WEST LIBERTY. Kiplinger.
WHEELING. Meyer.

WISCONSIN

APPLETON. Berry.
BELOIT. Conwell, Huffer.
CUDAHY. Sedlak.
EAU CLAIRE. Otteson.
LA CROSSE. Adkins, Malin.
MADISON.
Univ. of Wisconsin. Allen, Andree, Arnold, Bruck, Chessin, Evans, Hart, Ingraham, Langer, MacDuffee, March, Rose, Simpson, Sokolnikoff, Spitzbart, Trump, Wagner.
MILWAUKEE. Beckwith, Bigelow, Boehmer, Clark, London, Mary Felice, Mary Gertrude, Norris.
Marquette Univ. Kennedy, Luteyn, Moeller, Otis, Pettit, Wilczewski.
Univ. of Wisconsin in Milwaukee. Bardell, Battig, Kenney, Marden, Parkinson, Vass, Wolf.
OSHKOSH. Beenken, Price.
PLATTEVILLE. Harrell.
RACINE. Jautz.
RIVER FALLS. Eide.
ST. FRANCIS. Fetterer.
SUPERIOR. Flogstad, Smith.
WAUKESHA. Dancey, Hopkins.
W ST ALLIS. Fitzpatrick.
WHITEFISH. Anderson.

WYMOMING

LAMONT. Bellamy.
LARAMIE.
Univ. of Wyoming. Alden, Barr, Neubauer, Rechard, Varineau.

CANADA

AURORA. Lane.
EDMONTON.
Univ. of Alberta. Campbell, Cook, Sheldon.
FREDERICTON. Miller.
HAMILTON. Findlay.
KINGSTON. Bradley.
Queen's Univ. Blyth, Jeffery, Mendelsohn, Miller.
LONDON. Cole, Kingston.
MONTREAL. Gauthier, Gough, Lalonde, Pelletier, Perry.
McGill Univ. Pall, Rosenthall, Wood.
OTTAWA. Dubé, Keyfitz.
QUEBEC. Pouliot, Roland, Tremblay.
SACKVILLE. Crawford, McEwen.
SASKATOON. Ferns.
TORONTO. Dobson, Grant.
Univ. of Toronto. Beatty, Brauer, Burk, Coxeter, Pounder, Robinson.
VANCOUVER.
Univ. of British Columbia. Buchanan, Gage, James, Jennings, Murdoch, Nowlan.
VICTORIA. Wallace.
WINNIPEG. Warren.
WOLFVILLE. Macphail.

FOREIGN MEMBERS

ARGENTINA

BUENOS AIRES. Baidaff, Barral-Souto.

BELGIUM

UCCLE. Errera.

BRITISH HONDURAS

BELIZE. Zimmerman.

BRITISH WEST INDIES

JAMAICA. Ball.

CHILE

SANTIAGO. Moreno.

CHINA

CANTON. MacDonald.

CUBA

HAVANA. Aleman, Gonzáles, Rodríguez.

ECUADOR

QUITO. Thullen.

ENGLAND

CAMBRIDGE. Hardy.
CHIPPING NORTON. O'Hara.
ENGLEFIELD GREEN. McCrea.
LONDON. Dalal, Todd.

FINLAND

HELSINGFORS. Ahlfors.

FRANCE

TENNIE. Thébault.

HUNGARY

BUDAPEST. Arany.

INDIA

BANGALORE. Rao.
SURAT. Shah.

IRELAND

DUBLIN. Broderick.

ITALY

BOLOGNA. Bortolotti.
NAPLES. Crudeli.
ROME. Enriques, Laboccetta.

MEXICO

MEXICO. Nápoles.

NETHERLANDS

THE HAGUE. Spruitenburg.

NEW ZEALAND

DUNEDIN. Martyn.

PANAMA

PANAMA CITY. Linares.

PERU

LIMA. Puga.

PORTUGAL

LISBON. Caraça.

ROUMANIA

TIMISOARA. Sergescu.

RUSSIA

MOSCOW. Kryloff.

SOUTH AFRICA

BLOEMFONTEIN. Arndt.

STRAITS SETTLEMENTS

SINGAPORE. Oppenheim.

SWITZERLAND

FRIBOURG. Bays.
NEUCHATEL. DuPasquier.

SYRIA

BEIRUT. Jurdak.

TURKEY

ISTANBUL. Lanckton, Miller.

URUGUAY

MONTEVIDEO. Calcagno.

VENEZUELA

CARACAS. Michalup.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)
(As amended to January 1, 1946)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission, endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by this membership in constituencies (hereinafter called "Regions") established by the Board.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Region shall elect biennially a Governor for a term of two years. Nominations shall be made by the Section or Sections of the Association existing within the Region, or, in the absence of such Sections, by a committee appointed for that purpose by the Governor representing the Region.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nominations by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members in the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the Board. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Board shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for eaching meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each member shall be Four Dollars ($4), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of one dollar.

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION

(The periods were for the calendar years except that after 1942 the terms of Regional Governors began and ended July 1.)

Honorary President for Life

H. E. Slaught, December 1933–May 1937

Presidents

E. R. Hedrick....1916
Florian Cajori....1917
E. V. Huntington....1918
H. E. Slaught....1919
D. E. Smith....1920
G. A. Miller....1921
R. C. Archibald....1922
R. D. Carmichael....1923
H. L. Rietz....1924
J. L. Coolidge....1925
Dunham Jackson....1926
W. B. Ford....1927–1928
J. W. Young....1929–1930
E. T. Bell....1931–1932
Arnold Dresden....1933–1934
D. R. Curtiss....1935–1936
A. J. Kempner....1937–1938
W. B. Carver....1939–1940
R. W. Brink....1941–1942
W. D. Cairns....1943–1944

Vice-Presidents

E. V. Huntington....1916
G. A. Miller....1916
D. N. Lehmer....1917, 1918
Oswald Veblen....1917
J. W. Young....1918, 1926
R. G. D. Richardson....1919
H. L. Rietz....1919
Helen A. Merrill....1920
E. J. Wilczynski....1920
R. C. Archibald....1921
R. D. Carmichael....1921, 1922
B. F. Finkel....1922
A. B. Chace....1923
L. P. Eisenhart....1923
J. L. Coolidge....1924
Dunham Jackson....1924, 1925
A. A. Bennett....1925, 1933, 1934
W. B. Ford....1926
A. J. Kempner....1927, 1928, 1935
Clara E. Smith....1927
F. D. Murnaghan....1928, 1939
E. T. Bell....1929, 1930
W. C. Graustein....1929, 1930, 1940
Arnold Dresden....1931
C. N. Moore....1931
W. H. Bussey....1932
G. C. Evans....1932
E. B. Stouffer....1933
E. P. Lane....1934
L. L. Dines....1935
N. A. Court....1936
T. C. Fry....1936
T. H. Hildebrandt....1937
E. J. Moulton....1937, 1938
H. E. Buchanan....1938
W. L. Hart....1939
R. W. Brink....1940
B. H. Brown....1941–1942
R. E. Langer....1941
Tomlinson Fort....1942–1943
C. C. MacDuffee....1943–1944
W. M. Whyburn....1944–1945

Secretary-Treasurer

(Appointed by the Board after 1918)

W. D. Cairns....1916–1942

Committee on Official Journal

(Appointed by the Board. Discontinued after 1939)

H. E. Slaught....1916–1937
R. D. Carmichael....1916–1918
W. H. Bussey....1916–1918, 1926–1931
R. C. Archibald....1919–1921
W. A. Hurwitz....1919–1921
A. A. Bennett....1922
H. P. Manning....1921–1922
W. B. Ford....1923–1925
J. L. Coolidge....1923
A. J. Kempner....1924–1939
W. B. Carver....1932–1936, 1937–1939
E. J. Moulton....1937–1939

Editors-in-Chief after 1939

E. J. Moulton....1940–1941

Additional Members of the Board

D. N. Lehmer....1916–1918, 1922–1924, 1930–1932
R. E. Moritz....1916–1918
K. D. Swartzel....1916
Oswald Veblen....1916, 1920–1922, 1926–1931
R. C. Archibald....1916–1917, 1923–1930
Florian Cajori....1916, 1918–1923, 1926–1930
M. B. Porter....1916–1917
J. W. Young....1916–1917, 1920–1922
B. F. Finkel....1916–1921, 1930–1935
E. H. Moore....1916–1921, 1923–1928
Alexander Ziwet....1916–1918

E. R. HEDRICK.......1917–1922, 1924–1929
1932–1937
J. N. VAN DER VRIES...........1916–1918
HELEN A. MERRILL............1917–1919
D. E. SMITH.........1917–1919, 1921–1926
1937–1939
ELIZABETH B. COWLEY..........1918–1920
G. A. MILLER........1918–1920, 1922–1924
E. J. WILCZYNSKI....1918–1919, 1922–1926
L. P. EISENHART.....1919–1922, 1925–1930
E. V. HUNTINGTON........1917, 1919–1927
1933–1935
E. L. DODD....................1920
R. D. CARMICHAEL........1920, 1924–1929
1939–1941
A. A. BENNETT..1921, 1930–1932, 1939–1941
H. L. RIETZ.........1921–1923, 1925–1930
1934–1936
C. F. GUMMER.................1921–1925
DUNHAM JACKSON..............1923–1929
CLARA E. SMITH...............1923–1925
A. B. CHACE..................1924–1925
J. L. COOLIDGE................1926–1931
E. T. BELL....................1927–1928
E. P. LANE....................1928–1933
W. B. FORD....................1929–1934
E. R. SMITH...................1929
W. L. HART....................1930–1935
LAO G. SIMONS.................1930–1931
L. L. DINES...................1931–1933
T. C. FRY.....................1931–1933
J. W. GLOVER..................1931–1933
H. E. BUCHANAN...............1932–1937
W. R. LONGLEY......1932–1934, 1936–1938
E. J. MOULTON......1933–1936, 1943–1945
R. W. BRINK...................1934–1939
D. R. CURTISS..1934, 1937–1939, 1940–1942
J. L. WALSH...................1934–1936
ARNOLD DRESDEN....1935–1940, 1943–1945
J. O. HASSLER.................1935–1936
F. D. MURNAGHAN.............1935–1937
G. C. EVANS...................1936–1941
MARY EMILY SINCLAIR..........1936–1938
J. M. THOMAS.......1937–1939, 1940–1941
MARIE J. WEISS...............1937–1938
WILLIAM BETZ.................1938–1940
A. B. COBLE...................1938–1940
J. H. WEAVER..................1938–1940
PHILIP FRANKLIN..............1940–1942
F. L. GRIFFIN.................1940–1942
MAYME I. LOGSDON.............1940–1942
G. T. WHYBURN................1940–1942
C. V. NEWSOM..................1940–1941
O. J. PETERSON................1940–1941
F. B. WILEY...................1940–1941
H. M. BACON...................1941–1943
H. J. ETTLINGER...............1941–1943
CORNELIUS GOUWENS............1941–1943
W. C. KRATHWOHL..............1941–1943
E. J. MCSHANE.................1941–1943
F. W. OWENS...................1941–1943
S. T. SANDERS.................1941–1943
W. L. AYRES...................1942–1944
R. L. WILDER..................1942–1944
SAUNDERS MAC LANE...........1943–1945
R. P. AGNEW...................1942–1944
L. M. BLUMENTHAL.............1942–1944
W. F. CHENEY, JR..............1942–1944
C. G. LATIMER.................1942–1944
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O. H. RECHARD.................1942–1944
H. A. ROBINSON................1942–1944
H. E. BRAY....................1943–1945
D. W. HALL....................1943–1945
C. G. JAEGER..................1943–1945
A. L. NELSON..................1943–1945
W. V. PARKER..................1943–1945
K. W. WEGNER..................1943–1945

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